East-West J. Numer. Math., Vol. 7, No. 2, pp. 133–158 (1999)

Variational crimes in a finite element discretization of 3D Stokes equations with nonstandard boundary conditions

P. KNOBLOCH*

Received November 1, 1998

Abstract — We investigate a finite element discretization of the Stokes equations with nonstandard boundary conditions defined in a bounded three–dimensional domain with a curved, piecewise smooth boundary. For tetrahedral triangulations of this domain, we prove, under general assumptions on the discrete problem and under usual regularity assumptions on the weak solution, that the convergence order of the discrete solutions is 1 or $\frac{3}{2}$. Examples of finite element spaces leading to these convergence orders are given.

Keywords: Stokes equations, nonstandard boundary conditions, finite element method, approximation of boundary.

AMS(MOS) subject classification: 65N30, 35Q30.

1. FORMULATION OF THE PROBLEM

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, the boundary of which consists of the closures of the sets Γ^D and Γ^N, and let us consider the following Stokes equations:

$$\begin{aligned} -\nu\,\Delta\,\boldsymbol{u} + \nabla\,p &= \boldsymbol{f} && \text{in } \Omega, && (1.1)\\ \operatorname{div}\boldsymbol{u} &= 0 && \text{in } \Omega, && (1.2)\\ \boldsymbol{u} &= \boldsymbol{u}_b && \text{on } \Gamma^D, && (1.3)\\ \boldsymbol{u}\cdot\boldsymbol{n} &= 0 && \text{on } \Gamma^N, && (1.4)\\ (\boldsymbol{I} - \boldsymbol{n}\otimes\boldsymbol{n})\,(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}})\,\boldsymbol{n} &= \boldsymbol{\varphi} && \text{on } \Gamma^N. && (1.5)\end{aligned}$$

The equations describe a slow motion of a viscous incompressible fluid (e.g. of molten glass) and the used symbols have the following meanings: $\boldsymbol{u}$ is the velocity, p is the pressure, ν is the kinematic viscosity, which is assumed to be constant and positive, $\boldsymbol{f}$ is an outer volume force, $\boldsymbol{n}$ is the unit outward normal vector to the boundary of the domain Ω, $\boldsymbol{I}$ is the identity tensor and $\boldsymbol{\varphi}$ is a surface force acting in the tangential direction to Γ^N. Such a surface force can arise due to variations of the surface tension on Γ^N (the so–called Marangoni effect). The surface tension is often linearly proportional to the temperature θ so that we have $\boldsymbol{\varphi} = \alpha\,\nabla_s\,\theta$, where α is a constant and ∇_s is the surface gradient.

We assume that the boundary $\partial\Omega$ of Ω is Lipschitz–continuous and that the sets Γ^D and Γ^N are open in $\partial\Omega$, disjoint and such that $\operatorname{meas}_2(\Gamma^D) > 0$, $\operatorname{meas}_2(\Gamma^N) \geq 0$ and

*Institute of Numerical Mathematics, Faculty of Mathematics and Physics, Charles University, Malostranské náměsti 25, 118 00 Praha 1, Czech Republic, e–mail: knobloch@karlin.mff.cuni.cz
This work was supported by the Deutsche Forschungsgemeinschaft (DFG) and by the Czech Grant Agency under the grant No. 201/99/0267.

$\partial\Omega = \overline{\Gamma^D \cup \Gamma^N}$. Further, we assume that

$$\boldsymbol{f} \in L^2(\Omega)^3 \,, \tag{1.6}$$

$$\boldsymbol{u}_b \in H^{\frac{1}{2}}(\partial\Omega)^3 \,, \qquad \boldsymbol{u}_b \cdot \boldsymbol{n} = 0 \;\text{ on } \Gamma^N, \qquad \int_{\partial\Omega} \boldsymbol{u}_b \cdot \boldsymbol{n} \,\mathrm{d}\sigma = 0 \,, \tag{1.7}$$

$$\boldsymbol{\varphi} \in H^{\frac{1}{2}}(\partial\Omega)^3 \,, \qquad \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \;\text{ on } \Gamma^N. \tag{1.8}$$

The third condition in (1.7) and the second condition in (1.8) are necessary conditions for the solvability of (1.1)–(1.5).

The aim of this paper is to investigate properties of a finite element discretization of (1.1)–(1.5) defined using a tetrahedral triangulation of Ω. We are interested in cases when the boundary of Ω is curved and hence the discrete problems are defined in domains which only approximate the original computational domain Ω. The approximation of the computational domain is one of the so–called variational crimes and it is well known that, for polyhedral approximations of Ω, the best possible convergence order of finite element discretizations of simple elliptic problems is $\frac{3}{2}$ (cf. e.g. [11], Section 4.4). Therefore, we shall look for conditions which guarantee a linear convergence or a convergence order $\frac{3}{2}$. For proving these convergence results, we shall need further assumptions on the regularity of the boundary of Ω. First, we introduce a property of subsets of $\partial\Omega$, analogous to the Lipschitz property of domains in $\mathbb{R}^2$. For this, we employ local Cartesian coordinate systems from the definition of the Lipschitz–continuity of $\partial\Omega$ (see the beginning of Section 4). In each of these local coordinate systems, a part of $\partial\Omega$ is represented by the graph of a function $f \in C^{0,1}([-a,a]^2)$, where $a > 0$ is a constant common to all the coordinate systems.

Definition 1.1. A relatively open set $\Gamma \subset \partial\Omega$ is of class $C^{0,1*}$ if there exists a finite number of the above–mentioned local coordinate systems such that the graphs of the functions f on $(-a,a)^2$ cover the whole boundary $\partial\Omega$ and, in each of these coordinate systems, the projection of the respective part of Γ into the square $(-a,a)^2$ is a set with a Lipschitz–continuous boundary.

We suppose that the boundary of Ω consists of closures of disjoint relatively open C^2 surfaces $\Gamma_i \subset \partial\Omega$, $i = 1,\dots,K$, which are of class $C^{0,1*}$, have piecewise C^2 boundaries and satisfy, for some $K^D \in \{1,\dots,K\}$,

$$\Gamma^D = \operatorname{int} \bigcup_{i=1}^{K^D} \overline{\Gamma_i} \,, \qquad \Gamma^N = \operatorname{int} \bigcup_{i=K^D+1}^{K} \overline{\Gamma_i} \,.$$

In addition, we assume that each surface Γ_i can be extended to a C^2 surface $\widetilde{\Gamma}_i$ satisfying $\operatorname{dist}(\partial\widetilde{\Gamma}_i, \partial\Gamma_i) > 0$.

The $C^{0,1*}$ regularity of the sets Γ_i guarantees that no zero angle between two parts of $\partial\Gamma_i$ can arise in a point where $\partial\Gamma_i$ is not C^2. That is necessary for constructing regular triangulations of Γ_i.

The plan of the paper is as follows. In Section 2, we derive a weak formulation of (1.1)–(1.5), prove its unique solvability and show the equivalence between classical solutions and smooth weak solutions. In Section 3, we introduce triangulations of Ω and of the sets Γ^D and Γ^N. The elements of a given triangulation make up a domain Ω_h approximating the domain Ω (the index h represents the maximal diameter of elements of the respective triangulation). Further, we introduce general finite element spaces approximating the spaces from the weak formulation and define a finite element discretization of (1.1)–(1.5).

Section 4 is devoted to the derivation of various auxiliary results. First, we introduce decompositions of the sets $\partial\Omega$ and $\Omega \setminus \Omega_h \cup \Omega_h \setminus \Omega$ into sets assigned to the faces of the triangulation making up the boundary $\partial\Omega_h$ of Ω_h. These decompositions make it possible to investigate the convergence behaviour of differences between various integrals over a part of $\partial\Omega$ and the corresponding integrals over a part of $\partial\Omega_h$. The techniques we use generalize those ones applied in [3] for investigating variational crimes in the two–dimensional case. Further, we prove that an extension operator $\mathrm{P}_h : H^1(\Omega_h) \to H^1(\mathbb{R}^3)$ defined by means of the Nikolskij method satisfies the inequality

$$\|\mathrm{P}_h\, v\|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega} \leq C\, h^{\frac{1}{2}}\, \|v\|_{1,\Omega_h} \qquad \forall\ v \in V_h\,, \tag{1.9}$$

where $V_h \subset H^1(\Omega_h)$ is a general finite element space and the constant C is independent of h. This inequality plays an important role in the investigation of various effects caused by an approximation of Ω, which will be also seen in this paper. In the two–dimensional case, an operator P_h satisfying (1.9) can be defined for piecewise polynomial functions using the so–called natural extension (cf. [13], [3]). However, a generalization of the natural extension to the three–dimensional case is connected with many technical difficulties and, up to now, it has been performed only under restrictive assumptions on the domain Ω (cf. [8]). Finally, in this section, we shall also give a simple proof of the discrete Korn inequality.

In Section 5, we prove optimal convergence results for the discretization of (1.1)–(1.5) and, in Section 6, we give examples of finite element spaces satisfying the assumptions introduced in Section 3 and so leading to a convergence order 1 or $\frac{3}{2}$.

In contrast with the most authors who usually consider rather restrictive assumptions on Ω such as convexity and C^2 or C^3 regularity of the boundary, all our investigations are performed only under the assumptions on Ω given above (resp., for proving the convergence order $\frac{3}{2}$, under the additional assumption that the sets $\widetilde{\Gamma}_i$, $i = K^D\!+\!1, \ldots, K$, are C^3 surfaces). Further, we introduce general conditions for an optimal approximation of the boundary condition (1.4). The formulation of a discrete analogue of (1.4) in case of an approximation of Ω is very delicate and it was shown in [12] that an icorrect formulation of this condition can lead to discrete solutions which generally do not converge to the weak solution. In [12], a discretization of (1.1)–(1.5) with $\Gamma^D = \emptyset$ and $\boldsymbol{\varphi} = \mathbf{0}$ was investigated for a piecewise quadratic approximation of the velocity and a piecewise linear approximation of the pressure and, in consequence of (1.4), only a convergence order $\frac{1}{2}$ could be shown. Therefore, our results, which give a convergence order $\frac{3}{2}$ for the mentioned discrete functions, represent a substantial improvement of the results of [12].

Throughout the paper we use standard notations which can be found e.g. in [5]. We only mention a few of them. We denote by $\|\cdot\|_{k,p,\Omega}$ and $|\cdot|_{k,p,\Omega}$ the usual norm and seminorm, respectively, in the Sobolev space $W^{k,p}(\Omega)$ and, for $p = 2$, we drop the second index and use the notations $\|\cdot\|_{k,\Omega}$, $|\cdot|_{k,\Omega}$ and $H^k(\Omega) \equiv W^{k,2}(\Omega)$. The space of functions $v \in L^2(\Omega)$ satisfying $\int_\Omega v\,\mathrm{d}x = 0$ is denoted by $L^2_0(\Omega)$. The notations C, $\widetilde{C}$ and $\bar{C}$ are used to denote generic constants independent of h.

2. WEAK FORMULATION

To derive a weak formulation of (1.1)–(1.5), we first assume that $\boldsymbol{u} \in C^2(\overline{\Omega})^3$ and $p \in C^1(\overline{\Omega})$. We define the space

$$\mathbf{V} = \{\boldsymbol{v} \in H^1(\Omega)^3\,;\ \boldsymbol{v} = \mathbf{0}\ \text{ on }\ \Gamma^D,\ \ \boldsymbol{v}\cdot\boldsymbol{n} = 0\ \text{ on }\ \Gamma^N\}$$

and multiply the equations (1.1) and (1.2) by arbitrary functions $\boldsymbol{v} \in \mathbf{V}$ and $q \in L^2_0(\Omega)$, respectively. Then we integrate them over Ω, use the identities

$$-\boldsymbol{v} \cdot \Delta \boldsymbol{u} = \frac{1}{2} (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}}) \cdot (\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}}) - \operatorname{div}[(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})\, \boldsymbol{v}] + \boldsymbol{v} \cdot \nabla(\operatorname{div} \boldsymbol{u}) \,, \tag{2.1}$$

$$\boldsymbol{v} \cdot \nabla p = \operatorname{div}(p\, \boldsymbol{v}) - p \operatorname{div} \boldsymbol{v} \,, \tag{2.2}$$

which are valid for any functions $\boldsymbol{u} \in H^2(\Omega)^3$, $\boldsymbol{v} \in H^1(\Omega)^3$, $p \in H^1(\Omega)$, apply the Gauss integral theorem and substitute the conditions (1.2) and (1.5). Then we obtain

$$\frac{\nu}{2} \int_\Omega (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}}) \cdot (\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}}) \,\mathrm{d}x - \int_\Omega p \operatorname{div} \boldsymbol{v} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x + \nu \int_{\Gamma^N} \boldsymbol{\varphi} \cdot \boldsymbol{v} \,\mathrm{d}\sigma \,,$$

$$\int_\Omega q \operatorname{div} \boldsymbol{u} \,\mathrm{d}x = 0 \,.$$

All these integrals are also well defined for $\boldsymbol{u} \in H^1(\Omega)^3$, $p \in L^2(\Omega)$ and hence, denoting

$$a(\boldsymbol{u}, \boldsymbol{v}) = \frac{\nu}{2} \int_\Omega (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}}) \cdot (\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}}) \,\mathrm{d}x \,,$$

$$b(\boldsymbol{v}, p) = - \int_\Omega p \operatorname{div} \boldsymbol{v} \,\mathrm{d}x \,, \qquad \langle \boldsymbol{g}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x + \nu \int_{\Gamma^N} \boldsymbol{\varphi} \cdot \boldsymbol{v} \,\mathrm{d}\sigma \,,$$

we can introduce the following weak formulation.

Definition 2.1. Let $\widetilde{\boldsymbol{u}}_b \in H^1(\Omega)^3$ be any function satisfying

$$\widetilde{\boldsymbol{u}}_b|_{\partial\Omega} = \boldsymbol{u}_b \,. \tag{2.3}$$

Then the functions $\boldsymbol{u}$, p are a *weak solution* of the problem (1.1)–(1.5) if

$$\boldsymbol{u} - \widetilde{\boldsymbol{u}}_b \in \mathbf{V} \,, \qquad p \in L^2_0(\Omega) \tag{2.4}$$

and

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \langle \boldsymbol{g}, \boldsymbol{v} \rangle \qquad \forall\, \boldsymbol{v} \in \mathbf{V} \,, \tag{2.5}$$

$$b(\boldsymbol{u}, q) = 0 \qquad \forall\, q \in L^2_0(\Omega) \,. \tag{2.6}$$

Remark 2.1. The weak solution does not depend on the particular choice of the function $\widetilde{\boldsymbol{u}}_b$ satisfying (2.3). Since the pressure p is determined by (1.1), resp. by (2.5), up to an arbitrary additive constant, we consider only pressures with zero mean value.

Remark 2.2. In view of (2.6) and (1.7), we have $b(\boldsymbol{u}, q) = 0 \;\; \forall\, q \in L^2(\Omega)$ and hence any weak solution satisfies the condition $\operatorname{div} \boldsymbol{u} = 0$.

It follows from the derivation of the weak formulation that any classical solution $\boldsymbol{u} \in C^2(\overline{\Omega})^3$, $p \in C^1(\overline{\Omega}) \cap L^2_0(\Omega)$ of (1.1)–(1.5) satisfies (2.5) and (2.6). Since $\boldsymbol{u}$, p obviously also fulfil (2.4), any classical solution of the problem (1.1)–(1.5) (with $p \in L^2_0(\Omega)$) is a weak solution of this problem. The following theorem shows that, for sufficiently smooth weak solutions, the contrary also holds.

Theorem 2.1. *Any weak solution of (1.1)–(1.5) satisfying $\boldsymbol{u} \in H^2(\Omega)^3$, $p \in H^1(\Omega)$ fulfils the equations (1.1)–(1.5) a.e. in Ω, resp. a.e. in $\partial\Omega$. Any weak solution of (1.1)–(1.5) satisfying $\boldsymbol{u} \in C^2(\overline{\Omega})^3$, $p \in C^1(\overline{\Omega})$ is a classical solution of (1.1)–(1.5).*

Proof. For $\boldsymbol{u} \in H^2(\Omega)^3$, $p \in H^1(\Omega)$, it follows from (2.5), (2.1), (2.2) and Remark 2.2 that, for any $\boldsymbol{v} \in \mathbf{V}$,

$$\int_\Omega \boldsymbol{v} \cdot [-\nu\, \Delta\, \boldsymbol{u} + \nabla\, p - \boldsymbol{f}]\, \mathrm{d}x = \nu \int_{\Gamma^N} \boldsymbol{v} \cdot [\boldsymbol{\varphi} - (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})\, \boldsymbol{n}]\, \mathrm{d}\sigma\,. \tag{2.7}$$

Particularly, (2.7) holds for any $\boldsymbol{v} \in H^1_0(\Omega)^3$ with a vanishing right–hand side, which implies (1.1) in view of Proposition 1.1 from [9], p. 56. Consider any $i \in \{K^D+1, \ldots, K\}$. Since $\widetilde{\Gamma}_i$ is a C^2 surface, there exists $\boldsymbol{m} \in C^1(\mathbb{R}^3)^3$ such that $\boldsymbol{m} = \boldsymbol{n}$ on Γ_i. For any $\boldsymbol{v} \in H^1(\Omega)^3$ with $\boldsymbol{v} = \boldsymbol{0}$ on $\partial\Omega \setminus \Gamma_i$, we have $(\boldsymbol{I} - \boldsymbol{m} \otimes \boldsymbol{m})\, \boldsymbol{v} \in \mathbf{V}$ and hence it follows from (2.7), (1.1) and (1.8) that

$$\int_{\Gamma_i} \boldsymbol{v} \cdot [\boldsymbol{\varphi} - (\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n})(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})\, \boldsymbol{n}]\, \mathrm{d}\sigma = 0 \quad \forall\, \boldsymbol{v} \in H^1(\Omega)^3\,,\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega \setminus \Gamma_i\,.$$

Applying the mentioned Proposition 1.1 from [9], p. 56, in local coordinate systems describing Γ_i, we obtain

$$(\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n})(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})\, \boldsymbol{n} = \boldsymbol{\varphi} \quad \text{on} \quad \Gamma_i\,.$$

The validity of (1.2)–(1.4) follows from Remark 2.2 and (2.4) and hence the first part of the theorem holds. If $\boldsymbol{u} \in C^2(\overline{\Omega})^3$, then $(\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n})(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})\, \boldsymbol{n}$ is piecewise continuous on $\partial\Omega$ and since $\boldsymbol{\varphi} \in H^{\frac{1}{2}}(\partial\Omega)^3$, we infer that (1.5) holds at any point of Γ^N. Finally, for $\boldsymbol{u} \in C^2(\overline{\Omega})^3$, $p \in C^1(\overline{\Omega})$, the first part of the theorem immediately implies that the equations (1.1)–(1.4) hold in the classical sense. $\square$

The solvability of the weak formulation is investigated in the following theorem.

Theorem 2.2. *The problem (2.4)–(2.6) has a unique solution.*

Proof. The uniqueness readily follows from the Korn inequality

$$\bar{C}\, \|\boldsymbol{v}\|^2_{1,\Omega} \le a(\boldsymbol{v}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \mathbf{V}\,,$$

valid with a constant $\bar{C} > 0$ (cf. [10], p. 97, Lemma 3.1), and from the inf–sup condition

$$\sup_{\boldsymbol{v} \in \mathbf{V}, \boldsymbol{v} \neq \boldsymbol{0}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{1,\Omega}} \ge \beta\, \|q\|_{0,\Omega} \qquad \forall\, q \in L^2_0(\Omega)\,, \tag{2.8}$$

which holds with a constant $\beta > 0$. The inf–sup condition (2.8) immediately follows from the inf–sup condition for $\mathbf{V} = H^1_0(\Omega)^3$ established in [5], p. 81. For proving the existence of a solution we first use the fact that, according to Lemma 4.1 from [5], p. 58, the inf–sup condition (2.8) implies the existence of $\widetilde{\boldsymbol{u}}_0 \in \mathbf{V}$ such that

$$b(\widetilde{\boldsymbol{u}}_0, q) = -b(\widetilde{\boldsymbol{u}}_b, q) \qquad \forall\, q \in L^2_0(\Omega)\,.$$

Denoting

$$\widetilde{\mathbf{V}} = \{\boldsymbol{v} \in \mathbf{V}\,;\ b(\boldsymbol{v}, q) = 0 \ \ \forall\, q \in L^2_0(\Omega)\}\,,$$

it follows from (2.4) and (2.6) that $\boldsymbol{u} = \widetilde{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}_0 + \widetilde{\boldsymbol{u}}_b$ with $\widetilde{\boldsymbol{u}} \in \widetilde{\mathbf{V}}$. Substituting this expression for $\boldsymbol{u}$ into (2.5) and considering only test functions $\boldsymbol{v} \in \widetilde{\mathbf{V}}$, we obtain the problem

$$a(\widetilde{\boldsymbol{u}}, \boldsymbol{v}) = \langle \boldsymbol{g}, \boldsymbol{v} \rangle - a(\widetilde{\boldsymbol{u}}_0 + \widetilde{\boldsymbol{u}}_b, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \widetilde{\mathbf{V}}$$

which has a unique solution $\tilde{\boldsymbol{u}} \in \widetilde{\mathbf{V}}$ according to the Lax–Milgram theorem (cf. [2], p. 29, Theorem 1.3). Denoting

$$\langle h, \boldsymbol{v} \rangle = -a(\boldsymbol{u}, \boldsymbol{v}) + \langle \boldsymbol{g}, \boldsymbol{v} \rangle \,,$$

we have $h \in \mathbf{V}'$ and $\langle h, \boldsymbol{v} \rangle = 0 \;\; \forall\, \boldsymbol{v} \in \widetilde{\mathbf{V}}$. Therefore, again in view of the inf–sup condition (2.8) and Lemma 4.1 from [5], p. 58, there exists $p \in L^2_0(\Omega)$ such that

$$b(\boldsymbol{v}, p) = \langle h, \boldsymbol{v} \rangle \qquad \forall\, \boldsymbol{v} \in \mathbf{V} \,,$$

which completes the proof. □

Remark 2.3. Note that we have not used the property $\int_{\partial\Omega} \boldsymbol{u}_b \cdot \boldsymbol{n} \, \mathrm{d}\sigma = 0$ in the proof of Theorem 2.2. This fact is important for proving the solvability of the discrete problem introduced in the next section, where such a property is not available.

3. DISCRETE PROBLEM

We assume that we are given a family $\boldsymbol{\mathcal{T}} = \{\mathcal{T}_h\}$ of triangulations of the domain Ω having the following properties. Given $h > 0$, the set $\mathcal{T}_h$ consists of a finite number of closed tetrahedra called elements and denoted by T, $\mathrm{diam}(T) \le h$ for any $T \in \mathcal{T}_h$, all vertices of any $T \in \mathcal{T}_h$ belong to $\overline{\Omega}$, and any two different elements $T_1, T_2 \in \mathcal{T}_h$ are either disjoint or possess either a common vertex or a common edge or a common face. The domain Ω will be approximated by a polyhedral domain Ω_h defined by

$$\overline{\Omega_h} = \bigcup_{T \in \mathcal{T}_h} T \,.$$

We assume that the boundary of Ω_h is Lipschitz–continuous. The faces of the elements of $\mathcal{T}_h$ (being open two–dimensional sets) will be denoted by T' and faces belonging to the boundary of Ω_h will be called boundary faces. We assume that, for any boundary face T', there exists $i \in \{1, \ldots, K\}$ such that all three vertices of T' belong to $\overline{\Gamma_i}$. Further, we assume that the set of the vertices of any triangulation $\mathcal{T}_h \in \boldsymbol{\mathcal{T}}$ contains all the points, in which the boundary of some of the sets Γ_i is not C^2. The family of the triangulations is assumed to be regular, i.e., there exists a number $\sigma > 0$ such that, for any $\mathcal{T}_h \in \boldsymbol{\mathcal{T}}$ and any $T \in \mathcal{T}_h$,

$$\frac{h_T}{\varrho_T} \le \sigma \,, \tag{3.1}$$

where

$$h_T = \mathrm{diam}(T) \,, \qquad\qquad \varrho_T = \sup_{B \subset T \text{ is a ball}} \mathrm{diam}(B) \,.$$

It is easy to see that, for any face T' belonging to some of the triangulations $\mathcal{T}_h \in \boldsymbol{\mathcal{T}}$, we then have

$$\frac{h_{T'}}{\varrho_{T'}} \le \sigma \,,$$

where $h_{T'}$ and $\varrho_{T'}$ are defined analogously as h_T and ϱ_T, respectively. As usual, the parameters h in $\{\mathcal{T}_h\}$ are supposed to constitute a set whose only accumulation point is zero. Saying "any $h > 0$", we shall still mean "any $\mathcal{T}_h \in \boldsymbol{\mathcal{T}}$". Since the set of the parameters h is bounded, we can introduce a bounded domain $\widetilde{\Omega} \subset \mathbb{R}^3$ with a Lipschitz–continuous boundary such that $\overline{\Omega} \subset \widetilde{\Omega}$ and $\overline{\Omega_h} \subset \widetilde{\Omega}$ for any $h > 0$.

For any $i \in \{1,\dots,K\}$, the set Γ_i will be approximated by a relatively open set $\Gamma_{ih} \subset \partial\Omega_h$ consisting of boundary faces whose all vertices belong to $\overline{\Gamma_i}$. Such a definition would generally lead to sets Γ_{ih} which are not disjoint since, for an arbitrarily small h, it can happen that all vertices of a boundary face belong to $\overline{\Gamma_i} \cap \overline{\Gamma_j}$ for some $i \neq j$. Therefore, we need some additional condition on the boundary faces which should belong to Γ_{ih}. For any boundary face T', we denote by $x^c_{T'}$ the barycentre of T' and by $x^b_{T'}$ the nearest point to $x^c_{T'}$ lying on $\partial\Omega$ and satisfying $(x^c_{T'} - x^b_{T'}) \perp T'$. Then any boundary face T' having its vertices on $\overline{\Gamma_i}$ is considered to belong to Γ_{ih} only if $x^b_{T'} \in \Gamma_i$, which makes sense if h is sufficiently small (cf. Section 4). The definition of Γ_{ih} for large values of h is not important since we are only interested in the behaviour of the discrete solutions for $h \to 0$. However, we shall still assume that $\Gamma_{ih} \cap \Gamma_{jh} = \emptyset$ for any $i \neq j$ and that $\bigcup_{i=1}^K \overline{\Gamma_{ih}} = \partial\Omega_h$. The sets Γ^D and Γ^N will be approximated by relatively open sets Γ^D_h and Γ^N_h, respectively, defined by

$$\Gamma^D_h = \operatorname{int} \bigcup_{i=1}^{K^D} \overline{\Gamma_{ih}}\,, \qquad \Gamma^N_h = \operatorname{int} \bigcup_{i=K^D+1}^{K} \overline{\Gamma_{ih}}\,.$$

We denote by $\widehat{T}$ the standard reference element having the vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Then, for any tetrahedron T, there exists a regular affine mapping $F_T : \widehat{T} \to T$ which maps $\widehat{T}$ onto T (cf. e.g. [2]). For any $T \in \mathcal{T}_h$ and any $v \in L^1(T)$, we introduce the notation

$$\widehat{v}_T = v \circ F_T \tag{3.2}$$

and we define general finite element spaces

$$V_h = \{v \in C(\overline{\Omega_h})\,;\ \widehat{v}_T \in \widehat{V} \quad \forall\, T \in \mathcal{T}_h\}\,, \tag{3.3}$$
$$M_h = \{v \in L^2(\Omega_h)\,;\ \widehat{v}_T \in \widehat{M} \quad \forall\, T \in \mathcal{T}_h\}\,, \tag{3.4}$$

where $\widehat{V} \subset W^{1,\infty}(\widehat{T})$ and $\widehat{M} \subset L^\infty(\widehat{T})$ are fixed *finite–dimensional* spaces. We assume that we are given spaces

$$\mathbf{V}_h \subset \{\boldsymbol{v} \in [V_h]^3\,;\ \boldsymbol{v} = \mathbf{0} \ \text{ on } \Gamma^D_h\}\,, \qquad \mathrm{Q}_h \subset M_h \cap L^2_0(\Omega_h)$$

approximating the spaces $\mathbf{V}$ and $L^2_0(\Omega)$, respectively, and we make the following assumptions on their approximation properties:

A1: There exist an integer $l \in \{2,3\}$ and an operator $r_h \in \mathcal{L}(\mathbf{V} \cap H^l(\Omega)^3, \mathbf{V}_h)$ such that

$$\|\boldsymbol{v} - r_h\,\boldsymbol{v}\|_{1,\Omega_h} \le C\,h^{\frac{l}{2}}\,\|\boldsymbol{v}\|_{l,\widetilde{\Omega}} \qquad \forall\,\boldsymbol{v} \in H^l(\widetilde{\Omega})^3\,,\ \boldsymbol{v}|_\Omega \in \mathbf{V}\,. \tag{3.5}$$

If $l = 2$, then

$$\|\boldsymbol{v}_h \cdot \boldsymbol{n}_h\|_{0,\frac{4}{3},\Gamma^N_h} \le C\,h\,\|\boldsymbol{v}_h\|_{1,\Omega_h} \qquad \forall\,\boldsymbol{v}_h \in \mathbf{V}_h\,, \tag{3.6}$$

where $\boldsymbol{n}_h$ is the unit outward normal vector to $\partial\Omega_h$. If $l = 3$, then

$$\|\boldsymbol{v}_h \cdot \bar{\boldsymbol{n}}_h\|_{0,1,\Gamma^N_h} \le C\,h^{\frac{3}{2}}\,\|\boldsymbol{v}_h\|_{1,\Omega_h} \qquad \forall\,\boldsymbol{v}_h \in \mathbf{V}_h\,, \tag{3.7}$$

where $\bar{\boldsymbol{n}}_h \in L^2(\partial\Omega_h)^3$ is the piecewise linear interpolate of the unit outward normal vector $\boldsymbol{n}$ to $\partial\Omega$ (cf. (3.10) below).

A2: There exists an operator $s_h \in \mathcal{L}(L^2_0(\Omega) \cap H^{l-1}(\Omega), \mathrm{Q}_h)$ such that

$$\|q - s_h\, q\|_{0,\Omega_h} \leq C\, h^{\frac{l}{2}}\, \|q\|_{l-1,\widetilde{\Omega}} \qquad \forall\; q \in H^{l-1}(\widetilde{\Omega})\,,\; q|_\Omega \in L^2_0(\Omega)\,, \tag{3.8}$$

where l is the same as in A1.

A3: There exists a constant $\beta > 0$ independent of h such that

$$\sup_{\boldsymbol{v}_h \in \mathbf{V}_h, \boldsymbol{v}_h \neq \mathbf{0}} \frac{\displaystyle\int_{\Omega_h} q_h \operatorname{div} \boldsymbol{v}_h \,\mathrm{d}x}{\|\boldsymbol{v}_h\|_{1,\Omega_h}} \geq \beta\, \|q_h\|_{0,\Omega_h} \qquad \forall\; q_h \in \mathrm{Q}_h\,. \tag{3.9}$$

The piecewise linear interpolate $\bar{\boldsymbol{n}}_h \in L^2(\partial\Omega_h)^3$ of $\boldsymbol{n}$ used in (3.7) is defined, on any face $T' \subset \Gamma_{ih}$, $i = 1, \dots, K$, by

$$\bar{\boldsymbol{n}}_h(x) = \sum_{j=1}^{3} \lambda_j(x)\, \boldsymbol{n}(x^j)\,, \qquad x \in T'\,, \tag{3.10}$$

where $\boldsymbol{n}$ is considered as continuous extension of $\boldsymbol{n}|_{\Gamma_i}$ onto $\overline{\Gamma_i}$, x^1, x^2, x^3 are the vertices of the triangle T' and λ_1, λ_2, λ_3 are the corresponding barycentric coordinates.

In order to define a discretization of (1.1)–(1.5), we introduce the notations

$$a_h(\boldsymbol{u}, \boldsymbol{v}) = \frac{\nu}{2} \int_{\Omega_h} (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}}) \cdot (\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}})\,\mathrm{d}x\,,$$

$$b_h(\boldsymbol{v}, p) = -\int_{\Omega_h} p \operatorname{div} \boldsymbol{v}\,\mathrm{d}x\,, \qquad \langle \boldsymbol{g}^*_h, \boldsymbol{v}\rangle = \int_{\Omega_h} \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x + \nu \int_{\Gamma^N_h} \boldsymbol{\varphi} \cdot \boldsymbol{v}\,\mathrm{d}\sigma\,,$$

where $\boldsymbol{f}$ and $\boldsymbol{\varphi}$ are now considered as extended onto $\widetilde{\Omega}$, i.e.,

$$\boldsymbol{f} \in L^2(\widetilde{\Omega})^3\,, \qquad \boldsymbol{\varphi} \in H^1(\widetilde{\Omega})^3\,.$$

Such extensions exist and can be defined in various ways. Further, we introduce a function $\widetilde{\boldsymbol{u}}_{bh} \in H^1(\Omega_h)^3$ approximating a function $\widetilde{\boldsymbol{u}}_b \in H^1(\Omega)^3$ satisfying (2.3). If $\widetilde{\boldsymbol{u}}_b \in H^l(\Omega)^3$, where l is the integer from the assumption A1, then the function $\widetilde{\boldsymbol{u}}_{bh}$ is usually defined as $\widetilde{\boldsymbol{u}}_{bh} = \widetilde{r}_h\, \widetilde{\boldsymbol{u}}_b$, where $\widetilde{r}_h \in \mathcal{L}(H^l(\Omega)^3, H^1(\Omega_h)^3)$ is an operator which maps into some finite element space and satisfies

$$\|\boldsymbol{v} - \widetilde{r}_h\, \boldsymbol{v}\|_{1,\Omega_h} \leq C\, h^{\frac{l}{2}}\, \|\boldsymbol{v}\|_{l,\widetilde{\Omega}} \qquad \forall\; \boldsymbol{v} \in H^l(\widetilde{\Omega})^3\,.$$

The operator $\widetilde{r}_h$ can be constructed using the techniques from Section 6.

Since typical finite element functions are piecewise polynomial, we can assume that an exact evaluation of the bilinear forms a_h and b_h requires only a small number of computational operations. That, however, cannot be assumed in case of the functional $\boldsymbol{g}^*_h$ and therefore, we replace it by some easily computable functional $\boldsymbol{g}_h \in [\mathbf{V}_h]'$ approximating $\boldsymbol{g}^*_h$. For example, if $\boldsymbol{f}$ and $\boldsymbol{\varphi}$ are continuous, we can obtain $\boldsymbol{g}_h$ by evalutaing $\boldsymbol{g}^*_h$ using *numerical integration* (cf. [2], Chapter IV) or by replacing $\boldsymbol{f}$ and $\boldsymbol{\varphi}$ by piecewise polynomial functions. In the latter case, defining $\boldsymbol{g}_h$ using piecewise linear interpolates of $\boldsymbol{f}$ and $\boldsymbol{\varphi}$, we have $\|\boldsymbol{g}_h - \boldsymbol{g}^*_h\|_{[\mathbf{V}_h]'} \leq C\, h^{\frac{3}{2}}\, (|\boldsymbol{f}|_{2,\widetilde{\Omega}} + |\boldsymbol{\varphi}|_{2,\widetilde{\Omega}})$ for any $\boldsymbol{f}, \boldsymbol{\varphi} \in H^2(\widetilde{\Omega})^3$ and, moreover, $\boldsymbol{g}_h$ depends only on the values of $\boldsymbol{f}|_{\overline{\Omega}}$ and $\boldsymbol{\varphi}|_{\overline{\Gamma^N}}$. The estimate follows from the properties of Lagrange interpolation operators and from the inequality

$\|u\|_{0,T'} \le C\,h^{\frac{3}{2}}\,(h_T^{-2}\,\|u\|_{0,T} + h_T^{-1}\,|u|_{1,T})$ valid for any $u \in H^1(\widetilde{\Omega})$ and any $T \in \mathcal{T}_h$ with a face T'.

Now we can introduce a family of discrete problems corresponding to the weak formulation (2.4)–(2.6).

Definition 3.1. The functions $\boldsymbol{u}_h$, p_h are a *discrete solution* of the problem (1.1)–(1.5) if

$$\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_{bh} \in \mathbf{V}_h\,, \qquad p_h \in \mathrm{Q}_h \tag{3.11}$$

and

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = \langle \boldsymbol{g}_h, \boldsymbol{v}_h \rangle \qquad \forall\ \boldsymbol{v}_h \in \mathbf{V}_h\,, \tag{3.12}$$

$$b_h(\boldsymbol{u}_h, q_h) = 0 \qquad \forall\ q_h \in \mathrm{Q}_h\,. \tag{3.13}$$

It can be shown in the same way as in the proof of Theorem 2.2 that the problem (3.11)–(3.13) has a unique solution.

4. AUXILIARY RESULTS

According to the definition of the Lipschitz–continuity of $\partial\Omega$, there exist positive real numbers a, b such that, for each $x^0 \in \partial\Omega$, the Cartesian coordinate system can be rotated and translated to x^0 in such a way that there exists $f \in C^{0,1}([-a,a]^2)$ satisfying for any $\overline{x} \in (-a,a)^2$

$$(\overline{x}, f(\overline{x})) \in \partial\Omega\,, \tag{4.1}$$

$$f(\overline{x}) < x_3 < f(\overline{x}) + b \quad \Rightarrow \quad (\overline{x}, x_3) \in \Omega\,, \tag{4.2}$$

$$f(\overline{x}) - b < x_3 < f(\overline{x}) \quad \Rightarrow \quad (\overline{x}, x_3) \notin \overline{\Omega}\,, \tag{4.3}$$

where we used the abbreviation $\overline{x}$ for (x_1, x_2). Thus, we can choose a finite number M of local coordinate systems and functions $f_1, \ldots, f_M \in C^{0,1}([-a,a]^2)$ satisfying (4.1)–(4.3) such that $\partial\Omega$ is covered by the graphs $\{(\overline{x}, f_r(\overline{x}))\,;\ \overline{x} \in (-a/4, a/4)^2\}$, $r = 1, \ldots, M$. Further, there exist functions $f_{ri} \in C^2([-a,a]^2)$, $r = 1, \ldots, M$, $i = 1, \ldots, K$, such that, for any $\overline{x} \in (-a,a)^2$, we have

$$(\overline{x}, f_r(\overline{x})) \in \overline{\Gamma_i} \quad \Rightarrow \quad f_r(\overline{x}) = f_{ri}(\overline{x})\,, \qquad r = 1, \ldots, M\,, \quad i = 1, \ldots, K\,,$$

i.e., the part of Γ_i described in the r–th local coordinate system is a part of the surface

$$\Gamma_{ri} = \{(\overline{x}, f_{ri}(\overline{x}))\,;\ \overline{x} \in (-a,a)^2\}\,.$$

It is appropriate to define the constants

$$C_1 = \max_{\substack{r=1,\ldots,M,\\ i=1,\ldots,K}} |f_{ri}|_{1,\infty,(-a,a)^2}\,, \qquad C_2 = \max_{\substack{r=1,\ldots,M,\\ i=1,\ldots,K}} |f_{ri}|_{2,\infty,(-a,a)^2}\,.$$

The domain $\widetilde{\Omega}$ introduced in Section 3 will be assumed to satisfy

$$\Omega \subset \widetilde{\Omega}\,, \qquad \operatorname{dist}(\partial\widetilde{\Omega}, \partial\Omega) > \max\{a, b\}\,.$$

We shall consider only triangulations with $h < \min\{a/8, b/8, 1/(C_2\,\sigma)\}$, which assures that $\overline{\Omega_h} \subset \widetilde{\Omega}$. In each local coordinate system, we introduce a continuous piecewise linear function f_{rh} defined on $[-a/2, a/2]^2$ and describing the corresponding part of $\partial\Omega_h$. All M graphs of these functions cover the whole boundary of Ω_h and, according to [7], p. 89, we have

$$|f_{rh}|_{1,\infty,(-a/2,a/2)^2} \leq 6\,\sigma\, C_1\,(1 + {C_1}^2)\,, \qquad r = 1, \dots, M\,. \tag{4.4}$$

Therefore, applying the Nikolskij method (cf. [9], p. 75, Theorem 3.9), we can define an extension operator $\mathrm{P}_h \in \mathcal{L}(H^1(\Omega_h), H^1_0(\widetilde{\Omega}))$ satisfying

$$\|\mathrm{P}_h\, v\|_{1,\widetilde{\Omega}} \leq C\, \|v\|_{1,\Omega_h} \qquad \forall\, v \in H^1(\Omega_h)$$

with a constant C independent of h (cf. Theorem 4.1). Using the operator P_h and the continuous imbedding $H^1(\widetilde{\Omega}) \subset L^p(\widetilde{\Omega})$, $p \in [1, 6]$, (cf. [9], p. 69, Theorem 3.4), we infer that, for any $p \in [1, 6]$, there exists a constant C independent of h such that

$$\|v\|_{0,p,\Omega_h} \leq C\, \|v\|_{1,\Omega_h} \qquad \forall\ v \in H^1(\Omega_h)\,. \tag{4.5}$$

Further, due to (4.5) and (4.4), we can prove, in the same way as for Theorem 4.2 in [9], p. 84, that, for any $p \in [1, 4]$, there exists a constant C independent of h such that

$$\|v\|_{0,p,\partial\Omega_h} \leq C\, \|v\|_{1,\Omega_h} \qquad \forall\ v \in H^1(\Omega_h)\,. \tag{4.6}$$

Choose any $i \in \{1, \dots, K\}$ and let $T' \subset \Gamma_{ih}$ be a boundary face. We denote by $K(T')$ the circle lying in the plane determined by T' and having a radius $h_{T'}$ and the centre in the barycentre of T'. Since $h < a/8$, we can find at least one of the M local coordinate systems introduced above, say the r–th one, such that the projection of $K(T')$ into the plane (x_1, x_2) lies in $(-a/2, a/2)^2$. Moreover, if h is sufficiently small, we deduce that the intersection $\widetilde{\Gamma}_{T'}$ of Γ_{ri} with the cylinder

$$\{x \in \mathbb{R}^3\,;\ x = \widetilde{x} + \alpha\, \boldsymbol{n}_K,\, \widetilde{x} \in K(T'),\, \alpha \in (-h, h),\, \boldsymbol{n}_K \perp K(T'),\, |\boldsymbol{n}_K| = 1\}$$

can be represented by the graph of a function $\varphi_{T'} \in C^2(K(T'))$ satisfying

$$|\varphi_{T'}(\overline{x})| \leq C\, h^2_{T'}\,, \qquad |\nabla\varphi_{T'}(\overline{x})| \leq C\, h_{T'} \qquad \forall\ \overline{x} \in K(T')\,, \tag{4.7}$$

where the constant C is independent of T' and h.

The vertices of a triangulation divide the boundaries $\partial\Gamma_i$ of the surfaces Γ_i, $i = 1, \dots, K$, into C^2 curves s_l, each of which contains exactly two vertices of the triangulation being the end points of s_l. The index l denotes the edge connecting these two vertices. It can be shown that there exists a constant C independent of l and h such that

$$s_l \subset \{x \in \mathbb{R}^3\,;\ |x - y| \leq C\, |x^{l1} - y|\, |x^{l2} - y|,\ y \in l\}\,, \tag{4.8}$$

where x^{l1}, x^{l2} are the end points of l. Therefore, for h sufficiently small and any $i \in \{1, \dots, K\}$, the edges l corresponding to all curves $s_l \subset \partial\Gamma_i$ make up the boundary $\partial\Gamma_{ih}$ of Γ_{ih}. If $l \subset \partial\Omega_h$ is an edge which does not belong to any of the boundaries $\partial\Gamma_{ih}$, then we construct a plane π_l containing l and halving the angle between the normal vectors to boundary faces adjacent to l and we define a curve s_l as the part of $\pi_l \cap \partial\Omega$ "between" the end points of l. Again, such curves s_l satisfy the relation (4.8) and hence, for any edge $l \subset \partial\Omega_h$, we have

$$\mathrm{dist}(x, l) \leq C\, |l|^2 \qquad \forall\ x \in s_l\,, \tag{4.9}$$

where $|l|$ denotes the length of l. Further, it follows from (4.8) that, for h sufficiently small, the intersection of any two different s_{l_1}, s_{l_2} can consist only of end points of l_1, l_2.

Therefore, any face $T' \subset \partial\Omega_h$ with edges l_1, l_2, l_3 can be associated with a relatively open set $\Gamma_{T'} \subset \partial\Omega$ whose boundary consists of s_{l_1}, s_{l_2}, s_{l_3}. Then

$$\Gamma_{T'_1} \cap \Gamma_{T'_2} = \emptyset \quad \forall\, T'_1 \neq T'_2 \qquad \text{and} \qquad \bigcup_{T' \subset \Gamma_{ih}} \overline{\Gamma_{T'}} = \overline{\Gamma_i}\,, \quad i = 1, \dots, K\,,$$

where "$T' \subset \Gamma_{ih}$" stands for "boundary faces contained in Γ_{ih}". Clearly, $\Gamma_{T'} \subset \widetilde{\Gamma}_{T'}$ for small h and hence $\Gamma_{T'}$ is described by $\varphi_{T'}$.

Our further task is to define open sets $G_{T'} \subset \Omega \setminus \Omega_h \cup \Omega_h \setminus \Omega$ related to the "space between T' and $\Gamma_{T'}$". This is easy if none of the edges l_1, l_2, l_3 of a face $T' \subset \partial\Omega_h$ belongs to the boundaries $\partial\Gamma_{ih}$, $i = 1, \dots, K$. For any edge l not belonging to the boundaries $\partial\Gamma_{ih}$, we introduce a set $\omega_l \subset \pi_l$, the boundary of which consists of l and s_l, and we define $G_{T'}$ as an open set with a boundary consisting of T', $\Gamma_{T'}$ and ω_{l_1}, ω_{l_2}, ω_{l_3}. If, e.g., l_1 belongs to some of the boundaries $\partial\Gamma_{ih}$, we have to proceed in another way. The reason is that, constructing $G_{T'}$ by spanning a surface between l_1 and s_{l_1}, we generally have $G_{T'} \not\subset \Omega \setminus \Omega_h \cup \Omega_h \setminus \Omega$.

We denote by $\mathcal{X}$ the set of all points in which some of the boundaries $\partial\Gamma_i$ is not C^2 and consider any edge $l \subset \partial\Gamma_{ih}$ (with some $i \in \{1, \dots, K\}$), the end points of which are not contained in $\mathcal{X}$. Let T'_1, T'_2 be the two boundary faces adjacent to l. We assume that h is sufficiently small so that l is the only edge of T'_1, T'_2 which belongs to the boundaries $\partial\Gamma_{ih}$, $i = 1, \dots, K$. Denoting by l_1, l_2, l_3, l_4 the edges of T'_1, T'_2 different from l, we can define an open set G_l, the boundary of which consists of T'_1, T'_2, $\Gamma_{T'_1}$, $\Gamma_{T'_2}$ and ω_{l_1}, ω_{l_2}, ω_{l_3}, ω_{l_4}. For any G_l we choose a fixed local coordinate system such that the projection l' of l into the plane (x_1, x_2) lies in $(-a/2, a/2)^2$ and we define the sets $G_{T'_1}$, $G_{T'_2}$ by cutting G_l with the set $\{(\overline{x}, x_3) \in \mathbb{R}^3\,;\ \overline{x} \in l'\}$.

Finally, let us consider any $x \in \mathcal{X}$ and let $\mathcal{F}_{hx}$ be the set of all boundary faces T', for which $x \in \overline{T'}$ and at least one edge belongs to some of the boundaries $\partial\Gamma_{ih}$. Again, we assume that h is small so that any edge of $T' \in \mathcal{F}_{hx}$ belonging to some $\partial\Gamma_{ih}$ contains x and no element of the triangulation contains two points from $\mathcal{X}$. Then we can define an open set G_x, the boundary of which consists of $\{T'\}_{T' \in \mathcal{F}_{hx}}$, $\{\Gamma_{T'}\}_{T' \in \mathcal{F}_{hx}}$ and $\{\omega_l\}_l$, where l are edges making up the boundary of $\bigcup_{T' \in \mathcal{F}_{hx}} \overline{T'}$. For any G_x, we denote by $\mathcal{L}_{hx}$ the set of all edges belonging to two faces from $\mathcal{F}_{hx}$ and we choose a fixed local coordinate system such that the projection of x into the plane (x_1, x_2) lies in $(-a/4, a/4)^2$. Cutting G_x with the sets $\{(\overline{x}, x_3) \in \mathbb{R}^3\,;\ \overline{x} \in l'\}$, where l' are projections of the edges $l \in \mathcal{L}_{hx}$ into the plane (x_1, x_2), we obtain the sets $\{G_{T'}\}_{T' \in \mathcal{F}_{hx}}$.

Thus, for any face $T' \subset \partial\Omega_h$ with h sufficiently small, we have constructed an open set $G_{T'} \subset \Omega \setminus \Omega_h \cup \Omega_h \setminus \Omega$ and it is easy to verify that

$$G_{T'_1} \cap G_{T'_2} = \emptyset \quad \forall\, T'_1 \neq T'_2 \qquad \text{and} \qquad \bigcup_{T' \subset \partial\Omega_h} \overline{G_{T'}} = \overline{\Omega \setminus \Omega_h \cup \Omega_h \setminus \Omega}\,.$$

We denote by $D_{T'}$ the projection of $\Gamma_{T'}$ into $K(T')$ and by $B_{T'}$ the intersection of T' with the projection of $\Gamma_{T'} \cap \overline{G_{T'}}$ into $K(T')$. Then $B_{T'} \subset T' \cap D_{T'}$ and

$$\{(\overline{x}, x_3) \in B_{T'} \times \mathbb{R}\,;\ \min\{0, \varphi_{T'}(\overline{x})\} < x_3 < \max\{0, \varphi_{T'}(\overline{x})\}\} \subset G_{T'}\,. \tag{4.10}$$

Clearly, $B_{T'} = T' \cap D_{T'}$ for any $T' \subset \partial\Omega_h$, the edges of which do not belong to the boundaries $\partial\Gamma_{ih}$, $i = 1, \dots, K$. For any edge l of such a face T', the angle α between the normal vectors to the two boundary faces adjacent to l satisfies $|\sin\alpha| \le C\, h_{T'}$, where $C = C(C_1, C_2, \sigma)$, and hence, due to the definition of s_l, we obtain

$$\mathrm{meas}_2((D_{T'} \setminus B_{T'}) \cup (T' \setminus B_{T'})) \le C\, h_{T'}^4\,.$$

If some of the edges of T' belongs to the boundaries $\partial\Gamma_{ih}$, then, using (4.7) and (4.9), we only obtain

$$\mathrm{meas}_2((D_{T'} \setminus B_{T'}) \cup (T' \setminus B_{T'})) \le C\, h_{T'}^3 \,.$$

Thus, we have

$$\sum_{T' \subset \partial\Omega_h} \mathrm{meas}_2((D_{T'} \setminus B_{T'}) \cup (T' \setminus B_{T'})) \le C \sum_{i=1}^{K} \sum_{\substack{T' \subset \Gamma_{ih}, \\ \overline{T'} \cap \partial\Gamma_{ih} = \text{edge(s)}}} h_{T'}^3 +$$

$$+ C \sum_{T' \subset \partial\Omega_h} h_{T'}^4 \le C\, \sigma\, h^2 \sum_{i=1}^{K} \mathrm{meas}_1(\partial\Gamma_i) + \widetilde{C}\, h^2 \sum_{T' \subset \partial\Omega_h} \mathrm{meas}_2(\Gamma_{T'}) \le \bar{C}\, h^2 \,. \quad (4.11)$$

In what follows, we shall establish a few estimates for differences between integrals over Γ_i and the corresponding integrals over Γ_{ih}. We shall need the following lemma.

Lemma 4.1. *Let $B \subset \mathbb{R}^2$ be a bounded domain and let $\varphi \in C^{0,1}(\mathbb{R}^2)$ be a given function. Let us define the set*

$$G = \{(\overline{x}, x_3) \in B \times \mathbb{R}\,;\; \min\{0, \varphi(\overline{x})\} < x_3 < \max\{0, \varphi(\overline{x})\}\} \,.$$

Then, for any $v \in H^1(\mathbb{R}^3)$ and $p \in [1, 2]$,

$$\int_B |v(\overline{x}, \varphi(\overline{x})) - v(\overline{x}, 0)|^p \, \mathrm{d}\overline{x} \le \Big(\sup_B |\varphi| \Big)^{p-1} \, |v|_{1,p,G}^p \,. \quad (4.12)$$

Proof. According to density and trace theorems, it suffices to prove (4.12) for smooth functions. Choose $v \in C^\infty(\mathbb{R}^3)$. Then, by the Hölder inequality,

$$\int_B |v(\overline{x}, \varphi(\overline{x})) - v(\overline{x}, 0)|^p \, \mathrm{d}\overline{x} = \int_B \left| \int_0^{\varphi(\overline{x})} \frac{\partial v}{\partial x_3}(\overline{x}, x_3) \, \mathrm{d}x_3 \right|^p \mathrm{d}\overline{x} \le$$

$$\le \int_B \left(\int_{\min\{0,\varphi(\overline{x})\}}^{\max\{0,\varphi(\overline{x})\}} \Big| \frac{\partial v}{\partial x_3}(\overline{x}, x_3) \Big| \, \mathrm{d}x_3 \right)^p \mathrm{d}\overline{x} \le$$

$$\le \int_G |\varphi(\overline{x})|^{p-1} \Big| \frac{\partial v}{\partial x_3}(\overline{x}, x_3) \Big|^p \, \mathrm{d}x_3 \, \mathrm{d}\overline{x} \le \Big(\sup_B |\varphi| \Big)^{p-1} \, |v|_{1,p,G}^p \,.$$

□

Lemma 4.2. *For any $i \in \{1, \ldots, K\}$, there exists a constant C independent of h such that*

$$\left| \int_{\Gamma_i} \theta\, v \, \mathrm{d}\sigma - \int_{\Gamma_{ih}} \theta\, v \, \mathrm{d}\sigma \right| \le C\, h\, \|\theta\|_{1,\widetilde{\Omega}} \, \|v\|_{1,\widetilde{\Omega}} \qquad \forall\, \theta, v \in H^1(\widetilde{\Omega}) \,. \quad (4.13)$$

Proof. For any face $T' \subset \partial\Omega_h$, we denote

$$\rho_{T'}(\overline{x}) = \sqrt{1 + \left(\frac{\partial \varphi_{T'}}{\partial x_1}(\overline{x}) \right)^2 + \left(\frac{\partial \varphi_{T'}}{\partial x_2}(\overline{x}) \right)^2} \,, \qquad \overline{x} \in K(T') \,.$$

Since $\sqrt{1+K} - 1 \le K/2$ for any $K \ge -1$, we obtain by (4.7)

$$|1 - \rho_{T'}(\overline{x})| \le C\, h_{T'}^2 \qquad \forall\, \overline{x} \in K(T') \,. \quad (4.14)$$

Then we have for any $T' \subset \Gamma_{ih}$

$$\left| \int_{T'} \theta\, v \,\mathrm{d}\sigma - \int_{\Gamma_{T'}} \theta\, v \,\mathrm{d}\sigma \right| \le \left| \int_{B_{T'}} \theta(\overline{x},0)\, v(\overline{x},0) - \theta(\overline{x},\varphi_{T'}(\overline{x}))\, v(\overline{x},\varphi_{T'}(\overline{x}))\, \rho_{T'}(\overline{x}) \,\mathrm{d}\overline{x} \right| + \\ + \left| \int_{T'\setminus B_{T'}} \theta(\overline{x},0)\, v(\overline{x},0) \,\mathrm{d}\overline{x} \right| + \left| \int_{D_{T'}\setminus B_{T'}} \theta(\overline{x},\varphi_{T'}(\overline{x}))\, v(\overline{x},\varphi_{T'}(\overline{x}))\, \rho_{T'}(\overline{x}) \,\mathrm{d}\overline{x} \right|,$$

$$\left| \int_{T'\setminus B_{T'}} \theta(\overline{x},0)\, v(\overline{x},0) \,\mathrm{d}\overline{x} \right| \le \|\theta\|_{0,4,T'}\, \|v\|_{0,4,T'}\, [\mathrm{meas}_2(T' \setminus B_{T'})]^{\frac{1}{2}} \,,$$

$$\left| \int_{D_{T'}\setminus B_{T'}} \theta(\overline{x},\varphi_{T'}(\overline{x}))\, v(\overline{x},\varphi_{T'}(\overline{x}))\, \rho_{T'}(\overline{x}) \,\mathrm{d}\overline{x} \right| \le C\, \|\theta\|_{0,4,\Gamma_{T'}}\, \|v\|_{0,4,\Gamma_{T'}}\, [\mathrm{meas}_2(D_{T'}\setminus B_{T'})]^{\frac{1}{2}},$$

$$\left| \int_{B_{T'}} \theta(\overline{x},0)\, v(\overline{x},0) - \theta(\overline{x},\varphi_{T'}(\overline{x}))\, v(\overline{x},\varphi_{T'}(\overline{x}))\, \rho_{T'}(\overline{x}) \,\mathrm{d}\overline{x} \right| \le \\ \le \left| \int_{B_{T'}} \theta(\overline{x},0)\, v(\overline{x},0)\, (1-\rho_{T'}(\overline{x})) \,\mathrm{d}\overline{x} \right| + \\ + \left| \int_{B_{T'}} [\, \theta(\overline{x},0) - \theta(\overline{x},\varphi_{T'}(\overline{x}))]\, v(\overline{x},0)\, \rho_{T'}(\overline{x}) \,\mathrm{d}\overline{x} \right| + \\ + \left| \int_{B_{T'}} \theta(\overline{x},\varphi_{T'}(\overline{x}))\, [\, v(\overline{x},0) - v(\overline{x},\varphi_{T'}(\overline{x}))]\, \rho_{T'}(\overline{x}) \,\mathrm{d}\overline{x} \right| \le \\ \le C\, h_{T'}^2\, \|\theta\|_{0,T'}\, \|v\|_{0,T'} + C\, h_{T'}\, |\theta|_{1,G_{T'}}\, \|v\|_{0,T'} + C\, h_{T'}\, \|\theta\|_{0,\Gamma_{T'}}\, |v|_{1,G_{T'}} \,.$$

The last estimate follows from (4.14), (4.12), (4.10) and (4.7). Since by (4.11) and (4.6)

$$\sum_{T'\subset\Gamma_{ih}} \|\theta\|_{0,4,T'}\, \|v\|_{0,4,T'}\, [\mathrm{meas}_2(T' \setminus B_{T'})]^{\frac{1}{2}} \le \\ \le \left[\sum_{T'\subset\Gamma_{ih}} \|\theta\|_{0,4,T'}^4 \right]^{\frac{1}{4}} \left[\sum_{T'\subset\Gamma_{ih}} \|v\|_{0,4,T'}^4 \right]^{\frac{1}{4}} \left[\sum_{T'\subset\Gamma_{ih}} \mathrm{meas}_2(T' \setminus B_{T'}) \right]^{\frac{1}{2}} \le \\ \le C\, h\, \|\theta\|_{0,4,\Gamma_{ih}}\, \|v\|_{0,4,\Gamma_{ih}} \le \widetilde{C}\, h\, \|\theta\|_{1,\widetilde{\Omega}}\, \|v\|_{1,\widetilde{\Omega}} \,,$$

$$\sum_{T'\subset\Gamma_{ih}} \|\theta\|_{0,4,\Gamma_{T'}}\, \|v\|_{0,4,\Gamma_{T'}}\, [\mathrm{meas}_2(D_{T'} \setminus B_{T'})]^{\frac{1}{2}} \le \\ \le C\, h\, \|\theta\|_{0,4,\Gamma_i}\, \|v\|_{0,4,\Gamma_i} \le \widetilde{C}\, h\, \|\theta\|_{1,\widetilde{\Omega}}\, \|v\|_{1,\widetilde{\Omega}} \,,$$

$$\sum_{T'\subset\Gamma_{ih}} h_{T'}^2\, \|\theta\|_{0,T'}\, \|v\|_{0,T'} \le h^2\, \|\theta\|_{0,\Gamma_{ih}}\, \|v\|_{0,\Gamma_{ih}} \le C\, h^2\, \|\theta\|_{1,\widetilde{\Omega}}\, \|v\|_{1,\widetilde{\Omega}} \,,$$

$$\sum_{T'\subset\Gamma_{ih}} h_{T'}\, |\theta|_{1,G_{T'}}\, \|v\|_{0,T'} \le h\, |\theta|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega}\, \|v\|_{0,\Gamma_{ih}} \le C\, h\, |\theta|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega}\, \|v\|_{1,\widetilde{\Omega}} \,,$$

$$\sum_{T'\subset\Gamma_{ih}} h_{T'}\, \|\theta\|_{0,\Gamma_{T'}}\, |v|_{1,G_{T'}} \le h\, \|\theta\|_{0,\Gamma_i}\, |v|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega} \le C\, h\, \|\theta\|_{1,\widetilde{\Omega}}\, |v|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega} \,,$$

we obtain

$$\sum_{T'\subset\Gamma_{ih}} \left| \int_{\Gamma_{T'}} \theta\, v \,\mathrm{d}\sigma - \int_{T'} \theta\, v \,\mathrm{d}\sigma \right| \le C\, h\, \|\theta\|_{1,\widetilde{\Omega}}\, \|v\|_{1,\widetilde{\Omega}} \,,$$

which gives (4.13). □

Lemma 4.3. *For any $i \in \{1,\ldots,K\}$, there exists a constant C independent of h such that, for any $\theta \in L^\infty(\widetilde\Omega) \cap H^1(\widetilde\Omega)$, $v \in H^1(\widetilde\Omega)$,*

$$\left|\int_{\Gamma_i} \theta\, v \,\mathrm{d}\sigma - \int_{\Gamma_{ih}} \theta\, v\,\mathrm{d}\sigma\right| \leq C\,h^{\frac{3}{2}}\,\|\theta\|_{0,\infty,\widetilde\Omega}\,\|v\|_{1,\widetilde\Omega} + \\ + C\,h\,|\theta|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega}\,\|v\|_{1,\widetilde\Omega} + C\,h\,\|\theta\|_{0,\infty,\widetilde\Omega}\,|v|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega}\,. \qquad (4.15)$$

Proof. The basic idea of the proof is the same as for Lemma 4.2. However, we have to use the estimates

$$\left|\int_{T'\setminus B_{T'}} \theta(\overline{x},0)\,v(\overline{x},0)\,\mathrm{d}\overline{x}\right| \leq \|\theta\|_{0,\infty,\widetilde\Omega}\,\|v\|_{0,4,T'}\,[\mathrm{meas}_2(T'\setminus B_{T'})]^{\frac{3}{4}}\,,$$

$$\left|\int_{D_{T'}\setminus B_{T'}} \theta(\overline{x},\varphi_{T'}(\overline{x}))\,v(\overline{x},\varphi_{T'}(\overline{x}))\,\rho_{T'}(\overline{x})\,\mathrm{d}\overline{x}\right| \leq C\,\|\theta\|_{0,\infty,\widetilde\Omega}\,\|v\|_{0,4,\Gamma_{T'}}\,[\mathrm{meas}_2(D_{T'}\setminus B_{T'})]^{\frac{3}{4}}\,.$$

□

Using the above techniques, we can also prove the following results.

Lemma 4.4. *For any $i \in \{1,\ldots,K\}$, there exists a constant C independent of h such that*

$$\|v\|_{0,1,\Gamma_i} \leq C\,h^{\frac{3}{2}}\,\|v\|_{1,\widetilde\Omega} + C\,h\,|v|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega} \qquad \forall\, v \in H^1(\widetilde\Omega)\,,\ v|_{\Gamma_{ih}} = 0\,, \qquad (4.16)$$

$$\|v\|_{0,\frac{4}{3},\Gamma_i} \leq C\,h\,\|v\|_{1,\widetilde\Omega} \qquad \forall\, v \in H^1(\widetilde\Omega)\,,\ v|_{\Gamma_{ih}} = 0\,, \qquad (4.17)$$

$$\left|\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{0,\frac{4}{3},\Gamma_i} - \|\boldsymbol{v}\cdot\boldsymbol{n}_h\|_{0,\frac{4}{3},\Gamma_{ih}}\right| \leq C\,h\,\|\boldsymbol{v}\|_{1,\widetilde\Omega} \qquad \forall\, \boldsymbol{v} \in H^1(\widetilde\Omega)^3\,, \qquad (4.18)$$

where $\boldsymbol{n}_h$ is the unit outward normal vector to $\partial\Omega_h$. If the extension $\widetilde\Gamma_i$ of Γ_i is a C^3 surface and $\bar{\boldsymbol{n}}_h$ is the piecewise linear interpolate of the normal vector to $\partial\Omega$ given by (3.10), then we have for any $\boldsymbol{v} \in H^1(\widetilde\Omega)^3$

$$\left|\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{0,1,\Gamma_i} - \|\boldsymbol{v}\cdot\bar{\boldsymbol{n}}_h\|_{0,1,\Gamma_{ih}}\right| \leq C\,h^{\frac{3}{2}}\,\|\boldsymbol{v}\|_{1,\widetilde\Omega} + C\,h\,|\boldsymbol{v}|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega}\,. \qquad (4.19)$$

For general $v \in H^1(\widetilde\Omega)$, the convergence order of the bounds in (4.15), (4.16) and (4.19) is only 1. However, the convergence investigations in the following section require to apply these estimates only to extensions of functions belonging to the space V_h defined in (3.3) or to functions from $W^{1,\infty}(\widetilde\Omega)$. In this case, the convergence order of the bounds is $\frac{3}{2}$, which follows from Theorem 4.1 and the relation (4.29) below. For proving this theorem and Theorem 4.2, we shall use the fact that, in consequence of the finite–dimensionality of the spaces $\widehat V$ and $\widehat M$ used in the definitions of the spaces V_h and M_h, there exists a constant C depending only on $\widehat V$, $\widehat M$ and σ (cf. [6], Lemmas 3.2 and 3.3) such that, for any $T \in \mathcal{T}_h$ and any measurable set $G \subset T$, we have

$$\|v\|^2_{1,G} \leq C\,\mathrm{meas}_3(G)\,h_T^{-3}\,\|v\|^2_{1,T} \qquad \forall\, v \in V_h\,, \qquad (4.20)$$

$$\|v\|^2_{0,G} \leq C\,\mathrm{meas}_3(G)\,h_T^{-3}\,\|v\|^2_{0,T} \qquad \forall\, v \in M_h\,. \qquad (4.21)$$

Theorem 4.1. *There exists an operator $\mathrm{P}_h \in \mathcal{L}(H^1(\Omega_h), H^1_0(\widetilde\Omega))$ satisfying*

$$\|\mathrm{P}_h\,v\|_{1,\widetilde\Omega} \leq C\,\|v\|_{1,\Omega_h} \qquad \forall\, v \in H^1(\Omega_h) \qquad (4.22)$$

and

$$\|\mathrm{P}_h\, v\|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega} \le C\, h^{\frac{1}{2}}\, \|v\|_{1,\Omega_h} \qquad \forall\; v \in V_h\,, \tag{4.23}$$

where the constant C is independent of h.

Proof. As already mentioned at the beginning of this section, we shall construct the operator P_h by means of the Nikolskij method. For $r = 1, \dots, M$, we define, in the local coordinate systems, the sets

$$\mathrm{U}_r = \{(\overline{x}, x_3) \in \mathbb{R}^3\,;\ \overline{x} \in (-a/2, a/2)^2,\ f_r(\overline{x}) - b/2 < x_3 < f_r(\overline{x}) + b/2\}\,,$$
$$\widetilde{\mathrm{U}}_r = \{(\overline{x}, x_3) \in \mathbb{R}^3\,;\ \overline{x} \in (-3a/8, 3a/8)^2,\ f_r(\overline{x}) - b/4 < x_3 < f_r(\overline{x}) + b/4\}$$

and we introduce an open set $\mathrm{U}_{M+1} \subset \Omega$ such that

$$\Omega \subset \bigcup_{r=1}^{M+1} \mathrm{U}_r \qquad \text{and} \qquad \widetilde{\mathrm{U}}_r \cap \mathrm{U}_{M+1} = \emptyset\,, \quad r = 1, \dots, M\,.$$

In view of the decomposition of $\Omega \setminus \Omega_h \cup \Omega_h \setminus \Omega$ into the sets $G_{T'}$, it is obvious that, for h sufficiently small,

$$\Omega \setminus \Omega_h \cup \Omega_h \setminus \Omega \subset \bigcup_{r=1}^{M} \widetilde{\mathrm{U}}_r\,, \tag{4.24}$$

which implies

$$\mathrm{U}_{M+1} \subset \Omega_h \qquad \text{and} \qquad \Omega_h \subset \Omega \cup \bigcup_{r=1}^{M} \widetilde{\mathrm{U}}_r\,.$$

Note also that, for small h and any $r \in \{1, \dots, M\}$, we have

$$\mathrm{U}_r \cap \Omega_h = \{(\overline{x}, x_3) \in \mathbb{R}^3\,;\ \overline{x} \in (-a/2, a/2)^2,\ f_{rh}(\overline{x}) < x_3 < f_r(\overline{x}) + b/2\}\,. \tag{4.25}$$

We define an operator $\mathrm{P}_{M+1} : H_0^1(\mathrm{U}_{M+1}) \to H_0^1(\widetilde{\Omega})$ by

$$\mathrm{P}_{M+1}\, v = \begin{cases} v & \text{in } \mathrm{U}_{M+1}\,, \\ 0 & \text{elsewhere,} \end{cases}$$

where $v \in H_0^1(\mathrm{U}_{M+1})$. For $r = 1, \dots, M$, we denote

$$V_{rh} = \{v \in H^1(\mathrm{U}_r \cap \Omega_h)\,;\ v = 0 \ \text{ on } \ \partial(\mathrm{U}_r \cap \Omega_h) \setminus \partial\Omega_h\}$$

and define operators $\mathrm{P}_{rh} : V_{rh} \to H_0^1(\widetilde{\Omega})$ in the following way. We extend any $v \in V_{rh}$ by zero in

$$\{(\overline{x}, x_3) \in \mathbb{R}^3\,;\ \overline{x} \in (-a/2, a/2)^2,\ x_3 \ge f_r(\overline{x}) + b/2\}$$

and set

$$(\mathrm{P}_{rh}\, v)(\overline{x}, x_3) = \begin{cases} v(\overline{x}, x_3) & \text{for } \overline{x} \in (-a/2, a/2)^2,\ x_3 \ge f_{rh}(\overline{x})\,, \\ v(\overline{x}, 2\, f_{rh}(\overline{x}) - x_3) & \text{for } \overline{x} \in (-a/2, a/2)^2,\ x_3 \le f_{rh}(\overline{x})\,, \\ 0 & \text{elsewhere.} \end{cases}$$

For small h, we have $\Omega_h \cap \operatorname{supp}(\mathrm{P}_{rh}\, v) \subset \mathrm{U}_r$. According to [9], p. 27, Proposition 2.3, there exist functions $\psi_r \in C_0^\infty(\mathrm{U}_r)$, $r = 1, \dots, M+1$, (the so–called partition of unity) such that

$$\sum_{r=1}^{M+1} \psi_r(x) = 1 \qquad \forall\; x \in \Omega \cup \bigcup_{r=1}^{M} \widetilde{\mathrm{U}}_r\,.$$

The extension operator P_h can be now defined by

$$\mathrm{P}_h\, v = \mathrm{P}_{M+1}(v\,\psi_{M+1}) + \sum_{r=1}^{M} \mathrm{P}_{rh}(v\,\psi_r)\,, \qquad v \in H^1(\Omega_h)\,.$$

Then $\mathrm{P}_h \in \mathcal{L}(H^1(\Omega_h), H^1_0(\widetilde{\Omega}))$ and, for any $v \in H^1(\Omega_h)$, we have $\mathrm{P}_h\, v|_{\Omega_h} = v$ and

$$\|\mathrm{P}_h\, v\|_{1,\widetilde{\Omega}} \leq 6 \left[\sum_{r=1}^{M} (1 + |f_{rh}|_{1,\infty,(-a/2,a/2)^2})\, \|\psi_r\|_{1,\infty,\mathrm{U}_r} + \|\psi_{M+1}\|_{1,\infty,\mathrm{U}_{M+1}} \right] \|v\|_{1,\Omega_h}\,,$$

which, in view of (4.4), gives (4.22).

Now, consider any $r \in \{1,\ldots,M\}$ and $\overline{x} \in (-a/2,a/2)^2$. Then there exists $i \in \{1,\ldots,K\}$ such that $x \equiv (\overline{x}, f_r(\overline{x})) \in \overline{\Gamma_i}$. Let $T' \subset \Gamma_{ih}$ be a face for which $x \in \overline{\Gamma_{T'}}$ and let T'' be the projection of T' into the (x_1,x_2)–plane of the r–th local coordinate system. According to (4.7) and (4.9), we have $\mathrm{dist}(\overline{x}, T'') \leq \mathrm{dist}(x, T') \leq C\, h^2_{T'}$. Denoting by $\overline{x}^1$, $\overline{x}^2$, $\overline{x}^3$ the vertices of T'', we obtain using the Taylor formula

$$f_{ri}(\overline{x}^j) = f_{ri}(\overline{x}) + \nabla f_{ri}(\overline{x}) \cdot (\overline{x}^j - \overline{x}) + O(h^2_{T'})\,, \quad j = 1,2,3,$$

and we infer that

$$|f_r(\overline{x}) - f_{rh}(\overline{x})| = |f_{ri}(\overline{x}) - f_{rh}(\overline{x})| \leq \bar{C}\, h^2_{T'}\,. \tag{4.26}$$

Due to the regularity assumption (3.1), this inequality also holds with that face T', for which $(\overline{x}, f_{rh}(\overline{x})) \in \overline{T'}$.

For any $T' \subset \partial\Omega_h$ with $T' \cap \mathrm{U}_r \neq \emptyset$, we denote

$$\mathrm{M}_{rh}(T') = \{(\overline{x}, x_3) \in \overline{\mathrm{U}_r}\,;\ \overline{x} \in \overline{T''},\ f_{rh}(\overline{x}) \leq x_3 \leq f_{rh}(\overline{x}) + \bar{C}\, h^2_{T'}\}\,,$$

where T'' again is the projection of T' into the plane (x_1,x_2), and we define the set

$$\mathrm{M}_{rh} = \bigcup_{T' \subset \partial\Omega_h,\, T' \cap \mathrm{U}_r \neq \emptyset} \mathrm{M}_{rh}(T')\,.$$

Given $v \in V_h$ and denoting $v_r = v\,\psi_r$, we have for any $\overline{x} \in (-a/2,a/2)^2$ and $x_3 < f_{rh}(\overline{x})$

$$|(\nabla \mathrm{P}_{rh}\, v_r)(\overline{x}, x_3)| \leq \sqrt{2}\,(1 + 2\,|f_{rh}|_{1,\infty,(-a/2,a/2)^2})\, |(\nabla v_r)(\overline{x}, 2\,f_{rh}(\overline{x}) - x_3)|\,.$$

Hence, due to (4.4), we obtain for any $\overline{x} \in T'' \cap (-a/2,a/2)^2$ with $f_r(\overline{x}) < f_{rh}(\overline{x})$

$$\int_{f_r(\overline{x})}^{f_{rh}(\overline{x})} |(\nabla \mathrm{P}_{rh}\, v_r)(\overline{x}, x_3)|^2\, \mathrm{d}x_3 \leq C \int_{f_{rh}(\overline{x})}^{2\,f_{rh}(\overline{x}) - f_r(\overline{x})} |(\nabla v_r)(\overline{x}, x_3)|^2\, \mathrm{d}x_3 \leq$$

$$\leq C \int_{f_{rh}(\overline{x})}^{f_{rh}(\overline{x}) + \bar{C}\, h^2_{T'}} |(\nabla v_r)(\overline{x}, x_3)|^2\, \mathrm{d}x_3\,,$$

which implies

$$|\mathrm{P}_{rh}\, v_r|_{1,\Omega\setminus\Omega_h} \leq C\, |v_r|_{1,\mathrm{M}_{rh}} \leq \widetilde{C}\, \|v\|_{1,\mathrm{M}_{rh}}\,.$$

Since $|\mathrm{P}_{rh}\, v_r|_{1,\Omega_h\setminus\Omega} = |v_r|_{1,\Omega_h\setminus\Omega} \leq |v_r|_{1,\mathrm{M}_{rh}}$, we get

$$|\mathrm{P}_{rh}\, v_r|_{1,\Omega\setminus\Omega_h \cup \Omega_h\setminus\Omega} \leq C\, \|v\|_{1,\mathrm{M}_{rh}}\,.$$

In view of (3.1), there exists a constant $C > 0$ independent of h such that, for any face $T' \subset \partial\Omega_h$,

$$O(T') \equiv \{x \in \mathbb{R}^3\,;\ \mathrm{dist}(x, T') < C\, h_{T'}\} \subset (\widetilde{\Omega} \setminus \overline{\Omega_h}) \cup \bigcup_{T \in \mathcal{T}_h,\, T \cap \overline{T'} \neq \emptyset} T\,.$$

Since, for h sufficiently small, $\mathrm{M}_{rh}(T') \subset O(T')$, we infer that any $T \in \mathcal{T}_h$ satisfies $T \cap \mathrm{M}_{rh}(T') \neq \emptyset$ if and only if $T \cap \overline{T'} \neq \emptyset$. Therefore, we obtain for any $T \in \mathcal{T}_h$

$$\mathrm{meas}_3(T \cap \mathrm{M}_{rh}) = \sum_{\substack{T' \subset \partial\Omega_h,\\ T' \cap \mathrm{U}_r \neq \emptyset}} \mathrm{meas}_3(T \cap \mathrm{M}_{rh}(T')) < \bar{C} \sum_{\substack{T' \subset \partial\Omega_h,\\ T \cap \overline{T'} \neq \emptyset}} h_{T'}^4 \leq \widetilde{C}\, h_T^4\,, \tag{4.27}$$

where the last inequality again follows from (3.1). Thus, according to (4.20), we have $\|v\|^2_{1,T\cap\mathrm{M}_{rh}} \leq C\, h_T\, \|v\|^2_{1,T}$ and hence $\|v\|^2_{1,\mathrm{M}_{rh}} = \sum_{T\in\mathcal{T}_h} \|v\|^2_{1,T\cap\mathrm{M}_{rh}} \leq C\, h\, \|v\|^2_{1,\Omega_h}$. Therefore, we deduce that

$$|\mathrm{P}_h\, v|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega} \leq C\, h^{\frac{1}{2}}\, \|v\|_{1,\Omega_h}\,.$$

It remains to investigate $\|\mathrm{P}_h\, v\|_{0,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega}$. Let $G \subset \widetilde{\Omega}$ be any measurable set and let $p \in [1,6]$. Then, by the Hölder inequality and the Sobolev imbedding theorems,

$$\|v\|_{0,p,G} \leq [\,\mathrm{meas}_3(G)]^{\frac{1}{p}-\frac{1}{6}}\, \|v\|_{0,6,\widetilde{\Omega}} \leq C(\widetilde{\Omega})\, [\,\mathrm{meas}_3(G)]^{\frac{1}{p}-\frac{1}{6}}\, \|v\|_{1,\widetilde{\Omega}} \quad \forall\, v \in H^1(\widetilde{\Omega})\,. \tag{4.28}$$

Further, owing to (4.24), (4.25) and (4.26), we have

$$\mathrm{meas}_3(\Omega \setminus \Omega_h \cup \Omega_h \setminus \Omega) \leq C\, h^2\,. \tag{4.29}$$

Thus, using (4.28) and (4.22), we obtain

$$\|\mathrm{P}_h\, v\|_{0,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega} \leq C\, h^{\frac{2}{3}}\, \|v\|_{1,\Omega_h} \qquad \forall\, v \in H^1(\Omega_h)\,,$$

which proves the theorem. □

Theorem 4.2. *There exists a constant C independent of h such that*

$$\|q\|_{0,\Omega_h\setminus\Omega} \leq C\, h^{\frac{1}{2}}\, \|q\|_{0,\Omega_h} \qquad \forall\, q \in M_h\,. \tag{4.30}$$

Proof. Since $\mathrm{U}_r \cap (\Omega_h \setminus \Omega) \subset \mathrm{M}_{rh}$ and (4.24) holds, we obtain $\Omega_h \setminus \Omega \subset \bigcup_{r=1}^M \mathrm{M}_{rh}$ and hence $(\mathrm{int}\, T) \setminus \Omega \subset \bigcup_{r=1}^M T \cap \mathrm{M}_{rh}$ for any $T \in \mathcal{T}_h$. Applying (4.27), we get $\mathrm{meas}_3(T \setminus \Omega) \leq C h_T^4$ and, in view of (4.21), we deduce that $\|q\|^2_{0,T\setminus\Omega} \leq C\, h\, \|q\|^2_{0,T}$, which implies (4.30).□

The above results make it possible to prove that the bilinear form a_h is uniformly elliptic on the space $\mathbf{V}_h$, i.e., that the so–called discrete Korn inequality holds. Another proof of the discrete Korn inequality, performed under more general assumptions than used here, can be found in [6].

Theorem 4.3. *There exists a positive constant C independent of h such that*

$$C\, \|\boldsymbol{v}\|^2_{1,\Omega_h} \leq a_h(\boldsymbol{v}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \mathbf{V}_h\,. \tag{4.31}$$

Proof. According to the Korn inequality (cf. [10], p. 97, Lemma 3.1), the constant C from (4.31) exists for any $h > 0$ and it suffices to show that this constant can be chosen independently of h. Using a Korn inequality on Ω and Lemma 4.2, we obtain for any $\boldsymbol{u} \in H^1(\widetilde{\Omega})^3$

$$\bar{C}\,\|\boldsymbol{u}\|_{1,\Omega}^2 \le a(\boldsymbol{u},\boldsymbol{u}) + \frac{\nu}{2}\int_{\Gamma^D} |\boldsymbol{u}|^2\,\mathrm{d}\sigma \le a_h(\boldsymbol{u},\boldsymbol{u}) + \frac{\nu}{2}\int_{\Gamma_h^D} |\boldsymbol{u}|^2\,\mathrm{d}\sigma + \\ + 2\,\nu\,|\boldsymbol{u}|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega}^2 + C\,h\,\|\boldsymbol{u}\|_{1,\widetilde{\Omega}}^2$$

with $\bar{C} > 0$. Since $\|\boldsymbol{u}\|_{1,\Omega_h}^2 \le \|\boldsymbol{u}\|_{1,\Omega}^2 + \|\boldsymbol{u}\|_{1,\Omega_h\setminus\Omega}^2$, we get, setting $\boldsymbol{u} = \mathrm{P}_h\,\boldsymbol{v}$ with $\boldsymbol{v} \in \mathbf{V}_h$ and applying (4.22) and (4.23),

$$\bar{C}\,\|\boldsymbol{v}\|_{1,\Omega_h}^2 \le a_h(\boldsymbol{v},\boldsymbol{v}) + C\,h\,\|\boldsymbol{v}\|_{1,\Omega_h}^2\,,$$

which yields the theorem. □

Remark 4.1. In fact, the statements of Lemmas 4.2–4.4 and Theorems 4.1 and 4.2 were proven only for $h \le h_0$, where h_0 is small. However, since there is only a finite number of triangulations $\mathcal{T}_h$ with $h > h_0$, we deduce that Lemmas 4.2–4.4 and Theorems 4.1 and 4.2 really hold for any $\mathcal{T}_h \in \boldsymbol{\mathcal{T}}$.

5. CONVERGENCE RESULTS

For proving optimal convergence results for the discrete problem (3.11)–(3.13), we shall need the following lemma.

Lemma 5.1. *If $\boldsymbol{f} \in L^3(\widetilde{\Omega})^3$, then we have*

$$|\langle \boldsymbol{g}, \mathrm{P}_h\,\boldsymbol{v}_h\rangle - \langle \boldsymbol{g}_h^*, \boldsymbol{v}_h\rangle| \le C\,h\,\|\boldsymbol{v}_h\|_{1,\Omega_h} \qquad \forall\ \boldsymbol{v}_h \in \mathbf{V}_h \tag{5.1}$$

and, if $\boldsymbol{f} \in L^{12}(\widetilde{\Omega})^3$ and $\boldsymbol{\varphi} \in W^{1,\infty}(\widetilde{\Omega})^3$, then

$$|\langle \boldsymbol{g}, \mathrm{P}_h\,\boldsymbol{v}_h\rangle - \langle \boldsymbol{g}_h^*, \boldsymbol{v}_h\rangle| \le C\,h^{\frac{3}{2}}\,\|\boldsymbol{v}_h\|_{1,\Omega_h} \qquad \forall\ \boldsymbol{v}_h \in \mathbf{V}_h \tag{5.2}$$

with a constant C independent of h.

Proof. For any $\boldsymbol{v}_h \in \mathbf{V}_h$, we have

$$|\langle \boldsymbol{g}, \mathrm{P}_h\,\boldsymbol{v}_h\rangle - \langle \boldsymbol{g}_h^*, \boldsymbol{v}_h\rangle| \le \int_{\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega} |\boldsymbol{f}\cdot\mathrm{P}_h\,\boldsymbol{v}_h|\,\mathrm{d}x + \\ + \nu\left|\int_{\Gamma^N} \boldsymbol{\varphi}\cdot\mathrm{P}_h\,\boldsymbol{v}_h\,\mathrm{d}\sigma - \int_{\Gamma_h^N} \boldsymbol{\varphi}\cdot\mathrm{P}_h\,\boldsymbol{v}_h\,\mathrm{d}\sigma\right|$$

and hence, using the Hölder inequality, (4.13), (4.22), (4.28) and (4.29), we obtain for $\boldsymbol{f} \in L^3(\widetilde{\Omega})^3$

$$|\langle \boldsymbol{g}, \mathrm{P}_h\,\boldsymbol{v}_h\rangle - \langle \boldsymbol{g}_h^*, \boldsymbol{v}_h\rangle| \le \|\boldsymbol{f}\|_{0,3,\widetilde{\Omega}}\,\|\mathrm{P}_h\,\boldsymbol{v}_h\|_{0,\frac{3}{2},\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega} + \\ + C\,h\,\|\boldsymbol{\varphi}\|_{1,\widetilde{\Omega}}\,\|\boldsymbol{v}_h\|_{1,\Omega_h} \le \widetilde{C}\,h\,(\|\boldsymbol{f}\|_{0,3,\widetilde{\Omega}} + \|\boldsymbol{\varphi}\|_{1,\widetilde{\Omega}})\,\|\boldsymbol{v}_h\|_{1,\Omega_h}\,,$$

which gives (5.1). The inequality (5.2) follows analogously — we use (4.15) instead of (4.13) and apply (4.23). □

Theorem 5.1. *Let $k \in \{2,3\}$ be a given integer and let the assumptions A1 and A2 hold with $l = k$. If $k = 2$, let $\boldsymbol{f} \in L^3(\widetilde{\Omega})^3$. If $k = 3$, let $\boldsymbol{f} \in L^{12}(\widetilde{\Omega})^3$, $\boldsymbol{\varphi} \in W^{1,\infty}(\widetilde{\Omega})^3$ and let the extensions $\widetilde{\Gamma}_i$ of Γ_i, $i = K^D + 1, \ldots, K$, be C^3 surfaces. Let $\widetilde{\boldsymbol{u}}_b \in H^k(\widetilde{\Omega})^3$ be a function satisfying (2.3) and let*

$$\|\widetilde{\boldsymbol{u}}_b - \widetilde{\boldsymbol{u}}_{bh}\|_{1,\Omega_h} \leq C\, h^{\frac{k}{2}}\,, \qquad \|\boldsymbol{g}_h - \boldsymbol{g}_h^*\|_{[\mathbf{V}_h]'} \leq C\, h^{\frac{k}{2}} \tag{5.3}$$

with a constant C independent of h. Finally, let the weak solution $\boldsymbol{u}$, p of (1.1)–(1.5) satisfy $\boldsymbol{u} \in H^k(\Omega)^3$, $p \in H^{k-1}(\Omega)$. Then, for arbitrary extension operators $\mathrm{P}_j : H^j(\Omega) \to H^j(\widetilde{\Omega})$, $j = k-1, k$, there exists a constant $\widetilde{C}$ independent of h such that the discrete solution $\boldsymbol{u}_h$, p_h of (1.1)–(1.5) satisfies the inequality

$$\|\mathrm{P}_k\, \boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega_h} + \|\mathrm{P}_{k-1}\, p - p_h\|_{0,\Omega_h} \leq \widetilde{C}\, h^{\frac{k}{2}}\,.$$

Proof. For simplicity, we shall denote by $\boldsymbol{u}$ and p the extensions $\mathrm{P}_k\, \boldsymbol{u}$ and $\mathrm{P}_{k-1}\, p$, respectively. According to Theorem 2.1, the functions $\boldsymbol{u}$, p satisfy the equations (1.1)–(1.5). Thus, in the same way as used for deriving the weak formulation, we deduce that, for any $\boldsymbol{v} \in H^1(\widetilde{\Omega})^3$,

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \langle \boldsymbol{g}, \boldsymbol{v} \rangle + \nu \int_{\Gamma^D} \boldsymbol{v} \cdot (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})\, \boldsymbol{n}\, \mathrm{d}\sigma\, + \\ + \nu \int_{\Gamma^N} (\boldsymbol{v} \cdot \boldsymbol{n}) \boldsymbol{n} \cdot (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})\, \boldsymbol{n}\, \mathrm{d}\sigma - \int_{\partial\Omega} p\, \boldsymbol{v} \cdot \boldsymbol{n}\, \mathrm{d}\sigma\,. \end{aligned}$$

Setting $\boldsymbol{v} = \mathrm{P}_h\, \boldsymbol{v}_h$ with any $\boldsymbol{v}_h \in \mathbf{V}_h$ and using (3.12), we get

$$\begin{aligned} & a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - a(\boldsymbol{u}, \mathrm{P}_h\, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) - b(\mathrm{P}_h\, \boldsymbol{v}_h, p) = \\ & \quad = \langle \boldsymbol{g}_h, \boldsymbol{v}_h \rangle - \langle \boldsymbol{g}, \mathrm{P}_h\, \boldsymbol{v}_h \rangle - \nu \int_{\Gamma^D} (\mathrm{P}_h\, \boldsymbol{v}_h) \cdot (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})\, \boldsymbol{n}\, \mathrm{d}\sigma\, - \\ & \quad\quad - \nu \int_{\Gamma^N} (\boldsymbol{n} \cdot \mathrm{P}_h\, \boldsymbol{v}_h) \boldsymbol{n} \cdot (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})\, \boldsymbol{n}\, \mathrm{d}\sigma + \int_{\partial\Omega} p\, \boldsymbol{n} \cdot (\mathrm{P}_h\, \boldsymbol{v}_h)\, \mathrm{d}\sigma\,. \end{aligned}$$

By (5.1)–(5.3), (4.16)–(4.19), (4.22), (4.23), (3.6) and (3.7), the terms on the right–hand side are bounded by $C\, h^{\frac{k}{2}}\, \|\boldsymbol{v}_h\|_{1,\Omega_h}$. Further, we obtain using (4.23), (4.28) and (4.29)

$$\begin{aligned} & |a(\boldsymbol{u}, \mathrm{P}_h\, \boldsymbol{v}_h) - a_h(\boldsymbol{u}, \boldsymbol{v}_h) + b(\mathrm{P}_h\, \boldsymbol{v}_h, p) - b_h(\boldsymbol{v}_h, p)| \leq \\ & \quad \leq C\, h^{\frac{1}{2}}\, (|\boldsymbol{u}|_{1,\Omega\setminus\Omega_h \cup \Omega_h\setminus\Omega} + \|p\|_{0,\Omega\setminus\Omega_h \cup \Omega_h\setminus\Omega})\, \|\boldsymbol{v}_h\|_{1,\Omega_h} \leq \widetilde{C}\, h^{\frac{k}{2}}\, \|\boldsymbol{v}_h\|_{1,\Omega_h}\,, \end{aligned}$$

which implies that

$$|a_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h - p)| \leq C\, h^{\frac{k}{2}}\, \|\boldsymbol{v}_h\|_{1,\Omega_h} \qquad \forall\ \boldsymbol{v}_h \in \mathbf{V}_h\,. \tag{5.4}$$

Let us define the sets

$$\begin{aligned} \mathbf{V}_h^b &= \{\widehat{\boldsymbol{v}}_h \in H^1(\Omega_h)^3\,;\ \widehat{\boldsymbol{v}}_h - \widetilde{\boldsymbol{u}}_{bh} \in \mathbf{V}_h\}\,, \\ \mathcal{V}_h^b &= \{\widehat{\boldsymbol{z}}_h \in \mathbf{V}_h^b\,;\ b_h(\widehat{\boldsymbol{z}}_h, q_h) = 0\ \ \forall\ q_h \in \mathrm{Q}_h\}\,. \end{aligned}$$

For any $\widehat{\boldsymbol{z}}_h \in \mathcal{V}_h^b$, we have $\boldsymbol{u}_h - \widehat{\boldsymbol{z}}_h \in \mathbf{V}_h$ and

$$b_h(\boldsymbol{u}_h - \widehat{\boldsymbol{z}}_h, p - p_h) = b_h(\boldsymbol{u}_h - \widehat{\boldsymbol{z}}_h, p - q_h) \qquad \forall\ q_h \in \mathrm{Q}_h\,.$$

Hence, by (4.31) and (5.4),

$$C\,\|\boldsymbol{u}_h - \widehat{\boldsymbol{z}}_h\|^2_{1,\Omega_h} \leq a_h(\boldsymbol{u}_h - \widehat{\boldsymbol{z}}_h, \boldsymbol{u}_h - \widehat{\boldsymbol{z}}_h) \leq C\,(h^{\frac{k}{2}} + \|p - q_h\|_{0,\Omega_h} + \\ + \|\boldsymbol{u} - \widehat{\boldsymbol{z}}_h\|_{1,\Omega_h})\,\|\boldsymbol{u}_h - \widehat{\boldsymbol{z}}_h\|_{1,\Omega_h} \qquad \forall\ \widehat{\boldsymbol{z}}_h \in \mathcal{V}^b_h\,,\ q_h \in \mathrm{Q}_h\,,$$

which gives by the triangular inequality

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega_h} \leq C\,(h^{\frac{k}{2}} + \|p - q_h\|_{0,\Omega_h} + \|\boldsymbol{u} - \widehat{\boldsymbol{z}}_h\|_{1,\Omega_h}) \quad \forall\ \widehat{\boldsymbol{z}}_h \in \mathcal{V}^b_h\,,\ q_h \in \mathrm{Q}_h\,. \tag{5.5}$$

Consider any $\widehat{\boldsymbol{v}}_h \in \mathbf{V}^b_h$. According to Lemma 4.1 from [5], p. 58, the assumption A3 implies that there exists $\widetilde{\boldsymbol{v}}_h \in \mathbf{V}_h$ such that

$$b_h(\widetilde{\boldsymbol{v}}_h, q_h) = \int_{\Omega_h} q_h \operatorname{div}(\widehat{\boldsymbol{v}}_h - \boldsymbol{u})\,\mathrm{d}x + \int_{\Omega_h\setminus\Omega} q_h \operatorname{div}\boldsymbol{u}\,\mathrm{d}x \qquad \forall\ q_h \in \mathrm{Q}_h$$

and

$$\begin{aligned} \beta\,\|\widetilde{\boldsymbol{v}}_h\|_{1,\Omega_h} &\leq \sqrt{3}\left(|\widehat{\boldsymbol{v}}_h - \boldsymbol{u}|_{1,\Omega_h} + |\boldsymbol{u}|_{1,\Omega_h\setminus\Omega} \sup_{q_h\in\mathrm{Q}_h,\,q_h\neq 0} \frac{\|q_h\|_{0,\Omega_h\setminus\Omega}}{\|q_h\|_{0,\Omega_h}}\right) \leq \\ &\leq \sqrt{3}\,(|\widehat{\boldsymbol{v}}_h - \boldsymbol{u}|_{1,\Omega_h} + C\,h^{\frac{k}{2}})\,, \end{aligned}$$

where we used (4.28), (4.29) and (4.30). Since $\operatorname{div}\boldsymbol{u} = 0$ in Ω, we infer that $b_h(\widetilde{\boldsymbol{v}}_h, q_h) = -b_h(\widehat{\boldsymbol{v}}_h, q_h)$ for any $q_h \in \mathrm{Q}_h$. Setting $\widehat{\boldsymbol{z}}_h = \widetilde{\boldsymbol{v}}_h + \widehat{\boldsymbol{v}}_h$, we have $\widehat{\boldsymbol{z}}_h \in \mathcal{V}^b_h$ and

$$\|\boldsymbol{u} - \widehat{\boldsymbol{z}}_h\|_{1,\Omega_h} \leq \|\boldsymbol{u} - \widehat{\boldsymbol{v}}_h\|_{1,\Omega_h} + \|\widetilde{\boldsymbol{v}}_h\|_{1,\Omega_h} \leq C\,(h^{\frac{k}{2}} + \|\boldsymbol{u} - \widehat{\boldsymbol{v}}_h\|_{1,\Omega_h})\,.$$

Thus,

$$\inf_{\hat{\boldsymbol{z}}_h\in\mathcal{V}^b_h} \|\boldsymbol{u} - \widehat{\boldsymbol{z}}_h\|_{1,\Omega_h} \leq C\,(h^{\frac{k}{2}} + \|\boldsymbol{u} - \widehat{\boldsymbol{v}}_h\|_{1,\Omega_h}) \qquad \forall\ \widehat{\boldsymbol{v}}_h \in \mathbf{V}^b_h\,,$$

which means that the inequality (5.5) holds for any $\widehat{\boldsymbol{z}}_h \in \mathbf{V}^b_h$, $q_h \in \mathrm{Q}_h$. Therefore, setting $\widehat{\boldsymbol{z}}_h = \widetilde{\boldsymbol{u}}_{bh} + r_h(\boldsymbol{u} - \widetilde{\boldsymbol{u}}_b)$ and $q_h = s_h\,p$ in (5.5) and applying (3.5), (3.8) and (5.3), we get

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega_h} \leq C\,h^{\frac{k}{2}}\,.$$

From (5.4), it now follows that, for any $\boldsymbol{v}_h \in \mathbf{V}_h$,

$$|b_h(\boldsymbol{v}_h, p_h - p)| \leq C\,h^{\frac{k}{2}}\,\|\boldsymbol{v}_h\|_{1,\Omega_h}$$

and hence, using the inequality

$$|b_h(\boldsymbol{v}_h, p_h - q_h)| \leq |b_h(\boldsymbol{v}_h, p_h - p)| + |b_h(\boldsymbol{v}_h, p - q_h)| \qquad \forall\ q_h \in \mathrm{Q}_h$$

and applying (3.9), we deduce that

$$\|p_h - q_h\|_{0,\Omega_h} \leq C\,(h^{\frac{k}{2}} + \|p - q_h\|_{0,\Omega_h}) \qquad \forall\ q_h \in \mathrm{Q}_h\,.$$

Using the triangular inequality and setting $q_h = s_h\,p$, we obtain $\|p - p_h\|_{0,\Omega_h} \leq C\,h^{\frac{k}{2}}$, which completes the proof. □

6. EXAMPLES OF FINITE ELEMENT SPACES SATISFYING A1–A3

In this section, we assume that the normal vector $\boldsymbol{n}$ is continuous on $\overline{\Gamma^N}$, which implies that $\bar{\boldsymbol{n}}_h$ (cf. (3.10)) is continuous on Γ_h^N.

For $l \geq 1$, we define the spaces

$$\begin{aligned} \mathrm{V}_{l,h} &= \{v \in C(\overline{\Omega_h})\,;\ v|_T \in P_l(T) \quad \forall\, T \in \mathcal{T}_h\}\,, \\ \mathrm{M}_{l-1,h} &= \{v \in L^2(\Omega_h)\,;\ v|_T \in P_{l-1}(T) \quad \forall\, T \in \mathcal{T}_h\}\,, \end{aligned}$$

where $P_l(T)$ is the space of all polynomials of degrees less than or equal to l defined on the element T. We introduce the usual Lagrange interpolation operator $r_{l,h} \in \mathcal{L}(H^2(\Omega_h), \mathrm{V}_{l,h})$ defined by

$$(r_{l,h}\, v)(x) = v(x) \qquad \forall\, x \in S_{l,T}\,,\ T \in \mathcal{T}_h\,,\ v \in H^2(\Omega_h)\,,$$

where $S_{l,T}$ is the principal lattice of order l of the tetrahedron T (cf. [2], p. 70, or [5], p. 99). According to [2], p. 124, Theorem 15.3, this operator satisfies for any $m \in \{1, \ldots, l\}$

$$\|v - r_{l,h}\, v\|_{1,\Omega_h} \leq C\, h^m\, |v|_{m+1,\Omega_h} \qquad \forall\, v \in H^{m+1}(\Omega_h)\,. \tag{6.1}$$

Using L^2 projections on elements of the triangulation (see e.g. [5], p. 102, Lemma A.5), we define an operator $s_{l-1,h} \in \mathcal{L}(L^1(\Omega_h), \mathrm{M}_{l-1,h})$ satisfying for any $m \in \{0, \ldots, l\}$

$$\|v - s_{l-1,h}\, v\|_{0,\Omega_h} \leq C\, h^m\, |v|_{m,\Omega_h} \qquad \forall\, v \in H^m(\Omega_h)\,. \tag{6.2}$$

Using these operators, we can establish the following auxiliary result.

Lemma 6.1. *Let $m \geq 1$ be a given integer and let $\mathrm{P} \in \mathcal{L}(H^m(\Omega), H^m(\widetilde{\Omega}))$ be some extension operator. Then there exists a constant C independent of h such that*

$$\|v - \mathrm{P}\,\mathrm{R}\, v\|_{k,\Omega_h} \leq C\, h^{m-k}\, \|v\|_{m,\widetilde{\Omega}} \qquad \forall\, v \in H^m(\widetilde{\Omega})\,,\ k = 0, 1, \tag{6.3}$$

where $\mathrm{R} : H^1(\widetilde{\Omega}) \to H^1(\Omega)$ is the restriction operator from $\widetilde{\Omega}$ onto Ω.

Proof. Consider any $v \in H^m(\widetilde{\Omega})$ and denote $g = v - \mathrm{P}\,\mathrm{R}\, v$. Then $g \in H^m(\widetilde{\Omega})$ and $g = 0$ in Ω. According to (4.30), we have $\|s_{m,h}\, g\|_{0,\Omega_h\setminus\Omega} \leq C\, h^{\frac{1}{2}}\, \|s_{m,h}\, g\|_{0,\Omega_h} \leq C\, h^{\frac{1}{2}}\, \|s_{m,h}\, g - g\|_{0,\Omega_h} + C\, h^{\frac{1}{2}}\, \|g\|_{0,\Omega_h\setminus\Omega}$ and hence $\|g\|_{0,\Omega_h\setminus\Omega} \leq \|g - s_{m,h}\, g\|_{0,\Omega_h} + \|s_{m,h}\, g\|_{0,\Omega_h\setminus\Omega} \leq C\, \|g - s_{m,h}\, g\|_{0,\Omega_h} + C\, h^{\frac{1}{2}}\, \|g\|_{0,\Omega_h\setminus\Omega}$. In the same way, using (4.23), we obtain for $m \geq 2$ the inequality $\|g\|_{1,\Omega_h\setminus\Omega} \leq C\, \|g - r_{m,h}\, g\|_{1,\Omega_h} + C\, h^{\frac{1}{2}}\, \|g\|_{1,\Omega_h\setminus\Omega}$. Thus, (6.3) follows applying (6.1) and (6.2). $\square$

Let $\mathrm{P} \in \mathcal{L}(H^1(\Omega), H^1(\widetilde{\Omega}))$ be some extension operator. Setting $s_h\, q = s_{0,h}(\mathrm{P}\, q) - \int_{\Omega_h} s_{0,h}(\mathrm{P}\, q)\, \mathrm{d}x/\mathrm{meas}_3(\Omega_h)$ for $q \in H^1(\Omega)$, we have for any $q \in H^1(\widetilde{\Omega})$ with $q|_\Omega \in L_0^2(\Omega)$

$$\begin{aligned} q|_{\Omega_h} - s_h\, q = q|_{\Omega_h} &- s_{0,h}(\mathrm{P}\,\mathrm{R}\, q) - \\ &- \int_{\Omega_h} q - s_{0,h}(\mathrm{P}\,\mathrm{R}\, q)\, \mathrm{d}x/\mathrm{meas}_3(\Omega_h) + \int_{\Omega_h} q\, \mathrm{d}x/\mathrm{meas}_3(\Omega_h)\,. \end{aligned}$$

Since $L_0^2(\Omega_h) \perp \mathbb{R}$ in $L^2(\Omega_h)$ and $|\int_{\Omega_h} q\, \mathrm{d}x| \leq \|q\|_{0,1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega} \leq C\, h^{\frac{5}{3}}\, \|q\|_{1,\widetilde{\Omega}}$, we obtain $\|q - s_h\, q\|_{0,\Omega_h} \leq \|q - s_{0,h}(\mathrm{P}\,\mathrm{R}\, q)\|_{0,\Omega_h} + C\, h^{\frac{5}{3}}\, \|q\|_{1,\widetilde{\Omega}}$. Applying (6.2) and (6.3), we get $\|q - s_h\, q\|_{0,\Omega_h} \leq C\, h\, \|q\|_{1,\widetilde{\Omega}}$ and since $s_h \in \mathcal{L}(L_0^2(\Omega) \cap H^1(\Omega), \mathrm{M}_{0,h} \cap L_0^2(\Omega_h))$, we see that, for $\mathrm{Q}_h \supset \mathrm{M}_{0,h} \cap L^2_0(\Omega_h)$, the assumption A2 holds with $l = 2$. Analogously, using the operator $r_{1,h}$, we infer that A2 holds with $l = 3$ for any space Q_h containing $\mathrm{M}_{1,h} \cap L^2_0(\Omega_h)$ or $\mathrm{V}_{1,h} \cap L^2_0(\Omega_h)$.

The space $\mathbf{V}_h$ is more difficult to construct. We denote

$$S_h^N = \{x \in \Gamma_h^N\, ;\ x \text{ is a vertex of } \mathcal{T}_h \text{ or midpoint of an edge of } \mathcal{T}_h\}$$

and we introduce a function $\boldsymbol{n}_h^* : S_h^N \to \mathbb{R}^3$ satisfying for some $\alpha \geq \frac{1}{2}$

$$|\boldsymbol{n}_h^*(x) - \bar{\boldsymbol{n}}_h(x)| \leq C\, h_T^\alpha \qquad \forall\ x \in S_h^N\,, \tag{6.4}$$

where T is any element containing the point x and C is independent of x, T and h. We shall investigate, under what conditions the spaces

$$\begin{aligned} \boldsymbol{V}_{1,h} &= \{\boldsymbol{v} \in [\mathrm{V}_{1,h}]^3\, ;\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_h^D\,,\ (\boldsymbol{v}\cdot\boldsymbol{n}_h^*)(x) = 0 \text{ at any vertex } x \in \Gamma_h^N\}\,,\\ \boldsymbol{V}_{2,h} &= \{\boldsymbol{v} \in [\mathrm{V}_{2,h}]^3\, ;\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_h^D\,,\ (\boldsymbol{v}\cdot\boldsymbol{n}_h^*)(x) = 0\ \forall\ x \in S_h^N\} \end{aligned}$$

satisfy the assumption A1. These investigations will be divided into several lemmas. Analogously to the notation $S_{l,T}$ introduced above, we shall denote by $S_{l,T'}$ the principle lattice of order l of the triangle T', i.e., $S_{1,T'}$ consists of the vertices of T' and $S_{2,T'}$ consists of the vertices of T' and the midpoints of edges of T'. Further, we shall need the relations (we use the notation introduced in (3.2))

$$C\, h_T^{\frac{3}{2}}\, \|\widehat{v}_T\|_{0,\widehat{T}} \leq \|v\|_{0,T} \leq h_T^{\frac{3}{2}}\, \|\widehat{v}_T\|_{0,\widehat{T}} \qquad \forall\ v \in L^2(T),\ T \in \mathcal{T}_h\,, \tag{6.5}$$

$$C\, h_T^{\frac{1}{2}}\, |\widehat{v}_T|_{1,\widehat{T}} \leq |v|_{1,T} \leq \widetilde{C}\, h_T^{\frac{1}{2}}\, |\widehat{v}_T|_{1,\widehat{T}} \qquad \forall\ v \in H^1(T),\ T \in \mathcal{T}_h\,, \tag{6.6}$$

which hold with positive constants C and $\widetilde{C}$ depending only on σ (cf. [2], Section 15).

Lemma 6.2. *There exists a constant C independent of h such that*

$$\|\boldsymbol{v}_h\cdot\boldsymbol{n}_h\|_{0,\frac{4}{3},\Gamma_h^N} \leq C\, h^{\min\{1,\alpha\}}\, \|\boldsymbol{v}_h\|_{1,\Omega_h} \qquad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_{1,h}\,, \tag{6.7}$$

where $\boldsymbol{n}_h$ is the unit outward normal vector to $\partial\Omega_h$.

Proof. For any $\boldsymbol{v}_h \in \boldsymbol{V}_{1,h}$ and any face $T' \subset \Gamma_h^N$, we have

$$\begin{aligned} \|\boldsymbol{v}_h\cdot\boldsymbol{n}_h\|_{0,\infty,T'} &= \max_{x\in S_{1,T'}\cap\Gamma_h^N} |\boldsymbol{v}_h(x)\cdot((\boldsymbol{n}_h|_{T'}) - \boldsymbol{n}_h^*(x))| \leq \\ &\leq \max_{x\in S_{1,T'}\cap\Gamma_h^N} |(\boldsymbol{n}_h|_{T'}) - \boldsymbol{n}_h^*(x)|\, \|\boldsymbol{v}_h\|_{0,\infty,T'}\,. \end{aligned}$$

Using (4.7), (4.14), (6.4) and (3.1) we obtain $\|\boldsymbol{v}_h\cdot\boldsymbol{n}_h\|_{0,\infty,T'} \leq C\, h_{T'}^{\min\{1,\alpha\}}\, \|\boldsymbol{v}_h\|_{0,\infty,T'}$ and since $\|\boldsymbol{v}_h\|_{0,\infty,T'} \leq C\, h_{T'}^{-\frac{3}{2}}\, \|\boldsymbol{v}_h\|_{0,\frac{4}{3},T'}$, we deduce that $\|\boldsymbol{v}_h\cdot\boldsymbol{n}_h\|_{0,\frac{4}{3},T'} \leq h_{T'}^{\frac{3}{2}}\, \|\boldsymbol{v}_h\cdot\boldsymbol{n}_h\|_{0,\infty,T'} \leq C\, h_{T'}^{\min\{1,\alpha\}}\, \|\boldsymbol{v}_h\|_{0,\frac{4}{3},T'}$. That, together with (4.6), implies (6.7). □

Lemma 6.3. *There exists a constant C independent of h such that*

$$\|\boldsymbol{v}_h\cdot\bar{\boldsymbol{n}}_h\|_{0,1,\Gamma_h^N} \leq C\, h^{\min\{\frac{3}{2},\alpha\}}\, \|\boldsymbol{v}_h\|_{1,\Omega_h} \qquad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_{2,h}\,.$$

Proof. We extend the function $\boldsymbol{n}_h^*$ to a function $\boldsymbol{n}_h^* \in [\mathrm{V}_{2,h}]^3$ satisfying (6.4) also in vertices and midpoints of edges lying on $\partial\Gamma_h^N$, where $\bar{\boldsymbol{n}}_h$ is considered as continuous extension of $\bar{\boldsymbol{n}}_h|_{\Gamma_h^N}$ onto $\overline{\Gamma_h^N}$. Then $\|\boldsymbol{v}_h\cdot\bar{\boldsymbol{n}}_h\|_{0,1,\Gamma_h^N} \le \|\boldsymbol{v}_h\cdot(\bar{\boldsymbol{n}}_h-\boldsymbol{n}_h^*)\|_{0,1,\Gamma_h^N} + \|\boldsymbol{v}_h\cdot\boldsymbol{n}_h^*\|_{0,1,\Gamma_h^N} \le C\,h^\alpha\,\|\boldsymbol{v}_h\|_{0,1,\Gamma_h^N} + \|\boldsymbol{v}_h\cdot\boldsymbol{n}_h^*\|_{0,1,\Gamma_h^N}$ for any $\boldsymbol{v}_h \in \boldsymbol{V}_{2,h}$. Thus, since $\|\boldsymbol{v}_h\cdot\boldsymbol{n}_h^*\|_{0,1,T'} \le h_{T'}^2\,\|\boldsymbol{v}_h\cdot\boldsymbol{n}_h^*\|_{0,\infty,T'}$ for any $T'\subset\Gamma_h^N$, it suffices to show that

$$\|\boldsymbol{v}_h\cdot\boldsymbol{n}_h^*\|_{0,\infty,T'} \le C\,h^{\min\{\frac12,\alpha-1\}}\,\|\boldsymbol{v}_h\|_{1,T}\,, \tag{6.8}$$

where T is the element possessing the face T'. Let $\widehat{T}'$ be the face of the reference element $\widehat{T}$, for which $F_T(\widehat{T}') = T'$. Since the seminorm $|\cdot|_{3,\widehat{T}'}$ is a norm on the space $\{p\in P_4(\widehat{T}')\,;\ p(\widehat{x}) = 0\ \ \forall\,\widehat{x}\in S_{2,\widehat{T}'}\}$, which contains the function $(\widehat{\boldsymbol{v}}_{hT}\cdot\widehat{\boldsymbol{n}}_{hT}^*)|_{\widehat{T}'}$ (we use the notation from (3.2)), the equivalence of norms on finite–dimensional spaces implies that

$$\begin{aligned}\|\widehat{\boldsymbol{v}}_{hT}\cdot\widehat{\boldsymbol{n}}_{hT}^*\|_{0,\infty,\widehat{T}'} &\le C\,|\widehat{\boldsymbol{v}}_{hT}\cdot\widehat{\boldsymbol{n}}_{hT}^*|_{3,\widehat{T}'} = C\inf_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbf{R}^3}|(\widehat{\boldsymbol{v}}_{hT}-\boldsymbol{\alpha})\cdot(\widehat{\boldsymbol{n}}_{hT}^*-\boldsymbol{\beta})|_{3,\widehat{T}'} \le\\ &\le \widetilde{C}\inf_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbf{R}^3}\|(\widehat{\boldsymbol{v}}_{hT}-\boldsymbol{\alpha})\cdot(\widehat{\boldsymbol{n}}_{hT}^*-\boldsymbol{\beta})\|_{0,1,\widehat{T}'} \le \widetilde{C}\inf_{\boldsymbol{\alpha}\in\mathbf{R}^3}\|\widehat{\boldsymbol{v}}_{hT}-\boldsymbol{\alpha}\|_{0,\widehat{T}'}\inf_{\boldsymbol{\beta}\in\mathbf{R}^3}\|\widehat{\boldsymbol{n}}_{hT}^*-\boldsymbol{\beta}\|_{0,\widehat{T}'}\,.\end{aligned}$$

According to the trace theorems and Theorem 14.1 from [2], p. 120, we have

$$\inf_{\boldsymbol{\alpha}\in\mathbf{R}^3}\|\widehat{\boldsymbol{v}}_{hT}-\boldsymbol{\alpha}\|_{0,\widehat{T}'} \le C\,|\widehat{\boldsymbol{v}}_{hT}|_{1,\widehat{T}}\,, \qquad \inf_{\boldsymbol{\beta}\in\mathbf{R}^3}\|\widehat{\boldsymbol{n}}_{hT}^*-\boldsymbol{\beta}\|_{0,\widehat{T}'} \le C\,|\widehat{\boldsymbol{n}}_{hT}^*|_{1,\widehat{T}'}$$

and hence, using (6.6) and

$$C\,|\widehat{v}_T|_{1,\widehat{T}'} \le |v|_{1,T'} \le \widetilde{C}\,|\widehat{v}_T|_{1,\widehat{T}'} \qquad \forall\,v\in H^1(T') \tag{6.9}$$

(cf. [2], Section 15), we get $\|\boldsymbol{v}_h\cdot\boldsymbol{n}_h^*\|_{0,\infty,T'} \le C\,h_{T'}^{-\frac12}\,|\boldsymbol{v}_h|_{1,T}\,|\boldsymbol{n}_h^*|_{1,T'}$. Let $i\in\{K^D+1,\dots,K\}$ be such that $T'\subset\Gamma_{ih}$. Since $\widetilde{\Gamma}_i$ is a C^2 surface, there exists $\boldsymbol{m}\in C^1(\mathrm{I\!R}^3)^3$ satisfying $\boldsymbol{m} = \boldsymbol{n}$ on Γ_i. Then $\bar{\boldsymbol{n}}_h|_{\Gamma_{ih}} = (r_{1,h}\,\boldsymbol{m})|_{\Gamma_{ih}}$ and, according to Theorem 15.3 from [2], p. 124, we have $|\boldsymbol{m}-r_{1,h}\,\boldsymbol{m}|_{1,T'} \le C\,h_{T'}\,|\boldsymbol{m}|_{1,\infty,\widetilde{\Omega}}$. Therefore, using (6.9) and (6.4), we obtain $|\boldsymbol{n}_h^*|_{1,T'} \le |\boldsymbol{n}_h^*-\bar{\boldsymbol{n}}_h|_{1,T'} + |\bar{\boldsymbol{n}}_h-\boldsymbol{m}|_{1,T'} + |\boldsymbol{m}|_{1,T'} \le C\,h_{T'}^{\min\{1,\alpha\}}$, which proves (6.8). □

Lemma 6.4. *There exists an operator $r_{1,h}^*\in\mathcal{L}(H^2(\Omega)^3,[\mathrm{V}_{1,h}]^3)$ such that $r_{1,h}^*\,\boldsymbol{v} = \mathbf{0}$ on Γ_h^D for any $\boldsymbol{v}\in H^2(\Omega)^3$,*

$$(\boldsymbol{n}_h^*\cdot r_{1,h}^*\,\boldsymbol{v})(x) = (\boldsymbol{v}\cdot\boldsymbol{n}_h^*)(x) \qquad \forall\,\boldsymbol{v}\in H^2(\Omega)^3 \text{ and for any vertex } x\in\Gamma_h^N \tag{6.10}$$

and

$$\|r_{1,h}^*\,\boldsymbol{v}\|_{1,\Omega_h} \le C\,h^{\alpha-\frac12}\,\|\boldsymbol{v}\|_{2,\widetilde{\Omega}} \qquad \forall\,\boldsymbol{v}\in H^2(\widetilde{\Omega})^3\,,\ \boldsymbol{v}|_\Omega\in\mathbf{V}\,. \tag{6.11}$$

Proof. We define an operator $r_{1,h}^*\in\mathcal{L}(H^2(\Omega)^3,[\mathrm{V}_{1,h}]^3)$ satisfying for any $\boldsymbol{v}\in H^2(\Omega)^3$

$$(r_{1,h}^*\,\boldsymbol{v})(x) = \begin{cases} ((\boldsymbol{v}\cdot\boldsymbol{n}_h^*)\,\boldsymbol{n}_h^*/|\boldsymbol{n}_h^*|^2)(x) & \text{at any vertex } x\in\Gamma_h^N,\\ \mathbf{0} & \text{at any vertex } x\in\overline{\Omega_h}\setminus\Gamma_h^N. \end{cases}$$

Then $r_{1,h}^*\,\boldsymbol{v} = \mathbf{0}$ on Γ_h^D and (6.10) holds. For any $\boldsymbol{v}\in H^2(\widetilde{\Omega})^3$ with $\boldsymbol{v}|_\Omega\in\mathbf{V}$ and for any $T\in\mathcal{T}_h$, we obtain using the definition of $\mathbf{V}$

$$\|r_{1,h}^*\,\boldsymbol{v}\|_{0,\infty,T} = \max_{x\in S_{1,T}}|(r_{1,h}^*\,\boldsymbol{v})(x)| = \max_{x\in S_{1,T}\cap\Gamma_h^N}|(\boldsymbol{v}\cdot(\boldsymbol{n}_h^*-\boldsymbol{n})/|\boldsymbol{n}_h^*|)(x)|\,,$$

which gives $\|r^*_{1,h}\,\boldsymbol{v}\|_{0,\infty,T} \le C\, h_T^{\alpha}\, \|\boldsymbol{v}\|_{2,\widetilde{\Omega}}$ in view of (6.4). Due to (6.5) and (6.6), we get $\|r^*_{1,h}\,\boldsymbol{v}\|_{1,T} \le C\, h_T^{\frac{1}{2}}\, \|r^*_{1,h}\,\boldsymbol{v}\|_{0,\infty,T}$ and hence we derive

$$\|r^*_{1,h}\,\boldsymbol{v}\|^2_{1,\Omega_h} = \sum_{T\in\mathcal{T}_h, T\cap\Gamma_h^N\neq\emptyset} \|r^*_{1,h}\,\boldsymbol{v}\|^2_{1,T} \le C\, h^{2\,\alpha-1}\, \|\boldsymbol{v}\|^2_{2,\widetilde{\Omega}} \sum_{T'\subset\Gamma_h^N}\ \sum_{T\in\mathcal{T}_h, T\cap\overline{T'}\neq\emptyset} h_T^2\,.$$

The assumption (3.1) implies that $\sum_{T\in\mathcal{T}_h, T\cap\overline{T'}\neq\emptyset} h_T^2 \le C\, h_{T'}^2$, which yields (6.11). $\square$

Lemma 6.5. *Let the extensions $\widetilde{\Gamma}_i$ of Γ_i, $i = K^D+1,\dots,K$, be C^3 surfaces. Then there exists an operator $r^*_{2,h} \in \mathcal{L}(H^2(\widetilde{\Omega})^3, [\mathrm{V}_{2,h}]^3)$ such that $r^*_{2,h}\,\boldsymbol{v} = \boldsymbol{0}$ on Γ_h^D for any $\boldsymbol{v} \in H^2(\widetilde{\Omega})^3$,*

$$(\boldsymbol{n}_h^* \cdot r^*_{2,h}\,\boldsymbol{v})(x) = (\boldsymbol{v}\cdot\boldsymbol{n}_h^*)(x) \qquad \forall\ \boldsymbol{v} \in H^2(\widetilde{\Omega})^3\,,\ x \in S_h^N \tag{6.12}$$

and

$$\|r^*_{2,h}\,\boldsymbol{v}\|_{1,\Omega_h} \le C\, h^{\min\{\frac{3}{2},\alpha-\frac{1}{2}\}}\, \|\boldsymbol{v}\|_{3,\widetilde{\Omega}} \qquad \forall\ \boldsymbol{v} \in H^3(\widetilde{\Omega})^3\,,\ \boldsymbol{v}|_\Omega \in \mathbf{V}\,. \tag{6.13}$$

Proof. We define an operator $r^*_{2,h} \in \mathcal{L}(H^2(\widetilde{\Omega})^3, [\mathrm{V}_{2,h}]^3)$ satisfying for any $\boldsymbol{v} \in H^2(\widetilde{\Omega})^3$

$$(r^*_{2,h}\,\boldsymbol{v})(x) = \begin{cases} ((\boldsymbol{v}\cdot\boldsymbol{n}_h^*)\,\boldsymbol{n}_h^*/|\boldsymbol{n}_h^*|^2)(x) & \forall\ x \in S_h^N\,,\\ \boldsymbol{0} & \forall\ x \in S_{2,T}\setminus S_h^N,\ T\in\mathcal{T}_h\,. \end{cases}$$

Then $r^*_{2,h}\,\boldsymbol{v} = \boldsymbol{0}$ on Γ_h^D and (6.12) holds. For any $\boldsymbol{v} \in H^3(\widetilde{\Omega})^3$ with $\boldsymbol{v}|_\Omega \in \mathbf{V}$ and for any $T \in \mathcal{T}_h$, we have

$$\|r^*_{2,h}\,\boldsymbol{v}\|_{0,\infty,T} \le 3 \max_{x\in S_{2,T}\cap S_h^N} |(\boldsymbol{v}\cdot(\boldsymbol{n}_h^* - \bar{\boldsymbol{n}}_h)/|\boldsymbol{n}_h^*|)(x)| + 3 \max_{x\in S_{2,T}\cap S_h^N} |(\boldsymbol{v}\cdot\bar{\boldsymbol{n}}_h/|\boldsymbol{n}_h^*|)(x)|\,.$$

Consider any $x \in S_h^N$ and let $i \in \{K^D+1,\dots,K\}$ be such that there exists a face $T' \subset \Gamma_{ih}$ satisfying $x \in \overline{T'}$. Since $\widetilde{\Gamma}_i$ is a C^3 surface, there exists $\boldsymbol{m} \in C^2(\mathbb{R}^3)^3$ such that $\boldsymbol{m} = \boldsymbol{n}$ on Γ_i. Further, according to (4.7) and (4.9), there exists $\widetilde{x} \in \Gamma_{T'}$ with $|x - \widetilde{x}| \le C\, h_{T'}^2$. Then $|\bar{\boldsymbol{n}}_h(x) - \boldsymbol{m}(\widetilde{x})| \le |(r_{1,h}\,\boldsymbol{m} - \boldsymbol{m})(x)| + |\boldsymbol{m}(x) - \boldsymbol{m}(\widetilde{x})| \le C\, h_{T'}^2\, \|\boldsymbol{m}\|_{2,\infty,\widetilde{\Omega}}$ and $|\boldsymbol{v}(x) - \boldsymbol{v}(\widetilde{x})| \le C\, h_{T'}^2\, |\boldsymbol{v}|_{1,\infty,\widetilde{\Omega}}$. Since $|\boldsymbol{m}(\widetilde{x})| = 1$, we obtain $|1 - |\bar{\boldsymbol{n}}_h(x)|| \le C\, h_{T'}^2$. Thus, applying (6.4) and (3.1) and using the fact that $(\boldsymbol{v}\cdot\boldsymbol{m})(\widetilde{x}) = 0$, we infer that $\|r^*_{2,h}\,\boldsymbol{v}\|_{0,\infty,T} \le C\, h_T^{\min\{2,\alpha\}}\, \|\boldsymbol{v}\|_{3,\widetilde{\Omega}}$ and (6.13) follows analogously as (6.11). $\square$

Lemma 6.6. *There exists an operator $r'_{2,h} \in \mathcal{L}(H^2(\widetilde{\Omega})^3, [\mathrm{V}_{2,h}]^3)$ such that $r'_{2,h}\,\boldsymbol{v} = \boldsymbol{0}$ on Γ_h^D for any $\boldsymbol{v} \in H^2(\widetilde{\Omega})^3$,*

$$(r'_{2,h}\,\boldsymbol{v})(x) = \boldsymbol{v}(x) \qquad \forall\ \boldsymbol{v} \in H^2(\widetilde{\Omega})^3\,,\ x \in S_h^N \tag{6.14}$$

and

$$\|\boldsymbol{v} - r'_{2,h}\,\boldsymbol{v}\|_{1,\Omega_h} \le C\, h^{\frac{3}{2}}\, \|\boldsymbol{v}\|_{3,\widetilde{\Omega}} \qquad \forall\ \boldsymbol{v} \in H^3(\widetilde{\Omega})^3\,,\ \boldsymbol{v}|_\Omega \in \mathbf{V}\,. \tag{6.15}$$

Proof. We define an operator $r'_{2,h} \in \mathcal{L}(H^2(\widetilde{\Omega})^3, [\mathrm{V}_{2,h}]^3)$ satisfying for any $\boldsymbol{v} \in H^2(\widetilde{\Omega})^3$ and any $T \in \mathcal{T}_h$

$$(r'_{2,h}\,\boldsymbol{v})(x) = \begin{cases} \boldsymbol{v}(x) & \forall\ x \in S_{2,T}\setminus\overline{\Gamma_h^D}\,,\\ \boldsymbol{0} & \forall\ x \in S_{2,T}\cap\overline{\Gamma_h^D}\,. \end{cases}$$

Then $r'_{2,h}\,\boldsymbol{v} = \mathbf{0}$ on Γ_h^D and (6.14) holds. Consider any $\boldsymbol{v} \in H^3(\widetilde{\Omega})^3$ with $\boldsymbol{v}|_\Omega \in \mathbf{V}$ and denote $\boldsymbol{u} = r_{2,h}\,\boldsymbol{v} - r'_{2,h}\,\boldsymbol{v}$. Then $\boldsymbol{u} \in [\mathrm{V}_{2,h}]^3$ and hence we have for any $T \in \mathcal{T}_h$

$$\|\boldsymbol{u}\|_{0,\infty,T} \le 3 \max_{x\in S_{2,T}} |\boldsymbol{u}(x)| = 3 \max_{x\in S_{2,T}\cap\overline{\Gamma_h^D}} |\boldsymbol{v}(x)|\,.$$

Choose any $x \in S_{2,T} \cap \overline{\Gamma_h^D}$ and let $T' \subset \Gamma_h^D$ be a face satisfying $x \in \overline{T'}$. According to (4.7) and (4.9), there exists $\tilde{x} \in \Gamma_{T'}$ with $|x - \tilde{x}| \le C\,h_{T'}^2$. Then $|\boldsymbol{v}(x)| = |\boldsymbol{v}(x) - \boldsymbol{v}(\tilde{x})| \le C\,h_{T'}^2\,|\boldsymbol{v}|_{1,\infty,\widetilde{\Omega}}$ and hence we infer using (3.1) that $\|\boldsymbol{u}\|_{0,\infty,T} \le C\,h_T^2\,\|\boldsymbol{v}\|_{3,\widetilde{\Omega}}$. Analogously as in the proof of Lemma 6.5, we obtain $\|\boldsymbol{u}\|_{1,\Omega_h} \le C\,h^{\frac{3}{2}}\,\|\boldsymbol{v}\|_{3,\widetilde{\Omega}}$ and (6.15) follows applying (6.1). □

The above results lead to the following theorem.

Theorem 6.1. *Let (6.4) hold with $\alpha \ge \frac{3}{2}$ and let $\mathbf{V}_h = \boldsymbol{V}_{1,h}$. Then the assumption A1 holds with $l = 2$. If (6.4) holds with $\alpha \ge 2$, the extensions $\widetilde{\Gamma}_i$ of Γ_i, $i = K^D+1,\dots,K$, are C^3 surfaces and $\mathbf{V}_h = \boldsymbol{V}_{2,h}$, then the assumption A1 holds with $l = 3$.*

Proof. We denote $r'_{1,h} = r_{1,h}$. Consider $l \in \{2,3\}$ and let $\mathrm{P} \in \mathcal{L}(H^l(\Omega)^3, H^l(\widetilde{\Omega})^3)$ be some extension operator. Setting $r_h\,\boldsymbol{v} = r'_{l-1,h}(\mathrm{P}\,\boldsymbol{v}) - r^*_{l-1,h}(\mathrm{P}\,\boldsymbol{v})$ for $\boldsymbol{v} \in H^l(\Omega)^3$, we have $r_h \in \mathcal{L}(\mathbf{V}\cap H^l(\Omega)^3, \boldsymbol{V}_{l-1,h})$ and (3.5) holds. □

In order to satisfy the assumption A3, we have, in the most cases, to enlarge the above spaces $\boldsymbol{V}_{1,h}$ and $\boldsymbol{V}_{2,h}$. For this, we introduce the following notations. For any $T \in \mathcal{T}_h$, we denote

$$\begin{aligned}\boldsymbol{\mathcal{P}}(T) &= \mathrm{span}\{\lambda_1\,\lambda_2\,\lambda_3\,\boldsymbol{n}_4,\ \lambda_2\,\lambda_3\,\lambda_4\,\boldsymbol{n}_1,\ \lambda_3\,\lambda_4\,\lambda_1\,\boldsymbol{n}_2,\ \lambda_4\,\lambda_1\,\lambda_2\,\boldsymbol{n}_3\}\,,\\ \mathcal{R}(T) &= \mathrm{span}\{\lambda_1\,\lambda_2\,\lambda_3\,\lambda_4\}\,,\end{aligned}$$

where λ_1, λ_2, λ_3, λ_4 are the barycentric coordinates with respect to the vertices x_1, x_2, x_3, x_4 of T, respectively, and $\boldsymbol{n}_i$ is a normal vector to the face of T lying opposite the vertex x_i, $i = 1,2,3,4$. Now we define the spaces

$$\begin{aligned}\boldsymbol{\mathcal{P}}_h &= \{\boldsymbol{v} \in H_0^1(\Omega_h)^3\,;\ \boldsymbol{v}|_T \in \boldsymbol{\mathcal{P}}(T)\ \ \forall\,T \in \mathcal{T}_h\}\,,\\ \boldsymbol{\mathcal{R}}_h &= \{\boldsymbol{v} \in H_0^1(\Omega_h)^3\,;\ \boldsymbol{v}|_T \in \mathcal{R}(T)^3\ \ \forall\,T \in \mathcal{T}_h\}\,.\end{aligned}$$

The following theorem gives several sufficient conditions for the validity of A3.

Theorem 6.2. *The assumption A3 is satisfied if $\mathrm{Q}_h = \mathrm{M}_{0,h} \cap L_0^2(\Omega_h)$ and $\mathbf{V}_h \supset [\mathrm{V}_{1,h}\cap H_0^1(\Omega_h)]^3 \oplus \boldsymbol{\mathcal{P}}_h$ or if $\mathrm{Q}_h = \mathrm{M}_{1,h}\cap L_0^2(\Omega_h)$ and $\mathbf{V}_h \supset [\mathrm{V}_{1,h}\cap H_0^1(\Omega_h)]^3 \oplus \boldsymbol{\mathcal{P}}_h \oplus \boldsymbol{\mathcal{R}}_h$ or if $\mathrm{Q}_h = \mathrm{V}_{1,h}\cap L_0^2(\Omega_h)$ and $\mathbf{V}_h \supset [\mathrm{V}_{1,h}\cap H_0^1(\Omega_h)]^3 \oplus \boldsymbol{\mathcal{R}}_h$. If, for any $T \in \mathcal{T}_h$, at least one vertex of T lies in Ω, then A3 also holds for $\mathrm{Q}_h = \mathrm{V}_{1,h}\cap L_0^2(\Omega_h)$ and $\mathbf{V}_h \supset [\mathrm{V}_{2,h}\cap H_0^1(\Omega_h)]^3$.*

Proof. The basic ideas of the proof are the same as for $\Omega_h = \Omega$ (see e.g. [5] and [1]). □

Using the results of this section, we can set up pairs of finite element spaces which, under the assumptions of Theorems 6.1 and 6.2, satisfy the assumptions A1–A3 with $l = 2$ or $l = 3$. Five of these pairs are presented in Table 1, where also the convergence orders are given to which these spaces lead provided the assumptions of Theorem 5.1 are fulfilled.

Table 1: Examples of the finite element spaces $\mathbf{V}_h$ and Q_h leading to the convergence order 1 or $\frac{3}{2}$

conv. order	$\mathbf{V}_h$	Q_h
1	$\boldsymbol{V}_{1,h} \oplus \boldsymbol{\mathcal{P}}_h$	$\mathrm{M}_{0,h} \cap L^2_0(\Omega_h)$
1	$\boldsymbol{V}_{1,h} \oplus \boldsymbol{\mathcal{R}}_h$	$\mathrm{V}_{1,h} \cap L^2_0(\Omega_h)$
3/2	$\boldsymbol{V}_{2,h} \oplus \boldsymbol{\mathcal{P}}_h \oplus \boldsymbol{\mathcal{R}}_h$	$\mathrm{M}_{1,h} \cap L^2_0(\Omega_h)$
3/2	$\boldsymbol{V}_{2,h} \oplus \boldsymbol{\mathcal{R}}_h$	$\mathrm{V}_{1,h} \cap L^2_0(\Omega_h)$
3/2	$\boldsymbol{V}_{2,h}$ [cf. Th. 6.2]	$\mathrm{V}_{1,h} \cap L^2_0(\Omega_h)$

REFERENCES

1. F. Brezzi, M. Fortin, *Mixed and Hybrid Finite Element Methods.* Springer–Verlag, New York, 1991.
2. P. G. Ciarlet, Basic error estimates for elliptic problems. In: *Handbook of Numerical Analysis, v. 2 – Finite Element Methods (pt. 1)* (Eds. P. G. Ciarlet and J. L. Lions), North–Holland, Amsterdam, 1991, pp. 17–351.
3. M. Feistauer and A. Ženíšek, Finite element solution of nonlinear elliptic problems. *Numer. Math.* (1987) **50**, 451–475.
4. G. J. Fix, M. D. Gunzburger, J. S. Peterson, On finite element approximations of problems having inhomogeneous essential boundary conditions. *Comput. Math. Appl.* (1983) **9**, 687–700.
5. V. Girault, P.–A. Raviart, *Finite Element Methods for Navier–Stokes Equations.* Springer–Verlag, Berlin, 1986.
6. P. Knobloch, Discrete Friedrichs' and Korn's inequalities in two and three dimensions. *East–West J. Numer. Math.* (1996) **4**, 35–51.
7. P. Knobloch, *Solvability and Finite Element Discretization of a Mathematical Model Related to Czochralski Crystal Growth.* PhD Thesis, Preprint MBI–96–5, Otto–von–Guericke–Universität, Magdeburg, 1996.
8. S. Korotov, M. Křížek, *Finite element analysis of variational crimes for a quasilinear elliptic problem in 3D.* Report 7/1997, University of Jyväskylä, Laboratory of Scientific Computing, 1997.
9. J. Nečas, *Les méthodes directes en théorie des équations elliptiques.* Academia, Praha, 1967.
10. J. Nečas and I. Hlaváček, *Mathematical Theory of Elastic and Elastico–Plastic Bodies: An Introduction.* Elsevier, Amsterdam, 1981.
11. G. Strang, G. J. Fix, *An Analysis of the Finite Element Method.* Prentice–Hall, Englewood Cliffs, New Jersey, 1973.
12. R. Verfürth, Finite element approximation of steady Navier–Stokes equations with mixed boundary conditions, M^2AN (1985) **19**, 461–475.
13. A. Ženíšek, Discrete forms of Friedrichs' inequalities in the finite element method. *RAIRO Anal. Numér.* (1981) **15**, 265–286.