A STREAMLINE DIFFUSION METHOD FOR NONCONFORMING FINITE ELEMENT APPROXIMATIONS APPLIED TO THE LINEARIZED INCOMPRESSIBLE NAVIER–STOKES EQUATION

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Recently, nonconforming approximations of streamline diffusion type have been studied for solving scalar convection–diffusion problems. Theoretical and numerical investigations for triangular and tetrahedral meshes have shown that the usual application of the SDFEM gives no sufficient stabilization. By adding appropriate jump terms it could be shown that for smooth solutions the method guarantees error estimates of the same quality as in the conforming case. In this paper the analysis is extended to the linearized Navier–Stokes equation.

1 Introduction

The streamline diffusion finite element method (SDFEM) for the solution of convection–diffusion problems has been successfully applied in the case of conforming finite element spaces. This method was proposed first by Hughes and Brooks [1] and applied to several classes of problems. Starting with the fundamental work by Nävert [2], it was mainly analyzed by Johnson and his co–workers [3,4,5]. Nowadays, the convergence properties are well understood in the conforming case [6,7,8].

For a scalar convection–diffusion problem, a first analysis of convergence properties for a class of nonconforming SDFEM reflecting the dependency of the perturbation parameter is given in [9]. If the standard SDFEM is applied, large oscillations can occur. The theoretical and numerical investigations have shown that by adding jump terms to the discretization one can preserve the same $O(h^{k+1/2})$ order of L^2–convergence as in the conforming case for piecewise polynomials of degree k. Since the error analysis essentially uses the fact that there exists a conforming finite element subspace of the nonconforming approximation space, the analysis in [9] cannot be applied to the nonconforming

Q_1^{rot} quadrilateral elements [10]. Therefore, new jump terms have been developed in [11] which also allow to handle mixed meshes consisting of triangles and quadrilaterals. The efficient solution of the algebraic systems of equations in the case of rectangular Q_1^{rot} discretizations of convection–diffusion problems is discussed in [12].

For the Navier–Stokes equation the convergence properties of streamline diffusion type methods using conforming finite element have been studied in [4,7]. However, up to now there is no comparable detailed analysis of the convergence properties of nonconforming finite element approximations of streamline diffusion type. In [13] error estimates for the P_1/P_0 discretization of the Navier–Stokes equation have been derived in the case of small data but the dependency of the error constant on the Reynolds number has not been specified. For a better understanding of the convergence properties, in this paper we extend the analysis given in [9] and [11] to the linearized Navier–Stokes equation.

The paper is organized as follows. In Section 2, notations and approximation properties of the considered nonconforming finite element spaces are introduced. In Section 3, the modified nonconforming discretization of the linearized Navier–Stokes equation is established and its consistency is addressed. Finally, Section 4 is devoted to the error estimates of the method.

2 Notations and preliminaries

For simplicity we will restrict our considerations to the two–dimensional case. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a polygonal boundary $\partial\Omega$. Let $0 < \varepsilon \ll 1$ be the inverse of the Reynolds number and let $b \in W^{1,\infty}(\Omega)^2$ and $f \in L^2(\Omega)^2$ be given functions. We assume that b is divergence–free, i.e. $\nabla \cdot b = 0$. Denoting $V = H_0^1(\Omega)^2$, $Q = L_0^2(\Omega)$, we consider the stationary linearized Navier–Stokes equation in its weak form:

Find $u \in V$ and $p \in Q$ such that

$$\varepsilon(\nabla u, \nabla v) + (b \cdot \nabla u, v) - (p, \nabla \cdot v) + (q, \nabla \cdot u) = (f, v) \quad \forall\, v \in V,\, q \in Q\,, \tag{1}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. It is well known that there exists a unique solution of (1).

For discretizing the problem (1), we will consider triangulations $\mathcal{T}_h$ of Ω consisting of elements K which are either triangles or convex quadrilaterals. The triangulations are assumed to possess the usual compatibility properties and to be shape regular in the sense of [11]. We denote by $\mathcal{E}_h$ the edges E of $\mathcal{T}_h$ and we assign to each edge a unit normal vector n_E. For a function v defined on Ω and an edge $E \not\subset \partial\Omega$ of $\mathcal{T}_h$, we define the jump $[|v|]_E$ and the average

$\langle|v|\rangle_E$ by

$$[|v|]_E = v|_{K_2} - v|_{K_1}\,, \qquad \langle|v|\rangle_E = \tfrac{1}{2}\,(v|_{K_1} + v|_{K_2})\,,$$

where K_1, K_2 are the two elements possessing the edge E chosen in such a way that n_E is pointed into K_2. For $E \subset \partial\Omega$, the jump and average are naturally defined by extending v by zero outside Ω.

We define the nonconforming finite element spaces

$$\begin{aligned} V_h\; &= \{v \in L^2(\Omega)^2;\; v|_K \in P(K)^2 \;\; \forall\; K \in \mathcal{T}_h,\;\; \textstyle\int_E [|v|]_E \,\mathrm{d}\sigma = 0 \;\; \forall\; E \in \mathcal{E}_h\}\,, \\ W_h &= \{v \in V_h;\; (\nabla\cdot v, 1)_K = 0 \;\; \forall\; K \in \mathcal{T}_h\}\,, \\ Q_h\; &= \{q \in L^2_0(\Omega);\; q|_K \in P_0(K) \;\; \forall\; K \in \mathcal{T}_h\}\,, \end{aligned}$$

where $L^2_0(\Omega) = \{q \in L^2(\Omega);\; (q,1) = 0\}$, $(\cdot,\cdot)_K$ is the inner product in $L^2(K)$ and $P(K) = P_1(K)$ if $\mathcal{T}_h$ consists of triangles, and $P(K) = Q_1^{rot}(K)$ if $\mathcal{T}_h$ consists of quadrilaterals. We refer to [10] for a definition of the rotated bilinear functions $Q_1^{rot}(K)$. In both the triangular and the quadrilateral cases, there exists operators $I_h : H^2(\Omega)^2 \cap H^1_0(\Omega)^2 \to V_h$ and $J_h : H^1(\Omega) \cap L^2_0(\Omega) \to Q_h$ such that, for any $K \in \mathcal{T}_h$,

$$|v - I_h v|_{l,K} \le C\, h_K^{2-l}\, |v|_{2,K} \qquad \forall\; v \in H^2(\Omega)^2 \cap H^1_0(\Omega)^2\,, \quad l = 0,1,2\,, \tag{2}$$

$$\|v - I_h v\|_{0,E} \le C\, h_E^{3/2}\, |v|_{2,K} \qquad \forall\; v \in H^2(\Omega)^2 \cap H^1_0(\Omega)^2,\;\; E \subset K\,, \tag{3}$$

$$|q - J_h q|_{l,K} \le C\, h_K^{1-l}\, |q|_{1,K} \qquad \forall\; q \in H^1(\Omega) \cap L^2_0(\Omega)\,, \quad l = 0,1\,, \tag{4}$$

and $I_h v \in W_h$ for any $v \in H^2(\Omega)^2 \cap H^1_0(\Omega)^2$ satisfying $\nabla\cdot v = 0$. In (2)–(4), we use the standard notations h_K for the diameter of K and h_E for the length of E.

3 Nonconforming streamline diffusion method

The nonconforming streamline diffusion method for the linearized Navier–Stokes equation (1) reads:

Find $u_h \in V_h$ and $p_h \in Q_h$ such that

$$a_h(u_h, v_h) + b_h(v_h, p_h) - b_h(u_h, q_h) = l_h(v_h) \qquad \forall\; v_h \in V_h,\, q_h \in Q_h\,, \tag{5}$$

where

$$\begin{aligned} a_h(u,v) = \sum_K \{&\varepsilon(\nabla u, \nabla v)_K + (b\cdot\nabla u, v)_K + \\ &+ (-\varepsilon\Delta u + b\cdot\nabla u, \delta_K\, b\cdot\nabla v)_K\} + \\ &+ \sum_E \int_E b\cdot n_E\,[|u|]_E\cdot\langle|v|\rangle_E\,\mathrm{d}\sigma + \sum_E \int_E \sigma_E\,[|u|]_E\cdot[|v|]_E\,\mathrm{d}\sigma\,, \end{aligned}$$

$$b_h(v,q) = -\sum_K (q, \nabla\cdot v)_K\,, \qquad l_h(v) = (f,v) + \sum_K (f, \delta_K\, b\cdot\nabla v)_K\,.$$

The parameter δ_K is positive and will be specified in the next section. The parameter σ_E is only assumed to be nonnegative. In view of the jump terms in the bilinear form a_h, we will be able to show the coercivity of a_h with respect to the mesh–dependent norm

$$|||v||| = \left(\sum_K \{\varepsilon\,|v|^2_{1,K} + \delta_K\,\|b\cdot\nabla v\|^2_{0,K}\} + \sum_E \sigma_E\,\|\,[|v|]_E\|^2_{0,E}\right)^{1/2}\,.$$

The error estimates will be established under the assumption that the solution of (1) satisfies $u\in H^2(\Omega)^2$ and $p\in H^1(\Omega)$. Thus, we have

$$-\varepsilon\Delta u + b\cdot\nabla u + \nabla p = f\,, \qquad \nabla\cdot u = 0$$

and hence, multiplying these equations by $v_h + \delta_K\, b\cdot\nabla v_h$ and q_h, respectively, integrating over K, applying Green's theorem, summing up over all elements and using the fact that $[|u|]_E = 0$ for any edge E, we infer that

$$a_h(u,v_h)+b_h(v_h,p)-b_h(u,q_h) = l_h(v_h)+d_h(p,v_h)+r^1_h(u,v_h)+r^2_h(p,v_h)\,, \quad (6)$$

where

$$\begin{aligned}
d_h(p,v_h) &= -\sum_K (\nabla p, \delta_K\, b\cdot\nabla v_h)_K\,,\\
r^1_h(u,v_h) &= -\sum_E \int_E \varepsilon\,\frac{\partial u}{\partial n_E}\cdot[|v_h|]_E\,\mathrm{d}\sigma\,,\\
r^2_h(p,v_h) &= \sum_E \int_E p\,[|v_h|]_E\cdot n_E\,\mathrm{d}\sigma\,.
\end{aligned}$$

Since $d_h(p_h,v_h) = 0$ for any $v_h\in V_h$ and $r^1_h(u,v_h)+r^2_h(p,v_h) = 0$ for $v_h\in V_h\cap H^1_0(\Omega)^2$, the discrete problem (5) is consistent in the sense that both the discrete and the weak solutions satisfy the same equations for tests functions $v_h\in V_h\cap H^1_0(\Omega)^2$.

4 Error estimates

In consequence of the shape regularity of the triangulations, there exists a constant μ independent of h such that

$$\|\Delta v_h\|_{0,K} \le \mu\, h_K^{-1}\,|v_h|_{1,K} \qquad \forall\ v_h\in V_h,\ K\in\mathcal{T}_h\,. \quad (7)$$

If $\mathcal{T}_h$ consists of triangles, then (7) holds with $\mu = 0$.

We assume that

$$
\begin{aligned}
&0 < \delta_K && \text{if } \mathcal{T}_h \text{ consists of triangles;}\\
&0 < \delta_K \leq \frac{h_K^2}{\varepsilon\,\mu^2} && \text{if } \mathcal{T}_h \text{ consists of quadrilaterals.}
\end{aligned}
$$

Lemma 1 *We have*

$$a_h(v_h, v_h) \geq \frac{1}{2}\,|||v_h|||^2 \qquad \forall\ v_h \in V_h\,. \tag{8}$$

Proof. Integrating the term $(b\cdot\nabla u, v)_K$ by parts, we obtain

$$(b\cdot\nabla u, v)_K = -(b\cdot\nabla v, u)_K + \int_{\partial K} b\cdot n_{\partial K}\,(u\cdot v)\,\mathrm{d}\sigma \quad \forall\ u,v\in H^1(K)^2\,, \tag{9}$$

which implies that, for any $v_h \in V_h$,

$$\sum_K (b\cdot\nabla v_h, v_h)_K + \sum_E \int_E b\cdot n_E\,[|v_h|]_E\cdot\langle|v_h|\rangle_E\,\mathrm{d}\sigma = 0$$

and therefore $a_h(v_h,v_h) = |||v_h|||^2 - \sum_K(\varepsilon\Delta v_h, \delta_K\, b\cdot\nabla v_h)_K$. In the triangular case, the second term on the right–hand side is equal to zero and hence (8) holds. In case of quadrilateral elements, we obtain using (7)

$$\sum_K(\varepsilon\Delta v_h, \delta_K\, b\cdot\nabla v_h)_K \leq \sum_K \frac{\delta_K}{2}\,\{\|\varepsilon\Delta v_h\|_{0,K}^2 + \|b\cdot\nabla v_h\|_{0,K}^2\} \leq \frac{1}{2}|||v_h|||^2\,,$$

which proves the lemma. ♣

Lemma 2 *The interpolation error $w = u - I_h\,u$ satisfies for any $u\in H^2(\Omega)^2$ and any $v_h \in V_h\setminus\{0\}$*

$$\frac{a_h(w,v_h)}{|||v_h|||} \leq C\left(\sum_K \lambda_K\,h_K^2\ |u|_{2,K}^2 + \sum_E \lambda_E\,(1+\sigma_E^2)\,h_E^3\,|u|_{2,K_E}^2\right)^{1/2},$$

where K_E is the union of the elements possessing the edge E and

$$\lambda_K = \varepsilon + \delta_K + \delta_K^{-1}\,h_K^2 + \varepsilon\,\delta_K^2\,h_K^{-2}\,, \qquad \lambda_E = \min\left(\frac{1}{\sigma_E}, \frac{h_E}{\varepsilon}\right)\,.$$

Proof. Using (9), we obtain

$$
\begin{aligned}
a_h(w,v_h) = &\sum_K\{\varepsilon(\nabla w, \nabla v_h)_K - (w, b\cdot\nabla v_h)_K + \\
&+ (-\varepsilon\Delta w + b\cdot\nabla w, \delta_K\, b\cdot\nabla v_h)_K\} - \\
&- \sum_E\int_E b\cdot n_E\,\langle|w|\rangle_E\cdot[|v_h|]_E\,\mathrm{d}\sigma + \sum_E\int_E \sigma_E\,[|w|]_E\cdot[|v_h|]_E\,\mathrm{d}\sigma\,.
\end{aligned}
$$

The terms in the sum over elements can be estimated by

$$(\varepsilon^{1/2}\,|w|_{1,K} + \delta_K^{-1/2}\,\|w\|_{0,K} + \varepsilon^{1/2}\,\delta_K\,\|b\|_{0,\infty,K}\,\|\Delta w\|_{0,K} + \\ + \delta_K^{1/2}\,\|b\|_{0,\infty,K}\,|w|_{1,K})(\varepsilon^{1/2}\,|v_h|_{1,K} + \delta_K^{1/2}\,\|b\cdot\nabla v_h\|_{0,K})\,.$$

Applying (2), summing up over all elements and using the Hölder inequality, we obtain a term which can be bounded by $C\,(\sum_K \lambda_K\,h_K^2\;|u|_{2,K}^2)^{1/2}\,|||v_h|||$. Further, using (3), the integral $\int_E \langle|w|\rangle_E \cdot [|v_h|]_E\,\mathrm{d}\sigma$ can be estimated by

$$C\,(\|w|_{K_1}\|_{0,E} + \|w|_{K_2}\|_{0,E})\,\|\,[|v_h|]_E\|_{0,E} \le \widetilde{C}\,h_E^{3/2}\,|u|_{2,K_E}\,\|\,[|v_h|]_E\|_{0,E}\,,$$

where K_1, K_2 are elements satisfying $K_1\cup K_2 = K_E$. As $[|v_h|]_E = [|v_h - z|]_E$ for any $z\in\mathbb{R}^2$, it is easy to show that $\|\,[|v_h|]_E\|_{0,E} \le C\,h_E^{1/2}\,|v_h|_{1,K_E}$. Thus, we infer that the integrals in the sums over edges are bounded by

$$C\,\lambda_E^{1/2}\,(1+\sigma_E)\,h_E^{3/2}\,|u|_{2,K_E}\;(\varepsilon^{1/2}\,|v_h|_{1,K_E} + {\sigma_E}^{1/2}\,\|\,[|v_h|]_E\|_{0,E})$$

and the lemma is proven. ♣

Lemma 3 *Let the solution of (1) satisfy $u\in H^2(\Omega)^2$, $p\in H^1(\Omega)$. If the triangulations consist of triangles, then we have for any $v_h\in W_h\setminus\{0\}$*

$$\frac{a_h(u-u_h,v_h)}{|||v_h|||} \le C\left(\sum_K \{\varepsilon\,h_K^2\;|u|_{2,K}^2 + \delta_K\,|p|_{1,K}^2\} + \sum_E \lambda_E\,h_E\,|p|_{1,K_E}^2\right)^{1/2}$$

and if they consist of quadrilaterals, we have

$$\frac{a_h(u-u_h,v_h)}{|||v_h|||} \le C\left(\sum_K \{\varepsilon\,h_K^2\;|u|_{2,K}^2 + (\delta_K + \varepsilon^{-1}\,h_K^2)\,|p|_{1,K}^2\}\right)^{1/2}.$$

Further, we have for any $v_h\in V_h\setminus\{0\}$

$$b_h(v_h,p-p_h) \le C\,\{(\delta_h + h)\,(\varepsilon\,|u|_{2,\Omega} + |p|_{1,\Omega}) + \mu_h^{1/2}\,|||u-u_h|||\}\,\|v_h\|_h\,,$$

where

$$\|v_h\|_h = \Big(\sum_K |v_h|_{1,K}^2\Big)^{1/2}, \qquad \delta_h = \max_K \delta_K\,,$$
$$\mu_h = \max_K\{\varepsilon + \delta_K + \min(\varepsilon^{-1},\delta_K^{-1})\} + \max_E\{h_E\,(\sigma_E + \sigma_E^{-1})\}\,.$$

Proof. Subtracting (5) from (6), we have for $q_h = 0$ and any $v_h\in V_h$

$$a_h(u-u_h,v_h) = -b_h(v_h,p-p_h) + d_h(p,v_h) + r_h^1(u,v_h) + r_h^2(p,v_h)\,.$$

Let $v_h \in W_h$. If $\mathcal{T}_h$ consists of triangles, we have $\nabla \cdot v_h|_K = 0$ for any $K \in \mathcal{T}_h$ and hence $b_h(v_h, p - p_h) = 0$. In case of quadrilateral triangulations, we use the fact that $b_h(v_h, p - p_h) = b_h(v_h, p - J_h p)$ and that, in view of (4),

$$b_h(v_h, p - J_h p) \leq \sqrt{2} \sum_K \|p - J_h p\|_{0,K} \, |v_h|_{1,K} \leq C \sqrt{\sum_K \frac{h_K^2}{\varepsilon} \, |p|_{1,K}^2} \, |||v_h||| \, .$$

Further, we employ Lemma 2 from [9] to obtain

$$\int_E \varepsilon \, \frac{\partial u}{\partial n_E} \cdot [|v_h|]_E \, \mathrm{d}\sigma \leq C \, \varepsilon^{1/2} \, h_E \, |u|_{2,K_E} \, \varepsilon^{1/2} \, |v_h|_{1,K_E} \, ,$$

$$\int_E p \, [|v_h|]_E \cdot n_E \, \mathrm{d}\sigma \leq C \, \varepsilon^{-1/2} \, h_E \, |p|_{1,K_E} \, \varepsilon^{1/2} \, |v_h|_{1,K_E} \, ,$$

$$\int_E p \, [|v_h|]_E \cdot n_E \, \mathrm{d}\sigma \leq C \, \sigma_E^{-1/2} \, h_E^{1/2} \, |p|_{1,K_E} \, \sigma_E^{1/2} \, \| \, [|v_h|]_E \|_{0,E} \, .$$

The term $(\nabla p, \delta_K \, b \cdot \nabla v_h)_K$ can be estimated by $\delta_K^{1/2} \, |p|_{1,K} \, \delta_K^{1/2} \, \|b \cdot \nabla v_h\|_{0,K}$ and hence, summing up over elements and edges, we obtain the first two estimates of the lemma. Now, let $v_h \in V_h$. Using the above relations, we get

$$d_h(p, v_h) + r_h^1(u, v_h) + r_h^2(p, v_h) \leq C \, \{(\delta_h + h) \, |p|_{1,\Omega} + \varepsilon \, h \, |u|_{2,\Omega}\} \, \|v_h\|_h$$

and hence, for proving the last estimate of the lemma, it suffices to bound $a_h(u - u_h, v_h)$ in terms of $|||u - u_h|||$ and $\|v_h\|_h$. We set $z = u - u_h$. The first term of $a_h(z, v_h)$ can be bounded by $\varepsilon^{1/2} \, |||z||| \, \|v_h\|_h$. Concerning the second term, we can use either $(b \cdot \nabla z, v_h)_K \leq C \, |z|_{1,K} \, \|v_h\|_{0,K}$ or $(b \cdot \nabla z, v_h)_K \leq \|b \cdot \nabla z\|_{0,K} \, \|v_h\|_{0,K}$, which leads to

$$(b \cdot \nabla z, v_h)_K \leq C \, \min\{\varepsilon^{-1/2}, \delta_K^{-1/2}\} \, (\varepsilon^{1/2} \, |z|_{1,K} + \delta_K^{1/2} \, \|b \cdot \nabla z\|_{0,K}) \, \|v_h\|_{0,K} \, .$$

Summing up and applying the discrete Friedrichs inequality $\|v_h\|_{0,\Omega} \leq C \, \|v_h\|_h$, we obtain a term which can be bounded by $C \, \mu_h^{1/2} \, |||z||| \, \|v_h\|_h$. For estimating the third term from $a_h(z, v_h)$, we use the fact that (2) and (7) imply that $\|\Delta z\|_{0,K} \leq C \, |u|_{2,K} + |z|_{1,K} / (\varepsilon \, \delta_K)^{1/2}$. Finally, applying Lemma 2 from [9], we have

$$\int_E b \cdot n_E \, [|u_h|]_E \cdot \langle |v_h| \rangle_E \, \mathrm{d}\sigma \leq C \, h_E^{1/2} \, \| \, [|u_h|]_E \|_{0,E} \, |(b \cdot n_E) \, v_h|_{1,K_E}$$

and an analogous estimate can also be obtain for the integrals from the last sum in $a_h(z, v_h)$. Since $[|z|]_E = -[|u_h|]_E$, we infer using the discrete Friedrichs inequality that the last two terms from $a_h(z, v_h)$ can be bounded by $C \, \max_E \{h_E(\sigma_E + \sigma_E^{-1})\}^{1/2} \, |||z||| \, \|v_h\|_h$. ♣

Theorem 1 *Let the solution of (1) satisfy $u \in H^2(\Omega)^2$, $p \in H^1(\Omega)$. Then the following error estimates hold*

$$|||u - u_h||| \le C \left(\sum_K \{\lambda_K \, h_K^2 \; |u|_{2,K}^2 + \delta_K \, |p|_{1,K}^2\} + \right.$$

$$\left. + \sum_E \{(\sigma_E + \lambda_E)\, h_E^3 \, |u|_{2,K_E}^2 + \lambda_E^* \, h_E \, |p|_{1,K_E}^2\} \right)^{1/2}, \qquad (10)$$

$$\|p - p_h\|_{0,\Omega} \le C\,(\delta_h + h)\,(\varepsilon\,|u|_{2,\Omega} + |p|_{1,\Omega}) + C\,\mu_h^{1/2}\,|||u - u_h|||\,, \qquad (11)$$

where $\lambda_E^* = \lambda_E$ if the triangulations consist of triangles, and $\lambda_E^* = \varepsilon^{-1}\,h_E$ if they consist of quadrilaterals.

Proof. Let us set $w = u - I_h u$ and $w_h = u_h - I_h u$. Then $w_h \in W_h$ and, according to Lemma 1, we have $|||w_h|||^2 \le 2\,a_h(w, w_h) - 2\,a_h(u - u_h, w_h)$. Hence

$$|||u - u_h||| \le |||w||| + 2\,\frac{a_h(w, w_h)}{|||w_h|||} - 2\,\frac{a_h(u - u_h, w_h)}{|||w_h|||}\,.$$

Applying (2) and (3), we immediately obtain

$$|||w||| \le C \left(\sum_K (\varepsilon + \delta_K)\, h_K^2 \; |u|_{2,K}^2 + \sum_E \sigma_E \, h_E^3 \, |u|_{2,K_E}^2 \right)^{1/2}$$

and (10) follows from Lemmas 2 and 3. Since the spaces V_h and Q_h satisfy the Babuška–Brezzi condition, we have, for some $\beta > 0$ independent of h,

$$\beta\,\|J_h p - p_h\|_{0,\Omega} \le \sup_{v_h \in V_h \setminus \{0\}} \frac{b_h(v_h, J_h p - p_h)}{\|v_h\|_h}$$

and hence

$$\beta\,\|p - p_h\|_{0,\Omega} \le (\beta + \sqrt{2})\,\|p - J_h p\|_{0,\Omega} + \sup_{v_h \in V_h \setminus \{0\}} \frac{b_h(v_h, p - p_h)}{\|v_h\|_h}\,,$$

which gives (11) in view of (4) and Lemma 3. ♣

Remark 1 Setting $\delta_K \sim h_K^2$ and $\sigma_E \sim h_E^{-1}$, Theorem 1 gives in the triangular case

$$|||u - u_h||| \le C\,h\,(|u|_{2,\Omega} + |p|_{1,\Omega})\,, \qquad (12)$$

where C is independent of u, p, ε and h.

Remark 2 Owing to the nonvanishing term $b_h(v_h, p-p_h)$ in the quadrilateral case (cf. the proof of Lemma 3), an estimate of the velocity error with an ε–uniform constant cannot be obtained for the SDFEM (5). However, adding the term $\sum_K (\nabla \cdot u, \nabla \cdot v)_K$ to the bilinear form a_h and setting $\delta_K \sim h_K^2$, $\sigma_E \sim h_E^{-1}$, we again have (12).

Remark 3 The pressure estimate (11) can be written, for the above choices of δ_K and σ_E, in the form

$$||p - p_h||_{0,\Omega} \leq C\, h\, (\varepsilon\, |u|_{2,\Omega} + |p|_{1,\Omega}) + C\, \varepsilon^{-1/2}\, |||u - u_h|||\,.$$

The dependence on ε is caused by the estimate of the term $(b \cdot \nabla z, v_h)_K$ in the proof of Lemma 3. The same difficulty appears in the conforming case.

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