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On non-nested multilevel solvers for the Stokes and Navier-Stokes equations

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Abstract

We present a general convergence analysis of multilevel solvers for finite element discretisations of mixed problems. As a basic new feature, our approach allows to use different finite element discretisations on each level of the multigrid hierarchy. Particularly, our theory enables to combine accurate higher order finite element discretisations on the finest level with low order finite element discretisations on all other levels. In this way, an accurate solution can be obtained for whose computation the efficiency of multilevel solvers for low order discretisations is exploited. The advantages of our approach are demonstrated on numerical results obtained for the incompressible Navier–Stokes equations.

Keywords: Multilevel method, finite element discretisation, mixed problems, Stokes equations, Navier–Stokes equations

1991 Mathematics subject classifications: 65N55, 65N30, 76D07

Introduction

Multigrid methods are well known as efficient solvers for discretisations of partial differential equations. Particularly, for finite element discretisations of selfadjoint elliptic partial differential equations, a convergence theory of multigrid methods using nested conforming finite element spaces is very well established; see for example the books [5, 14, 30] and the bibliographies therein. In the present paper, we focus our attention on the application of multigrid solvers to finite element discretisations of the Stokes and Navier–Stokes

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equations. First, we present a general theory of multilevel solvers for mixed problems (e.g., the Stokes equations) which allows to use non–nested spaces and to apply different discretisations on different levels of the multilevel hierarchy. Then, in a second part of the paper, we apply the ideas developed for mixed problems to the numerical solution of the incompressible Navier–Stokes equations.

The present paper extends and generalises former results on multigrid methods for elliptic problems [2, 14], non–conforming finite element approximations [1, 2, 4, 6, 7, 8, 9, 10, 11, 12, 19, 31] and mixed problems [1, 3, 7, 19, 26, 31, 33]. The generality of our approach makes it possible to develop new types of multilevel solvers which can be more efficient than the standard multigrid solvers. Particularly, we can use different finite element spaces on each level of the multigrid hierarchy. The motivation for using this type of multilevel solvers comes from general experiences that standard multigrid solvers are very efficient for low order discretisations. On the other hand, to obtain accurate numerical solutions, it is advisable to use higher order discretisations for which, however, standard multigrid solvers might be much less efficient. That leads to the idea of using the higher order discretisation on the highest level only and a low order discretisation on all other levels. This approach will be called multiple discretisation multilevel method. In this way, an accurate solution can be obtained for whose computation the efficiency of multilevel solvers for low order discretisations is exploited.

Since our theory enables the use of several discretisations which correspond to different levels but which are defined on the same geometrical grid, the number of levels in the multilevel hierarchy is generally larger than the number of the geometrical grids. For example, in the multiple discretisation multilevel method presented in Section 1.8, the number of levels is higher by 1 than the number of grids. Therefore, we use the notion 'multilevel solver' and not 'multigrid solver'.

The plan of the paper is as follows. In Section 1, we investigate the convergence properties of a multilevel method for solving mixed finite element discretisations in an abstract way. First, we formulate the Stokes equations as an example of a mixed method and then we introduce the abstract variational problem and the corresponding discrete problem. Then we describe the matrix representation, the prolongation operators and the smoother. Finally, the convergence of a two–level method and the W–cycle of a multilevel method is established. The general construction of prolongation operators satisfying the assumptions of our theory will be described briefly. We close Section 1 with the presentation of the multiple discretisation multilevel method. This method, applied to the Stokes equations, fits into the analysis presented in this section. In Section 2, we compare the standard multigrid method and the multiple discretisation multilevel method in the numerical solution of the 3d steady state incompressible Navier–Stokes equations describing the flow through a channel around a column with squared cross section. We obtained two main observations in the numerical tests. First, it is much more efficient and robust to use both types of multilevel methods as preconditioner in a flexible GMRES method than to use them as solver. Second, the multiple discretisation multilevel method behaves much better than the standard multigrid method if a finite element discretisation with continuous pressure approximation is used.

Throughout this paper, we denote by C a 'universal' constant which is independent of the mesh size and the level but whose value can differ from place to place.

1 The multilevel approach for mixed problems

1.1 The Stokes equations

The Stokes equations

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \qquad \nabla\cdot\mathbf{u} = 0 \qquad \text{in } \Omega$$

describe the velocity $\mathbf{u}$ and the pressure p of an incompressible viscous fluid contained in a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, under the assumption that non–linear convective effects can be neglected. A more realistic model are the incompressible Navier–Stokes equations (2.1). Nevertheless, the Stokes equations play an important role both in the theory and the numerical solution of the incompressible Navier–Stokes equations. Let us mention that, in the above equations, $\nu > 0$ is the kinematic viscosity of the fluid and $\mathbf{f}$ is an outer volume force.

Assuming that Ω is a bounded domain with Lipschitz continuous boundary and that, for simplicity,

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega,$$

we can introduce a weak formulation of the Stokes equations. To this end, we define the spaces $V = H_0^1(\Omega)^d$ and $Q = L_0^2(\Omega)$, and the bilinear forms

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\ \mathrm{d}\mathbf{x}, \qquad b(\mathbf{v},p) = -\int_\Omega p\nabla\cdot\mathbf{v}\ \mathrm{d}\mathbf{x}.$$

Now, the functions $\mathbf{u}$ and p are called a weak solution of the Stokes equations if $\mathbf{u} \in V$, $p \in Q$ and

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= \mathbf{f}(\mathbf{v}) && \forall\, \mathbf{v} \in V, \\ b(\mathbf{u},q) &= 0 && \forall\, q \in Q, \end{aligned}$$

where $\mathbf{f}$ is considered as a continuous linear functional on V, i.e., $\mathbf{f} \in V'$. It is well known [13] that the weak formulation is uniquely solvable and that the weak solution continuously depends on $\mathbf{f}$.

Note that, denoting $H = L^2(\Omega)^d$, we have $V \subset H \subset V'$, where the embeddings are continuous and dense. It is an example of the so–called Gelfand triple.

In what follows, we shall develop a multilevel theory for an abstract variational problem which includes the above weak formulation of the Stokes equations as a special case. This generalisation makes clear what assumptions are really necessary for the theoretical investigations and allows to apply our theory also to other mixed problems.

1.2 The abstract variational problem and its discretisation

Let V and Q be two Hilbert spaces and let $V \subset H \subset V'$ be the Gelfand triple. Given the symmetric bilinear form $a : V \times V \to \mathbb{R}$, the bilinear form $b : V \times Q \to \mathbb{R}$ and a functional $f \in V'$, we look for a solution $(u,p) \in V \times Q$ of

$$a(u,v) + b(v,p) = f(v) \qquad \forall\, v \in V, \tag{1}$$

$$b(u,q) = 0 \qquad \forall\, q \in Q. \tag{2}$$

We assume that

(H1) (Solvability) For all $f \in V'$, the problem (1), (2) admits a unique solution $(u,p) \in V \times Q$ with
$$\|u\|_V + \|p\|_Q \le C\|f\|_{V'}.$$

To define a discretisation of the above variational problem, we introduce sequences $V_l \subset H$ and $Q_l \subset Q$, $l = 0, 1, \ldots$, of (possibly non–conforming) finite element spaces approximating V and Q, respectively. Instead of the bilinear forms a and b, we use their discrete analogues $a_l : V_l \times V_l \to \mathbb{R}$ and $b_l : V_l \times Q_l \to \mathbb{R}$, respectively. We again assume that the bilinear forms a_l are symmetric. For $f \in H$ the discrete problem corresponding to (1), (2) reads:

Find $(u_l, p_l) \in V_l \times Q_l$ such that
$$a_l(u_l, v_l) + b_l(v_l, p_l) = (f, v_l) \qquad \forall\, v_l \in V_l, \tag{3}$$
$$b_l(u_l, q_l) = 0 \qquad \forall\, q_l \in Q_l, \tag{4}$$

where $(.,.)$ denotes the inner product in H.
We assume that

(H2) (Solvability and convergence) The problem (3), (4) admits a unique solution and the error estimate
$$\|u - u_l\|_H \le C h_l^2 \|f\|_H \tag{5}$$
holds, where h_l characterises how fine is the finite element mesh on which V_l and Q_l are defined.

This assumption is satisfied for the Stokes equations from Section 1.1 if the spaces V_l and Q_l possess certain approximation properties and satisfy an inf–sup condition, and if the boundary of Ω is of class C^2 or Ω is a plane convex polygon, see [13] for details.

1.3 The matrix representation

Let $\{\varphi_{l,i} : i \in I_l\}$ and $\{\psi_{l,j} : j \in J_l\}$ be bases of the spaces V_l and Q_l, respectively, where I_l, J_l denote the corresponding index sets. The unique representations
$$u_l = \sum_{i \in I_l} u_{l,i}\varphi_{l,i}\ , \qquad p_l = \sum_{j \in J_l} p_{l,j}\psi_{l,j}$$
define the finite element isomorphisms $\Phi_l : \mathcal{U}_l \to V_l$, $\Psi_l : \mathcal{P}_l \to Q_l$ between the vector spaces $\mathcal{U}_l = \mathbb{R}^{\dim V_l}$, $\mathcal{P}_l = \mathbb{R}^{\dim Q_l}$ of coefficient vectors $\underline{u}_l = (u_{l,i})_{i \in I_l}$, $\underline{p}_l = (p_{l,j})_{j \in J_l}$ and the finite element spaces V_l and Q_l, respectively. We introduce the finite element matrices A_l and B_l having the entries $a_{l,ij} = a_l(\varphi_{l,j}, \varphi_{l,i})$ and $b_{l,ij} = b_l(\varphi_{l,j}, \psi_{l,i})$. Now the discrete problem (3), (4) is equivalent to
$$\begin{pmatrix} A_l & B_l^T \\ B_l & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_l \\ \underline{p}_l \end{pmatrix} = \begin{pmatrix} \underline{f}_l \\ 0 \end{pmatrix} \tag{6}$$
with $f_{l,i} = (f, \varphi_{l,i})$. Note that A_l is a symmetric matrix. In the vector spaces $\mathcal{U}_l$ we will use the usual Euclidean norms scaled by suitable factors such that the following norm equivalence
$$C^{-1}\|\underline{v}_l\|_{\mathcal{U}_l} \le \|v_l\|_H \le C\|\underline{v}_l\|_{\mathcal{U}_l} \qquad \forall\, v_l \in V_l \tag{7}$$
is satisfied with a constant C independent of l.

1.4 The prolongation operators

Since generally $V_{l-1} \not\subset V_l$ and $Q_{l-1} \not\subset Q_l$, we have to introduce suitable prolongation operators $\iota_{u,l} : V_{l-1} \to V_l$ and $\iota_{p,l} : Q_{l-1} \to Q_l$. It turns out that the convergence analysis requires estimates for $\iota_{u,l}$ on the sum $V_{l-1} + V_l$. Thus, we define $\iota_{u,l} : \Sigma_l \to V_l$ on the (possibly larger) space Σ_l with $V_{l-1} + V_l \subset \Sigma_l \subset H$. We assume that the following properties of the mapping $\iota_{u,l}$ hold:

(H3) $\qquad \iota_{u,l}\, v = v \quad \forall v \in V_l,$

(H4) $\qquad \|\iota_{u,l}\, v\|_H \leq C\|v\|_H \quad \forall v \in \Sigma_l.$

For the results proven below, we do not need any assumption on $\iota_{p,l}$.

1.5 The smoother

We assume that a smoother $\mathcal{S}_l : V_l \times Q_l \times H \to V_l \times Q_l$ is given which should smooth the error of an approximate solution of the discrete problem (3), (4). We denote by $\mathcal{S}_l^m : V_l \times Q_l \times H \to V_l \times Q_l$ an operator representing m successive applications of $\mathcal{S}_l$, i.e., given two functions $u_l^0 \in V_l$, $p_l^0 \in Q_l$, the functions $(u_l^m, p_l^m) := \mathcal{S}_l^m(u_l^0, p_l^0, f_l)$ are defined by

$$(u_l^{j+1}, p_l^{j+1}) := \mathcal{S}_l(u_l^j, p_l^j, f_l), \quad j = 0, \ldots, m-1.$$

We assume that

(H5) (Smoothing property) There exists a constant C independent of l such that, for any functions $u_l^0 \in V_l$, $p_l^0 \in Q_l$ and any integer m, the functions $(u_l^m, p_l^m) := \mathcal{S}_l^m(u_l^0, p_l^0, f_l)$ satisfy

$$\|A_l(\underline{u}_l - \underline{u}_l^m) + B_l^T(\underline{p}_l - \underline{p}_l^m)\|_{\mathcal{U}_l} \leq \frac{C}{m} h_l^{-2} \|\underline{u}_l - \underline{u}_l^0\|_{\mathcal{U}_l}. \tag{8}$$

(H6) Denoting $(\widetilde{u}_l, \widetilde{p}_l) := \mathcal{S}_l(u_l^0, p_l^0, f_l)$ with arbitrary $u_l^0 \in V_l$, $p_l^0 \in Q_l$, we have

$$b_l(\widetilde{u}_l, q_l) = 0 \qquad \forall\, q_l \in Q_l.$$

An example of a smoother having the above properties is the so–called Braess–Sarazin type smoother [3, 26, 33]. For this smoother, one smoothing step of the form $(u_l^{j+1}, p_l^{j+1}) := \mathcal{S}_l(u_l^j, p_l^j, f_l)$ can be written, using the matrix representation from Section 1.3, as the basic iteration

$$\begin{pmatrix} \alpha_l D_l & B_l^T \\ B_l & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_l^{j+1} - \underline{u}_l^j \\ \underline{p}_l^{j+1} - \underline{p}_l^j \end{pmatrix} = \begin{pmatrix} \underline{f}_l \\ 0 \end{pmatrix} - \begin{pmatrix} A_l & B_l^T \\ B_l & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_l^j \\ \underline{p}_l^j \end{pmatrix}, \tag{9}$$

where α_l is a positive constant and the matrix D_l is a pre–conditioner of A_l such that the linear system (9) is more easily solvable than (6). The system (9) is uniquely solvable if D_l is symmetric and positive definite and $B_l B_l^T$ is non–singular. The matrix $B_l B_l^T$ is non–singular if the bilinear form b_l satisfies an inf–sup condition for the spaces V_l and Q_l. The following lemma formulates sufficient assumptions for the validity of the smoothing property (H5).

Lemma 1.1 *Let α_l be chosen such that*

$$\frac{1}{\delta}\lambda_{max}(A_l) < \alpha_l \lambda_{min}(D_l) \leq \alpha_l \|D_l\| \leq \gamma \lambda_{max}(A_l)$$

for some constants $\delta \in [1,2)$ and $\gamma > 0$ independent of l. Moreover, let the basis $\{\varphi_{l,i} : i \in I_l\}$ be chosen such that $\lambda_{max}(A_l) = \mathcal{O}(h_l^{-2})$. Then, the basic iteration (9) satisfies the smoothing property (8).

Proof. See [3, 26, 33]. ■

1.6 The two–level method

In this section, we analyse the two–level method for solving the discrete problem (3), (4) on level l using a coarse level correction on level $l-1$, $l \geq 1$.
Let $(u_l^0, p_l^0) \in V_l \times Q_l$ be an initial guess for the solution (u_l, p_l) of (3), (4). We apply m smoothing steps and obtain

$$(u_l^m, p_l^m) := \mathcal{S}_l^m(u_l^0, p_l^0, f_l).$$

Now, a correction for this new guess is computed on level $l-1$ by solving the problem

Find $(u_{l-1}^*, p_{l-1}^*) \in V_{l-1} \times Q_{l-1}$ such that

$$\begin{aligned} a_{l-1}(u_{l-1}^*, v_{l-1}) + b_{l-1}(v_{l-1}, p_{l-1}^*) &= (f, \iota_{u,l}\, v_{l-1}) - a_l(u_l^m, \iota_{u,l}\, v_{l-1}) \\ &\quad - b_l(\iota_{u,l}\, v_{l-1}, p_l^m) \qquad \forall\, v_{l-1} \in V_{l-1}, \\ b_{l-1}(u_{l-1}^*, q_{l-1}) &= 0 \qquad\qquad\qquad\qquad\qquad\qquad \forall\, q_{l-1} \in Q_{l-1}. \end{aligned}$$

Finally, the new approximation is obtained by

$$u_l^{new} := u_l^m + \iota_{u,l}\, u_{l-1}^*, \qquad p_l^{new} := p_l^m + \iota_{p,l}\, p_{l-1}^*.$$

To prove that this algorithm converges if it is repeated sufficiently many times, we will establish the so–called approximation property which describes the effect of the coarse–level correction. The basic idea for proving the approximation property comes from Braess and Verfürth [4] and consists in introducing the auxiliary problem

Find $(z, w) \in V \times Q$ such that

$$\begin{aligned} a(z,v) + b(v,w) &= (F_l, v) \qquad\qquad \forall v \in V, \\ b(z,q) &= 0 \qquad\qquad\qquad \forall q \in Q, \end{aligned}$$

where $F_l \in \Sigma_l$ is the Riesz representation of the residue defined by

$$(F_l, s) := (f, \iota_{u,l}\, s) - a_l(u_l^m, \iota_{u,l}\, s) - b_l(\iota_{u,l}\, s, p_l^m) \quad \forall s \in \Sigma_l.$$

Then, in view of (H3) and (H6), (u_{l-1}^*, p_{l-1}^*) and $(u_l - u_l^m, p_l - p_l^m)$ are discrete solutions of the auxiliary problem in the spaces $V_{l-1} \times Q_{l-1}$ and $V_l \times Q_l$, respectively.

Lemma 1.2 *Let $h_{l-1} \le Ch_l$ with a constant C independent of l. Then, the approximation property*

$$\|u_l - u_l^{new}\|_H \le Ch_l^2 \|A_l(\underline{u}_l - \underline{u}_l^m) + B_l^T(\underline{p}_l - \underline{p}_l^m)\|_{\mathcal{U}_l}$$

holds.

Proof. Applying (H3), (H4), the triangular inequality, and (5), we get

$$\begin{aligned} \|u_l - u_l^{new}\|_H &= \|u_l - (u_l^m + \iota_{u,l}\, u_{l-1}^*)\|_H = \|\iota_{u,l}\,(u_l - u_l^m - u_{l-1}^*)\|_H \\ &\le C\|u_l - u_l^m - u_{l-1}^*\|_H \le C(\|u_l - u_l^m - z\|_H + \|z - u_{l-1}^*\|_H) \\ &\le C(h_l^2 + h_{l-1}^2)\|F_l\|_H. \end{aligned} \tag{10}$$

It remains to state a relation between the residue $F_l \in \Sigma_l$ and the 'algebraic' residue $\underline{d}_l \in \mathcal{U}_l$ given by

$$\underline{d}_l := A_l(\underline{u}_l - \underline{u}_l^m) + B_l^T(\underline{p}_l - \underline{p}_l^m).$$

For this, we first consider the Riesz representation of the residue in V_l, i.e., $r_l \in V_l$ defined by

$$(r_l, v_l) := (f, v_l) - a_l(u_l^m, v_l) - b_l(v_l, p_l^m) \qquad \forall v_l \in V_l.$$

Since

$$\begin{aligned} (r_l, v_l) &= a_l(u_l - u_l^m, v_l) + b_l(v_l, p_l - p_l^m) \\ &= \Big(A_l(\underline{u}_l - \underline{u}_l^m) + B_l^T(\underline{p}_l - \underline{p}_l^m), \underline{v}_l\Big)_{\mathcal{U}_l} = (\underline{d}_l, \underline{v}_l)_{\mathcal{U}_l} \end{aligned}$$

for all $v_l \in V_l$, we have

$$(\underline{d}_l, \underline{v}_l)_{\mathcal{U}_l} = (r_l, v_l) = (r_l, \Phi_l \underline{v}_l) = (\Phi_l^* r_l, \underline{v}_l)_{\mathcal{U}_l} \qquad \forall \underline{v}_l \in \mathcal{U}_l,$$

from which the representation $\underline{d}_l = \Phi_l^* r_l$ follows. Now, using

$$\|F_l\|_H^2 = (F_l, F_l) = (r_l, \iota_{u,l}\, F_l) = ((\Phi_l^*)^{-1}\underline{d}_l, \iota_{u,l}\, F_l) = (\underline{d}_l, \Phi_l^{-1}\iota_{u,l}\, F_l)_{\mathcal{U}_l},$$

we get from (7) and (H4)

$$\|F_l\|_H^2 \le \|\underline{d}_l\|_{\mathcal{U}_l}\, \|\Phi_l^{-1}\iota_{u,l}\, F_l\|_{\mathcal{U}_l} \le C\|\underline{d}_l\|_{\mathcal{U}_l}\, \|\iota_{u,l}\, F_l\|_H \le C\|\underline{d}_l\|_{\mathcal{U}_l}\, \|F_l\|_H.$$

Dividing by $\|F_l\|_H$, we obtain

$$\|F_l\|_H \le C\|\underline{d}_l\|_{\mathcal{U}_l},$$

which together with (10) yields the statement of the lemma. ■

Combining the approximation property with the smoothing property, we get

Theorem 1.1 *Let $h_{l-1} \le Ch_l$ with a constant C independent of l. Then the two–level method converges for sufficiently many smoothing steps with respect to the H- and $\mathcal{U}_l$-norm. In particular, we have*

$$\|\underline{u}_l - \underline{u}_l^{new}\|_{\mathcal{U}_l} \le \frac{C_1}{m}\|\underline{u}_l - \underline{u}_l^0\|_{\mathcal{U}_l}, \qquad \|u_l - u_l^{new}\|_H \le \frac{C_2}{m}\|u_l - u_l^0\|_H, \tag{11}$$

where the constants C_1, C_2 are independent of l.

Proof. Taking into consideration the norm equivalence (7) and Lemma 1.2, we get

$$\|\underline{u}_l - \underline{u}_l^{new}\|_{\mathcal{U}_l} \leq C\|u_l - u_l^{new}\|_H \leq Ch_l^2\|A_l(\underline{u}_l - \underline{u}_l^m) + B_l^T(\underline{p}_l - \underline{p}_l^m)\|_{\mathcal{U}_l}.$$

Applying (H5), we obtain the first inequality in (11). The convergence in the H-norm follows from the norm equivalence (7). ∎

Note that in our context the notation 'two–level' does not necessarily mean that we consider two levels of mesh refinements. Thus, Theorem 1.1 is also applicable in cases where we have two different discretisations on the same mesh.
Once proven the convergence of the two–level method, the convergence of the W–cycle multilevel method follows in a standard way (see e.g. [2], [14]). This is also true for a combination of a finite number of different two-level algorithms provided that Theorem 1.1 holds true for each of these two–level methods.

1.7 Remarks on the construction of the prolongation operators

The above convergence results were derived under the assumptions (H1)–(H6). In the preceding sections, we have given examples how to satisfy (H1), (H2), (H5) and (H6) and now we would like to mention for which spaces V_l there exists a prolongation operator $\iota_{u,l}$ satisfying the assumptions (H3) and (H4). This question was already thoroughly investigated in [17] and therefore we shall present only the main ideas here. We shall consider $H = L^2(\Omega)^d$, like in Section 1.1.
We assume that we are given a family $\{\mathcal{T}_l\}_{l\geq 0}$ of triangulations of the domain Ω. Each triangulation consists of a finite number of mutually disjoint simply connected open cells K and we assume that, for any $l > 0$, the triangulation $\mathcal{T}_l$ is obtained from $\mathcal{T}_{l-1}$ by some 'refinement', i.e., each cell $K \in \mathcal{T}_{l-1}$ is either a member of $\mathcal{T}_l$ or it has been refined into child cells $K_i \in \mathcal{T}_l$. In particular, we allow $\mathcal{T}_{l-1} = \mathcal{T}_l$, which gives us the additional possibility to define different finite element spaces V_{l-1} and V_l on the same mesh. All cells are assumed to be shape–regular images of a finite number of reference cells under suitable one–to–one mappings. In addition, we suppose that the ratio of the diameters of any two neighbouring cells is bounded independently of l.
On the reference cells, we define finite–dimensional spaces which are mapped onto the cells of $\mathcal{T}_l$. On each cell $K \in \mathcal{T}_l$, we then introduce a finite–dimensional space $\mathbb{P}^l(K)$ which is a subspace of the mapped space. To specify degrees of freedom, we introduce an index set I_l whose elements are called nodes. These nodes may represent, e.g., vertices or edges of cells of $\mathcal{T}_l$. We denote by $I_l(K)$ the set of all nodes which are associated with a cell K. Finally, on each space $\mathbb{P}^l(K)$, we introduce linear functionals $\mathcal{N}_{K,i}$, $i \in I_l(K)$ (the so–called local nodal functionals) which are assumed to be unisolvent with the space $\mathbb{P}^l(K)$.
The spaces V_l are defined as usual as sets of all functions which belong to $\mathbb{P}^l(K)$ on each cell $K \in \mathcal{T}_l$ and which are 'continuous' in the sense of the equality of the corresponding local nodal functionals. Thus, we have

$$V_l = \big\{v \in L^2(\Omega)^d \,:\, v|_K \in \mathbb{P}^l(K) \quad \forall K \in \mathcal{T}_l, \\ N_{K,i}(v|_K) = N_{K',i}(v|_{K'}) \quad \forall K, K' \in \mathcal{T}_{l,i}, i \in I_l\big\}\,,$$

where $\mathcal{T}_{l,i}$ is the set of all cells $K \in \mathcal{T}_l$ sharing the node i. We denote by $\{\varphi_{l,i}\}_{i\in I_l}$ a basis of the space V_l and, for each cell K, we assume that we have $\varphi_{l,i}|_K = 0$ for $i \notin I_l(K)$ and that the functions $\{\varphi_{l,i}|_K\}_{i\in I_l(K)}$ form a basis of $\mathbb{P}^l(K)$ which is dual to the local nodal functionals, i.e.,

$$N_{K,i}(\varphi_{l,j}|_K) = \delta_{ij} \qquad \forall\, i,j \in I_l(K)\,,$$

where δ_{ij} denotes the Kronecker symbol. Finally, we assume that the functions $\varphi_{l,i}|_K$ can be represented as linear combinations, with coefficients bounded by 1, of functions obtained by transforming fixed functions defined on the respective reference cell onto the cell K. Note that usual finite element spaces V_l used for solving the Stokes equations possess all the above properties.

The definition of the spaces V_l shows that a natural choice for a global nodal functional is the arithmetic mean of the local nodal functionals

$$N_{l,i}(v) = \frac{1}{\mathrm{card}(\mathcal{T}_{l,i})} \sum_{K\in\mathcal{T}_{l,i}} N_{K,i}(v|_K).$$

Note that we again have the duality relation

$$N_{l,i}(\varphi_{l,j}) = \delta_{ij} \qquad \forall\, i,j \in I_l\,,$$

which implies that

$$v = \sum_{i\in I_l} N_{l,i}(v)\,\varphi_{l,i} \qquad \forall\, v \in V_l\,. \tag{12}$$

Now, let us turn to the question how the spaces Σ_l and the mappings $\iota_{u,l} : \Sigma_l \to V_l$ satisfying

$$V_{l-1} + V_l \subset \Sigma_l \subset H = L^2(\Omega)^d \tag{13}$$

and the assumptions (H3) and (H4) can be defined. Following the ideas developed in [29], we construct Σ_l as a discontinuous finite element space

$$\Sigma_l = \left\{v \in L^2(\Omega)^d \,:\, v|_K \in \mathbb{S}^l(K) \quad \forall K \in \mathcal{T}_l\right\},$$

where $\mathbb{S}^l(K)$ are finite–dimensional spaces satisfying

$$\mathbb{P}^{l-1}(\mathcal{F}(K))|_K + \mathbb{P}^l(K) \subset \mathbb{S}^l(K) \tag{14}$$

with $\mathcal{F}(K) \in \mathcal{T}_{l-1}$ being the parent cell of $K \in \mathcal{T}_l$ (we set $\mathcal{F}(K) = K$ for $K \in \mathcal{T}_l \cap \mathcal{T}_{l-1}$). Then (13) holds. We suppose that, for any $K \in \mathcal{T}_l$, the local nodal functionals $N_{K,i}$ are well defined on $\mathbb{S}^l(K)$, which usually means that the functions in $\mathbb{S}^l(K)$ are of the same type as those in $\mathbb{P}^l(K)$ (e.g., continuous). Then, we can define the transfer operator $\iota_{u,l} : \Sigma_l \to V_l$ simply by

$$\iota_{u,l}\, v = \sum_{i\in I_l} N_{l,i}(v)\,\varphi_{l,i}\,, \qquad v \in \Sigma_l\,.$$

In view of (12), we immediately obtain the validity of (H3). To assure (H4), we assume that each of the spaces $\mathbb{S}^l(K)$ can be obtained by transforming a fixed space of functions from one of the reference cells onto K. Together with this reference space, we also transform its basis functions and we require that the value of any local nodal functional $N_{K,i}$ on any of these transformed basis functions is bounded independently of K and l. The last assumption is automatically satisfied if the local nodal functionals $N_{K,i}$ are

defined by means of a finite number of functionals defined on the reference spaces, which is often the case.
A significant step in the above considerations was the assumption that there exist spaces $\mathbb{S}^l(K)$ obtained by transforming some reference spaces onto K and satisfying (14). Let us mention that this is always true for simplicial finite elements, see [17].

1.8 The multiple discretisation multilevel method

We will present now a multilevel method for saddle point problems arising from higher order finite element discretisations – the so–called multiple discretisation multilevel method, see Figure 1. This multilevel approach fits into the framework of the analysis presented in this section if, for example, it is applied to the Stokes equations and smoothers of Braess–Sarazin type are used. In this case, the analytical results have been confirmed by numerical studies in [17].

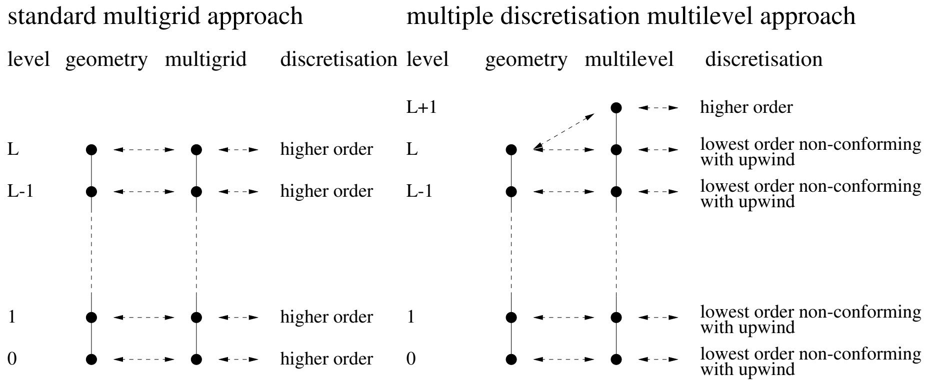

Figure 1: The standard and the multiple discretisation multilevel approach for higher order finite element discretisations of the Navier–Stokes equations.

In the numerical studies presented in this paper, we will focus on the saddle point problems which arise in the linearisation and discretisation of the stationary incompressible Navier–Stokes equations. It is known, e.g., see [18, 16], that higher order finite element discretisations might improve the accuracy of the computed solutions sometimes enormously in comparison to lowest order non–conforming finite element discretisations. This is not only true for norms of the velocity or pressure errors but also for coefficients which are relevant in applications, like drag or lift. But the saddle point problems which arise from higher order finite element discretisations are sometimes hard to solve by the standard coupled multigrid method, e.g., see also [18, 16]. In contrast, good experiences are available for coupled multigrid methods applied to saddle point problems coming from stabilised lowest order non–conforming finite element discretisations, e.g., see [15, 19]. Thus, it is a straightforward idea to construct a multilevel method for higher order finite element discretisations which is based on a multigrid method for stabilised non–conforming finite element discretisations of lowest order.
In the multiple discretisation multilevel method depicted in Figure 1, there are two discretisations applied on the finest geometric grid level L. One of them is the discretisation of interest, e.g., a higher order discretisation. This discretisation forms the highest level

$L+1$ of the multiple discretisation multilevel method. The second discretisation is a lowest order non–conforming finite element discretisation which is stabilised if necessary. This discretisation builds level L of the multiple discretisation multilevel method. On all coarser geometric grids, only a lowest order non–conforming discretisation (with stabilisation) is applied.

2 Comparison of the standard and the multiple discretisation multilevel method for the Navier–Stokes equations

2.1 The definition of the problem

We consider the flow through a channel around a column with a square cross section, see Figure 2. This example was defined as a benchmark problem within the DFG high priority research program *Flow Simulation with High–Performance Computers* by Schäfer and Turek [27].
The Navier–Stokes equations are given by

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega, \end{aligned}$$

where $\nu = 10^{-3}m^2/s$ and $\mathbf{f} = \mathbf{0}$ in the present example. The height of the channel is $H = 0.41m$ and the length of the edges of the square cross section is $D = 0.1m$. The inflow boundary conditions

$$\mathbf{u} = \begin{pmatrix} 16Uyz(H-y)(H-z)/H^4 \\ 0 \\ 0 \end{pmatrix} \qquad \text{at } x = 0,$$

with $U = 0.45m/s$ are prescribed and outflow boundary conditions (do–nothing boundary conditions) at $x = 2.5$. The flow has the Reynolds number 20 based on ν, D and the mean inflow $\overline{U} = 0.2m/s$.
The Navier–Stokes equations are linearised by a fixed point iteration. Given $(\mathbf{u}^n, p^n)$, the next iterate is computed by solving

$$\begin{aligned} -\nu\Delta\mathbf{u}^{n+1} + (\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1} + \nabla p^{n+1} &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u}^{n+1} &= 0 \quad \text{in } \Omega. \end{aligned} \tag{15}$$

All iterates of the velocity fulfil the boundary conditions. System (15) is discretised by a mixed finite element method leading to a saddle point problem of the form

$$\mathcal{A}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{16}$$

Experiences show that it is, in general, not necessary and even not advisable to solve (16) very accurately for an efficient solution of the Navier–Stokes equations. We applied the inf–sup stable, conforming pairs of finite element spaces Q_2/P_1^{disc} and Q_2/Q_1 in the computations to discretise (15) by the Galerkin discretisation. The only, but crucial,

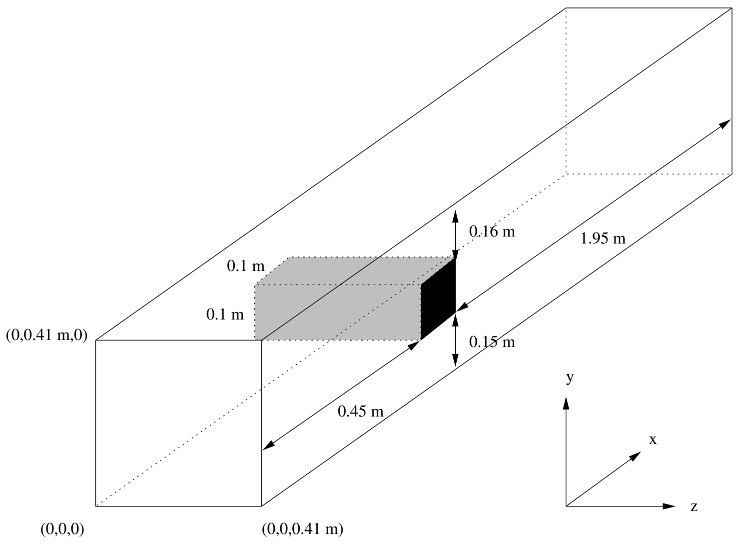

Figure 2: Channel with a square cross section.

difference of these spaces is that the former approximates the pressure by a discontinuous function and the latter by a continuous function. We use so–called mapped finite elements. That means, everything is defined on a reference mesh cell and the finite element in Ω is given by an appropriate reference map. The inf–sup condition for the mapped Q_2/P_1^{disc} finite element has been proven by Matthies and Tobiska [21] and the interpolation property has been studied by Matthies [20].
The initial grid (level 0) used in the computations is presented in Figure 3. This grid is refined uniformly.
One goal of the definition of the benchmark problem was a comparison of the computed drag and lift coefficient and the difference of the pressure in the points $(0.45, 0.2, 0.205)$ at the front of the column and the point $(0.55, 0.2, 0.205)$ at the back of the column. We give the results for these values obtained with the Q_2/P_1^{disc} finite element discretisation in Table 11. They fit well into the benchmark intervals defined in [27].

level	d.o.f.	drag	lift	pressure difference
0	15960	8.2560	8.9249e-2	1.8750e-1
1	117360	7.9767	8.0304e-2	1.7875e-1
2	899040	7.8644	7.3593e-2	1.7638e-1
3	7035840	7.8194	7.1251e-2	1.7598e-1
4	55666560	7.7960	7.0151e-2	1.7587e-1
reference interval		[7.7,7.9]	[6e-2,7e-2]	[1.72e-1,1.80e-1]

Table 11: Drag coefficient, lift coefficient and pressure difference computed with the Q_2/P_1^{disc} finite element discretisation.

In this section, we will compare the standard multigrid method and the multiple discretisation multilevel method presented in Figure 1. Both methods are applied as solvers for

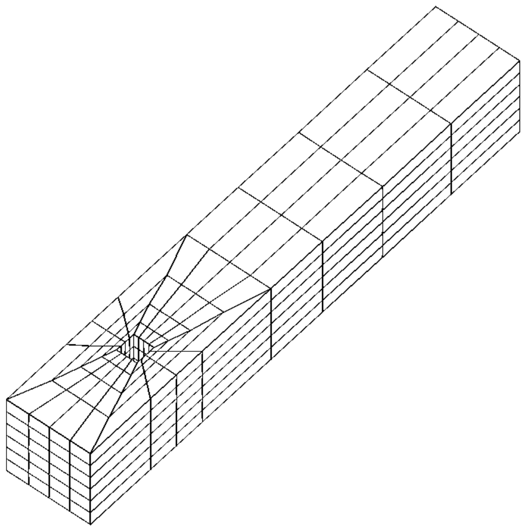

Figure 3: Initial grid (level 0).

(16) and also as preconditioner in the right preconditioned flexible GMRES (FGMRES) method by Saad [24, 25]. As lowest order non–conforming finite element discretisation, we use the point value oriented Q_1^{rot}/Q_0 pair of finite element spaces, see Rannacher and Turek [22] or Schieweck [28]. This finite element discretisation is stabilised with Samarskij upwinding, e.g., see Roos, Stynes and Tobiska [23].

2.2 The Vanka type smoothers

In contrast to the analysis in Section 1, smoothers are used in the computations which are based on the solution of local saddle point problems – so–called Vanka type smoothers, [32]. The reason for not using smoothers of Braess–Sarazin type is that we are not aware of any efficient method for solving the global saddle point problems (9) arising in each smoothing step. A comparison of Vanka type and Braess–Sarazin type smoothers in [19] shows the higher efficiency of Vanka type smoothers in coupled multigrid methods for lowest order non–conforming finite element discretisations of the stationary Navier–Stokes equations.

We will describe the Vanka type smoothers in more detail. They can be considered as block Gauss–Seidel methods. Let $\mathcal{V}^h$ and $\mathcal{Q}^h$ be the set of velocity and pressure degrees of freedom, respectively. These sets are decomposed into

$$\mathcal{V}^h = \cup_{j=1}^J \mathcal{V}_j^h, \quad \mathcal{Q}^h = \cup_{j=1}^J \mathcal{Q}_j^h. \tag{17}$$

The subsets are not required to be disjoint.

Let $\mathcal{A}_j$ be the block of the matrix $\mathcal{A}$ which is connected with the degrees of freedom of $\mathcal{W}_j^h = \mathcal{V}_j^h \cup \mathcal{Q}_j^h$, i.e. the intersection of the rows and columns of $\mathcal{A}$ with the global indices belonging to $\mathcal{W}_j^h$,

$$\mathcal{A}_j = \begin{pmatrix} A_j & B_j^T \\ B_j & 0 \end{pmatrix} \in \mathbb{R}^{\dim(\mathcal{W}_j^h)\times\dim(\mathcal{W}_j^h)}.$$

Similarly, we denote by $(\cdot)_j$ the restriction of a vector on the rows corresponding to the degrees of freedom in $\mathcal{W}_j^h$. Each smoothing step with a Vanka type smoother consists in a loop over all sets $\mathcal{W}_j^h$, where for each $\mathcal{W}_j^h$ a local system of equations connected with the degrees of freedom in this set is solved. The local solutions are updated in a Gauss–Seidel manner. The (full) Vanka smoother computes new velocity and pressure values for the degrees of freedom in $\mathcal{W}_j^h$ by

$$\begin{pmatrix} u \\ p \end{pmatrix}_j := \begin{pmatrix} u \\ p \end{pmatrix}_j + \mathcal{A}_j^{-1} \left(\begin{pmatrix} f \\ g \end{pmatrix} - \mathcal{A} \begin{pmatrix} u \\ p \end{pmatrix} \right)_j .$$

The general strategy for choosing the sets $\mathcal{V}_j^h$ and $\mathcal{Q}_j^h$ is as follows. First, pick some pressure degrees of freedom which define $\mathcal{Q}_j^h$. Second, $\mathcal{V}_j^h$ is defined by all velocity degrees of freedom which are connected with the pressure degrees of freedom from $\mathcal{Q}_j^h$ by non–zero entries in the matrix B.

We have applied two types of Vanka smoothers with respect to this strategy. The first one is called mesh cell oriented, because $\mathcal{Q}_j^h$ is defined by all pressure degrees of freedom which are connected to the mesh cell j. For this type of Vanka smoother, J coincides with the number of mesh cells. The mesh cell oriented Vanka smoother is applied only for the discretisation with discontinuous pressure approximation, i.e. for Q_2/P_1^{disc}. In this case, $\mathcal{V}_j^h$ consists of all velocity degrees of freedom which are connected to the same mesh cell j. This property is not given for the discretisation with continuous pressure approximation. For Q_2/Q_1, we use a decomposition in which $\mathcal{Q}_j^h$ is defined by a single pressure degree of freedom, $\dim \mathcal{Q}_j^h = 1$. This smoother is called pressure node oriented Vanka smoother. For this smoother, the number of subsets J in decomposition (17) is equal to the number of pressure degrees of freedom.

Using discontinuous discrete pressure, the local matrix of the mesh cell oriented Vanka smoother can be generated on the current mesh cell. A local system for the Q_2/P_1^{disc} finite element discretisation consists of 81 velocity degrees of freedom and 4 pressure degrees of freedom such that its total size is 85. These systems are solved by Gaussian elimination with column pivoting.

The size of the local systems for the pressure node oriented Vanka smoother for the Q_2/Q_1 finite element is, in general, 376. Only for pressure nodes on $\partial\Omega$, the systems are smaller. We solve the local systems approximately be GMRES. The iteration stops after having reduced the Euclidean norm of the residual by the factor 10. This strategy has been proven to be more efficient than Gaussian elimination in [16].

Since the Vanka type smoothers are block Gauss–Seidel methods, their performance depends on the ordering of the degrees of freedom. We did not study this dependence in our tests. In the computations, we ordered the degrees of freedom, with possibly few exceptions, from the inflow to the outflow.

In the pressure node oriented Vanka smoother for the Q_2/Q_1 finite element, it became necessary to damp the result of the smoother iteration to obtain convergence. Let (u_l, p_l) be the current iterate on the multigrid level l and $(\delta u_l, \delta p_l)$ be the update computed by one iteration of the smoother. Then, the new iterate is computed by $(u_l, p_l) + \omega_l(\delta u_l, \delta p_l)$. In the standard multigrid approach, we have chosen $\omega_l = 0.8$, $0 \le l \le L$, and in the multiple discretisation multilevel method $\omega_l = 1$, $0 \le l \le L$, $\omega_{L+1} = 0.8$.

The systems on the coarsest grid, level 0, are solved approximately by the same iterative scheme which is used as smoother. The iteration is stopped either after having reduced the Euclidean norm of the residual by the factor 10 or after 100 iterations.

2.3 The evaluation of the numerical results

We will present results for level $L = 3$ for the Q_2/P_1^{disc} finite element discretisation and for $L = 2$ for the Q_2/Q_1 finite element discretisation. Tables 12 and 13 give information on the number of degrees of freedom on different levels for both types of multilevel methods.

	Q_2/P_1^{disc}			Q_2/Q_1		
l	velocity	pressure	total	velocity	pressure	total
0	14040	1920	15960	14040	700	14740
1	102000	15360	117360	102000	4680	106680
2	776160	122880	899040	776160	34000	810160
3	6052800	983040	7035840			

Table 12: Degrees of freedom in the standard multigrid method.

	Q_2/P_1^{disc}			Q_2/Q_1		
l	velocity	pressure	total	velocity	pressure	total
0	4920	480	5400	4920	480	5400
1	36960	3840	40800	36960	3840	40800
2	286080	30720	316800	286080	30720	316800
3	2250240	245760	2496000	776160	34000	810160
4	6052800	983040	7035840			

Table 13: Degrees of freedom in the multiple discretisation multilevel method.

The initial iterate on the geometric grid level L is chosen to be the interpolation of the solution of level $L - 1$ in each test. We apply two stopping criteria for the iterative solution of the linear system (16) in each step of the fixed point iteration for solving the Navier–Stokes equations. Either, the iteration is stopped after the reduction of the Euclidean norm of the residual vector by the factor 10 or maximal 10 iterations (multigrid or FGMRES) are performed. The fixed point iteration itself is stopped if the Euclidean norm of the residual vector is less than 10^{-10}.
The efficiency of the solvers will be evaluated by computing times. Of course, the computing time is influenced by many factors which are computer and compiler dependent, but it is the most important measure in applications. All computations were performed on a computer with HP PA-RISC 8500 processor (440 MHz, 1760 Mflops/s peak). To simplify the evaluation of the results given in Tables 14 and 15, we emphasise the fastest computing time in each table and all other computing times which are not longer than 110% of the fastest time.
In the column 'mg cyc.' of Tables 14 and 15, we give the number of fixed point iterations to solve the non–linear system and the total number of iterations of the linear solver within these fixed point iterations. The abbreviation 'div.' indicates that the iteration diverges and 'no conv.' means that the stopping criterion of the fixed point iteration is not fulfilled after 50 steps. The computing times are given in seconds.

					FGMRES +			
	standard mg		mult. disc. ml		standard mg		mult. disc. ml	
cycle	mg cyc.	time	mg cyc.	time	mg cyc.	time	mg cyc.	time
$V(1,1)$	7/30	25089	no conv.	-	7/25	20969	8/62	44517
$V(2,2)$	7/17	27281	9/67	94396	7/17	27271	9/45	63533
$V(3,3)$	7/15	35586	8/44	92545	7/14	33213	8/36	75792
$F(1,1)$	7/19	20070	8/35	26193	7/18	*18798*	7/24	*18028*
$F(2,2)$	7/13	25045	8/24	35332	7/13	25003	7/18	26493
$F(3,3)$	7/11	30579	8/20	43866	7/11	30707	8/19	41658
$W(1,1)$	7/19	21282	8/33	24917	7/18	19960	7/24	*18205*
$W(2,2)$	7/13	26018	8/23	34021	7/13	25873	8/20	29654
$W(3,3)$	7/11	31739	8/20	44023	7/11	31678	8/19	41904

Table 14: Q_2/P_1^{disc} finite element discretisation, mesh cell oriented Vanka smoother.

For the Q_2/P_1^{disc} finite element discretisation, the fastest computing times are obtained with FGMRES, the multiple discretisation multilevel method $F(1,1)$– and $W(1,1)$–cycle as preconditioner and the standard multigrid method $F(1,1)$–cycle as preconditioner. The application of both multilevel methods as preconditioner in FGMRES is more efficient than using them as solver. Using the multilevel methods as solvers, the standard approach is superior to the multiple discretisation multilevel approach. Most of the solvers work satisfactorily, only the solvers with the multiple discretisation multilevel V–cycles behave worse than the other ones. Altogether, for the solution of the saddle point problems arising in the discretisation with discontinuous pressure approximation, none of both multilevel approaches was significantly better than the other one.

					FGMRES +			
	standard mg		mult. disc. ml		standard mg		mult. disc. ml	
cycle	mg cyc.	time	mg cyc.	time	mg cyc.	time	mg cyc.	time
$V(1,1)$	20/200	148063	no conv.	-	10/97	75376	12/116	72957
$V(2,2)$	div.	-	12/118	146402	10/96	140016	10/91	117474
$V(3,3)$	div.	-	11/107	206982	10/96	205560	10/91	170729
$F(1,1)$	no conv.	-	11/110	71586	10/93	82399	10/87	*55141*
$F(2,2)$	div.	-	10/94	121771	9/87	146979	10/85	107218
$F(3,3)$	div.	-	11/105	197389	9/86	222055	10/85	165312
$W(1,1)$	no conv.	-	10/100	65310	10/93	83755	10/86	*56201*
$W(2,2)$	div.	-	10/93	121603	9/87	147237	10/86	112092
$W(3,3)$	div.	-	11/105	204296	9/86	222373	10/86	162324

Table 15: Q_2/Q_1 finite element discretisation, pressure node oriented Vanka smoother.

The computing times using the Q_2/Q_1 finite element discretisation, Table 15, are extremely long compared to the Q_2/P_1^{disc} discretisation on a finer grid, see Table 14. The most efficient approach was again the use of FGMRES with the multiple discretisation multilevel $F(1,1)$– and $W(1,1)$–cycle as preconditioner. The use of both multilevel methods as preconditioner in FGMRES instead as solver improved the efficiency and also the robustness. The multiple discretisation multilevel method was for the discretisation with continuous pressure approximation clearly better than the standard multigrid method. The standard multigrid method failed almost completely as solver. There, the iterative scheme on the coarsest grid diverged. This behaviour can be certainly improved by a stronger damping. However, the need of choosing small damping factors is not desirable in applications because an appropriate size of the factors is often hard to determine. In addition, solvers which require strong damping often need many iterations to converge. Finally, the most important conclusions of the numerical study are summarised:

- There has been no significant differences in the behaviour of both multilevel methods applied to the solution of saddle point problems arising from the Q_2/P_1^{disc} finite element discretisation.
- For the Q_2/Q_1 finite element discretisation, the use of the multiple discretisation multilevel method improved the efficiency of the solver enormously compared to the standard multigrid method.
- Using both multilevel methods as preconditioners in FGMRES was much more efficient than using them as solvers. Also the robustness of the solution process was improved.
- In the studied example, the arising discrete systems could be solved fastest with the FGMRES method and the multiple discretisation multilevel method $F-$ and $W-$cycle preconditioner.

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