A Bubble–Type Stabilization of the Q_1/Q_1–Element for Incompressible Flows

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Abstract: Starting with the unstable finite Q_1/Q_1–element a general class of bubble type spaces V_h^2 is constructed and used for enlarging the space V_h^1 approximating the velocity. It is shown that the new pair of spaces satisfies the Babuška–Brezzi condition. Using the techniques of reduced discretizations and static condensation some new and old stabilized schemes are obtained.

Keywords: Babuška–Brezzi condition, stabilization, Stokes problem

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1 Introduction

In this paper we introduce a general class of stable finite element spaces suitable for a numerical solution of the Stokes equations

$$-\nu\,\Delta\,\boldsymbol{u} + \nabla p = \boldsymbol{f}\,, \qquad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega\,, \qquad \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega\,. \tag{1}$$

Here, $\boldsymbol{u}$ is the velocity and p is the pressure in a linear viscous fluid contained in a bounded domain $\Omega \subset \mathbb{R}^2$ with a polygonal boundary $\partial\Omega$. The parameter $\nu > 0$ is the kinematic viscosity and $\boldsymbol{f}$ is an external body force, e.g. the gravity. Denoting

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x\,, \qquad b(\boldsymbol{v},p) = -\int_\Omega p\operatorname{div}\boldsymbol{v}\,\mathrm{d}x\,,$$

the usual weak formulation of (1) reads: Given $\nu > 0$ and $\boldsymbol{f} \in H^{-1}(\Omega)^2$, find $\boldsymbol{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega)$ such that

$$\nu\,a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) - b(\boldsymbol{u},q) = \langle\boldsymbol{f},\boldsymbol{v}\rangle \qquad \forall\ \boldsymbol{v} \in H_0^1(\Omega)^2,\ q \in L_0^2(\Omega)\,, \tag{2}$$

where $L_0^2(\Omega)$ consists of $L^2(\Omega)$ functions having zero mean value on Ω. It can be shown that this problem has a unique solution (cf. [9], p. 80, Theorem 5.1).

A conforming finite element discretization of (2) reads: Find $\boldsymbol{u}_h \in \mathrm{V}_h$ and $p_h \in \mathrm{Q}_h$ satisfying

$$\nu\,a(\boldsymbol{u}_h,\boldsymbol{v}_h) + b(\boldsymbol{v}_h,p_h) - b(\boldsymbol{u}_h,q_h) = \langle\boldsymbol{f},\boldsymbol{v}_h\rangle \qquad \forall\ \boldsymbol{v}_h \in \mathrm{V}_h,\ q_h \in \mathrm{Q}_h\,, \tag{3}$$

where $\mathrm{V}_h \subset H_0^1(\Omega)^2$ and $\mathrm{Q}_h \subset L_0^2(\Omega)$ are finite element spaces based on a triangulation $\mathcal{T}_h$ of Ω. In this paper, we shall consider only triangulations consisting of rectangles T (cf. Section 2), thus, we implicitly assume that Ω can be decomposed into rectangles. We are interested in equal order spaces

$$\begin{aligned} \mathrm{V}_h^1 &= \{\boldsymbol{v} \in H_0^1(\Omega)^2;\, \boldsymbol{v}|_T \in Q_1(T)^2 \quad \forall\, T \in \mathcal{T}_h\}\,, \\ \mathrm{Q}_h &= \{q \in H^1(\Omega) \cap L_0^2(\Omega);\, q|_T \in Q_1(T) \quad \forall\, T \in \mathcal{T}_h\} \end{aligned}$$

for approximating the velocity and the pressure, respectively. However, this pair of spaces does not satisfy the Babuška–Brezzi condition

$$\exists\, \beta > 0 \,: \qquad \sup_{\boldsymbol{v}_h \in \mathrm{V}_h^1 \setminus \{\boldsymbol{0}\}} \frac{b(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,\Omega}} \geq \beta\, \|q_h\|_{0,\Omega} \qquad \forall\, q_h \in \mathrm{Q}_h,\, h > 0\,,$$

which often causes that the problem (3) with $\mathrm{V}_h = \mathrm{V}_h^1$ is not solvable or that its solution contains spurious oscillations. One way to suppress these oscillations and to assure the solvability is to add some extra terms to the discretization (3) (cf. e.g. [6], [10], [5]). Another way is to enlarge the space V_h^1 by a space V_h^2 so that the Babuška–Brezzi condition is satisfied. Here we start with the second possibility and construct a general class of spaces V_h^2 assuring the fulfilment of the Babuška–Brezzi condition. Then, by using the concept of reduced discretizations [11] and static condensation, we shall show that, for suitable spaces V_h^2, the V_h^1–component of $\boldsymbol{u}_h$ and the function p_h are solutions of the stabilized methods of [6] and [10].

In case of the mini element [1], which is defined by enriching continuous piecewise linear functions by cubic bubble functions, the close relation between stabilized methods and Galerkin methods with bubble functions was already established in [13] and [3]. In an abstract framework, this equivalence was investigated for linear problems in [2]. It was also realized that bubble functions can help to design new stabilized methods (cf. e.g. [7], [8]).

There is a lot of further papers devoted to investigations of discretizations stabilized using bubble functions, but most of them are restricted to *triangular* elements and to *linear* problems. In this paper, we deal with quadrilateral elements and, in addition, we consider more general functions than the standard bubble functions used before. Apart from investigating the relations to some well known stabilized methods, we shall also derive, eliminating a suitable space V_h^2 from the discretization, a new type of stabilization which can also be applied to the Navier–Stokes equations [12].

The space V_h^2 added to V_h^1 to satisfy the Babuška–Brezzi condition will be defined in a general way as

$$\mathrm{V}_h^2 = \mathrm{span}\{\varphi^i\, \boldsymbol{t}^i\}_{i=1}^{N_h}\,, \tag{4}$$

where $\varphi^i \in H_0^1(\Omega)$ and $\boldsymbol{t}^i \in \mathbb{R}^2$ are some suitable functions and vectors, respectively. The proof of the Babuška–Brezzi condition for the spaces $\mathrm{V}_h \equiv \mathrm{V}_h^1 \oplus \mathrm{V}_h^2$ and Q_h, which uses some ideas of [4] and a modification of the Verfürth trick [14], requires that the functions φ^i have localized supports and that, for any φ^i, there exists a point $A^i \in \overline{\Omega}$ such that

$$\int_\Omega \frac{\partial q_h}{\partial \boldsymbol{t}^i}\, \varphi^i \,\mathrm{d}x = \frac{\partial q_h}{\partial \boldsymbol{t}^i}(A^i) \int_\Omega \varphi^i \,\mathrm{d}x \qquad \forall\, q_h \in \mathrm{Q}_h\,,\ i = 1, \dots, N_h\,, \tag{5}$$

$$|q_h|^2_{1,T} \leq C\, h_T^2 \sum_{\substack{i=1,\\ A^i \in T}}^{N_h} \left| \frac{\partial q_h}{\partial \boldsymbol{t}^i}(A^i) \right|^2 \qquad \forall\, q_h \in \mathrm{Q}_h,\, T \in \mathcal{T}_h\,. \tag{6}$$

The plan of the paper is as follows. In Section 2, we introduce some notations and summarize the assumptions on the triangulations and the functions φ^i needed for proving the Babuška–Brezzi condition in Section 3. In Section 4, we give examples for the functions φ^i defining the supplement spaces V_h^2. In Section 5, we investigate discretizations obtained from (3) by eliminating the V_h^2–component of $\boldsymbol{u}_h$. In particular, we show the equivalence to the stabilized methods of [6] and [10] and establish a new type of stabilization.

2 Assumptions and notations

We assume that we are given a family $\{\mathcal{T}_h\}_{h>0}$ of triangulations of Ω consisting of closed rectangular elements T having the usual compatibility properties. Let $h_T \equiv \mathrm{diam}(T) \leq h$ for any $T \in \mathcal{T}_h$ and let the ratio of the longest and the shortest edge of each T satisfy

$$\frac{h_T^{max}}{h_T^{min}} \leq C_1$$

with some constant C_1 independent of h.

Before introducing the functions φ^i used for defining V_h^2 in (4), we define the points A^i mentioned in the introduction. Thus, for any triangulation $\mathcal{T}_h$, we define a set $\{A^i\}_{i=1}^{N_h}$ of points $A^i \in \overline{\Omega}$ different from the vertices of $\mathcal{T}_h$ and we assume that

$$\mathrm{card}\{A^i;\, A^i \in T\} \leq C_2 \qquad \forall\, T \in \mathcal{T}_h\,,$$

where the constant C_2 is independent of h. In order to be able to use the same function φ in (4) twice (each time with another $\boldsymbol{t}^i$), we admit $A^i = A^j$ for $i \neq j$. For any A^i, we denote by P^i the union of the elements containing A^i, i.e., P^i consists of one or two elements. Further, for any A^i, we introduce a unit vector $\boldsymbol{t}^i \in \mathbb{R}^2$ and if A^i belongs to an edge of some $T \in \mathcal{T}_h$, we require that $\boldsymbol{t}^i$ coincides with the direction of this edge. We choose a fixed Cartesian coordinate system with axes parallel to the edges of the triangulation and, for any $\boldsymbol{t}^i = (t_1^i, t_2^i)$, we define $\boldsymbol{n}^i = (t_2^i, t_1^i)$. For getting the property (6), we assume that, for any $T \in \mathcal{T}_h$, there exist indices $i, j, k \in \{1, \dots, N_h\}$ such that A^i, A^j, $A^k \in T$ and

$$S_h^{ijk} = (\boldsymbol{n}^i \times \boldsymbol{n}^j)\, \boldsymbol{n}^k \cdot A^k + (\boldsymbol{n}^j \times \boldsymbol{n}^k)\, \boldsymbol{n}^i \cdot A^i + (\boldsymbol{n}^k \times \boldsymbol{n}^i)\, \boldsymbol{n}^j \cdot A^j$$

satisfies

$$|S_h^{ijk}| \geq C_3\, h_T\,, \tag{7}$$

where the constant C_3 is independent of T and h, the vector product $\boldsymbol{a} \times \boldsymbol{b}$ is defined as $a_1\, b_2 - b_1\, a_2$ and $\boldsymbol{n}^i \cdot A^i$ is defined as $\boldsymbol{n}^i \cdot (A^i - \mathbf{0})$.

Remark 1 If, particularly, $\boldsymbol{n}^i = \boldsymbol{n}^j \perp \boldsymbol{n}^k$, then

$$|S_h^{ijk}| = |\boldsymbol{n}^i \cdot (A^i - A^j)|\,,$$

which illustrates the meaning of (7).

Finally, for each A^i, we introduce a function $\varphi^i \in H_0^1(\Omega)$ such that

$$\operatorname{supp}\varphi^i \subset P^i\,, \tag{8}$$

$$\boldsymbol{n}^i \cdot \int_T (x - A^i)\, \varphi^i(x)\, \mathrm{d}x = 0 \qquad \forall\; T \subset P^i\,, \tag{9}$$

$$\left| \int_\Omega \varphi^i\, \mathrm{d}x \right| \geq C_4\, |P^i|\,, \tag{10}$$

$$|\varphi^i|_{1,\Omega} \leq C_5\,, \tag{11}$$

where the constants C_4, $C_5 > 0$ are independent of h. We shall see (cf. Lemma 1 in the next section) that from (9) the property (5) can be concluded. The functions $\{\varphi^i\, \boldsymbol{t}^i\}_{i=1}^{N_h}$ are assumed to be linearly independent.

3 Babuška–Brezzi condition

In this section, we prove that, under the assumptions made in Section 2, the spaces $\mathrm{V}_h \equiv \mathrm{V}_h^1 \oplus \mathrm{V}_h^2$ and Q_h satisfy the Babuška–Brezzi condition with a constant independent of h. First, in Lemmas 1 and 2, we prove the validity of (5) and (6). Then, in Lemma 3, we establish a Babuška–Brezzi condition with a 'wrong' norm of q_h and, finally, in Theorem 1, we prove the desired Babuška–Brezzi condition applying the modified Verfürth trick.

Lemma 1 *We have*

$$\int_\Omega \frac{\partial q_h}{\partial \boldsymbol{t}^i}\, \varphi^i\, \mathrm{d}x = \frac{\partial q_h}{\partial \boldsymbol{t}^i}(A^i) \int_\Omega \varphi^i\, \mathrm{d}x \qquad \forall\; q_h \in \mathrm{Q}_h\,,\; i = 1, \dots, N_h\,.$$

Proof. Consider any $T \subset P^i$ and let $q_h(x) = \xi_0 + \xi_1\, x_1 + \xi_2\, x_2 + \xi_3\, x_1\, x_2$ for any $x \in T$, where $\xi_0, \dots, \xi_3$ are some real numbers. Denoting $A^i = (a_1, a_2)$, we have for $x \in T$

$$\frac{\partial q_h}{\partial x_1}(x) = \frac{\partial q_h}{\partial x_1}(A^i) + \xi_3\, (x_2 - a_2)\,, \qquad \frac{\partial q_h}{\partial x_2}(x) = \frac{\partial q_h}{\partial x_2}(A^i) + \xi_3\, (x_1 - a_1)\,.$$

Hence

$$\frac{\partial q_h}{\partial \boldsymbol{t}^i}(x) = \frac{\partial q_h}{\partial \boldsymbol{t}^i}(A^i) + \xi_3\, \boldsymbol{n}^i \cdot (x - A^i)$$

and the lemma follows using (9). □

Lemma 2 *For any $T \in \mathcal{T}_h$, we have*

$$|q_h|^2_{1,T} \le \frac{54}{C_3^2}\,|T| \sum_{\substack{i=1,\\ A^i\in T}}^{N_h} \left|\frac{\partial q_h}{\partial \boldsymbol{t}^i}(A^i)\right|^2 \qquad \forall\; q_h \in \mathrm{Q}_h\,. \tag{12}$$

Proof. Consider any $T \in \mathcal{T}_h$ and let A^1, A^2, $A^3 \in T$ be points satisfying (7). Let us set for $x \in T$ and $i = 1, 2, 3$

$$q^i(x) = \{\boldsymbol{n}^{i+1}\,(\boldsymbol{n}^{i-1}\cdot A^{i-1}) - \boldsymbol{n}^{i-1}\,(\boldsymbol{n}^{i+1}\cdot A^{i+1})\}\cdot(-x_1, x_2) + (\boldsymbol{n}^{i+1}\times\boldsymbol{n}^{i-1})\,x_1\,x_2\,,$$

with the convention that $i-1 \equiv 3$ for $i=1$ and $i+1 \equiv 1$ for $i=3$. An easy calculation shows that any $q_h \in \mathrm{Q}_h$ satisfies

$$q_h(x) = \xi_0 + \frac{1}{S_h^{1\,2\,3}} \sum_{i=1}^{3} \frac{\partial q_h}{\partial \boldsymbol{t}^i}(A^i)\, q^i(x) \qquad \forall\; x \in T\,,$$

where ξ_0 is a real number. It can be shown that $\left|\dfrac{\partial q^i}{\partial x_j}(x)\right| \le 3\,h_T$, $j = 1, 2$, and hence we obtain (12). □

Lemma 3 *We have*

$$\sup_{\boldsymbol{v}_h\in \mathrm{V}_h^2\setminus\{\boldsymbol{0}\}} \frac{b(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,\Omega}} \ge \frac{C_3 C_4}{C_5\sqrt{108\,C_1 C_2}} \sqrt{\sum_{T\in\mathcal{T}_h} h_T^2\,|q_h|^2_{1,T}} \qquad \forall\; q_h \in \mathrm{Q}_h\,. \tag{13}$$

Proof. Consider any $\boldsymbol{v}_h \in \mathrm{V}_h^2$. Then $\boldsymbol{v}_h = \sum_{i=1}^{N_h} \alpha^i\,\varphi^i\,\boldsymbol{t}^i$ and we have

$$|\boldsymbol{v}_h|^2_{1,\Omega} = \sum_{T\in\mathcal{T}_h} |\boldsymbol{v}_h|^2_{1,T} \le C_2 \sum_{T\in\mathcal{T}_h} \sum_{\substack{i=1,\\ A^i\in T}}^{N_h} |\alpha^i|^2\,|\varphi^i|^2_{1,T} \le C_2 C_5^2 \sum_{i=1}^{N_h} |\alpha^i|^2\,. \tag{14}$$

Applying Lemma 1, we obtain

$$b(\boldsymbol{v}_h, q_h) = \int_\Omega \boldsymbol{v}_h\cdot\nabla q_h\,\mathrm{d}x = \sum_{i=1}^{N_h} \alpha^i \int_\Omega \frac{\partial q_h}{\partial \boldsymbol{t}^i}\,\varphi^i\,\mathrm{d}x = \sum_{i=1}^{N_h} \alpha^i\, \frac{\partial q_h}{\partial \boldsymbol{t}^i}(A^i) \int_\Omega \varphi^i\,\mathrm{d}x\,.$$

Setting

$$\alpha^i = \frac{\partial q_h}{\partial \boldsymbol{t}^i}(A^i) \int_\Omega \varphi^i\,\mathrm{d}x\,,$$

it follows that

$$b(\boldsymbol{v}_h, q_h) = \sum_{i=1}^{N_h} |\alpha^i|^2 = \left[\sum_{i=1}^{N_h} |\alpha^i|^2\right]^{\frac12} \left[\sum_{i=1}^{N_h} \left|\frac{\partial q_h}{\partial \boldsymbol{t}^i}(A^i)\right|^2 \left|\int_\Omega \varphi^i\,\mathrm{d}x\right|^2\right]^{\frac12},$$

which implies (13) owing to (10), (12) and (14). □

Theorem 1 *There exists a constant* $C_6 > 0$ *independent of* h *such that the spaces* $\mathrm{V}_h = \mathrm{V}_h^1 \oplus \mathrm{V}_h^2$ *and* Q_h *satisfy the Babuška–Brezzi condition*

$$\sup_{\boldsymbol{v}_h \in \mathrm{V}_h \setminus \{\boldsymbol{0}\}} \frac{b(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,\Omega}} \geq C_6 \, \|q_h\|_{0,\Omega} \qquad \forall \; q_h \in \mathrm{Q}_h \,. \tag{15}$$

Proof. Applying the modified Verfürth trick presented in [5], pp. 255–256, we obtain

$$\sup_{\boldsymbol{v}_h \in \mathrm{V}_h \setminus \{\boldsymbol{0}\}} \frac{b(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,\Omega}} \geq C \, \|q_h\|_{0,\Omega} - \widetilde{C} \sqrt{\sum_{T \in \mathcal{T}_h} h_T^2 \, |q_h|_{1,T}^2} \qquad \forall \; q_h \in \mathrm{Q}_h$$

with some constant C, $\widetilde{C} > 0$ independent of h and (15) follows from Lemma 3. □

4 Examples of the finite element space V_h^2

In this section we give examples of functions φ^i satisfying the assumptions (8)–(11). Denoting by $\widehat{T} = [0,1]^2$ the reference element, we first introduce functions $\widehat{\varphi}_1 \in Q_2(\widehat{T}) \cap H_0^1(\widehat{T})$, $\widehat{\varphi}_2 \in Q_2(\widehat{T})$ and $\widehat{\varphi}_3 \in Q_3(\widehat{T}) \cap H_0^1(\widehat{T})$ defined for $\widehat{x} \in \widehat{T}$ by

$$\widehat{\varphi}_1(\widehat{x}) = \widehat{x}_1 \,(1 - \widehat{x}_1)\, \widehat{x}_2 \,(1 - \widehat{x}_2)\,, \qquad \widehat{\varphi}_2(\widehat{x}) = \widehat{x}_1 \,(1 - \widehat{x}_1)\,(1 - \widehat{x}_2)\,(\tfrac{1}{2} - \widehat{x}_2)\,,$$
$$\widehat{\varphi}_3(\widehat{x}) = (3 - 5\,\widehat{x}_2)\, \widehat{\varphi}_1(\widehat{x})\,.$$

It is easy to verify that

$$\int_{\widehat{T}} (\widehat{x} - C_{\widehat{T}})\, \widehat{\varphi}_1(\widehat{x}) \,\mathrm{d}\widehat{x} = 0\,, \qquad \int_{\widehat{T}} \widehat{x}_2 \, \widehat{\varphi}_2(\widehat{x}) \,\mathrm{d}\widehat{x} = 0\,, \qquad \int_{\widehat{T}} \widehat{x}_2 \, \widehat{\varphi}_3(\widehat{x}) \,\mathrm{d}\widehat{x} = 0\,, \tag{16}$$

where $C_{\widehat{T}} = (\frac{1}{2}, \frac{1}{2})$ is the barycentre of $\widehat{T}$.

We denote by $\mathcal{E}_h$ the set of all edges E of the triangulation $\mathcal{T}_h$, by S_E the midpoint of each edge E, by P_E the union of the elements possessing the edge E, by $\boldsymbol{t}_E$ a unit vector in the direction of E, by $\boldsymbol{n}_E$ the vector (t_{E2}, t_{E1}) and by C_T the barycentre of each element T. For any edge E of an element T, we define an rectangle R_T^E as depicted in Fig. 1 and we denote by R_E the union of the rectangles R_T^E possessing the edge E. Further, for any element T, we define an rectangle R_T (cf. Fig. 1) and rectangles R_T^1, R_T^2, R_T^3, R_T^4 with barycentres A_T^1, A_T^2, A_T^3, A_T^4, respectively (cf. Fig. 2). These points are defined as the images of the points

$$(\tfrac{3-\sqrt{3}}{6}, \tfrac{1}{2})\,, \qquad (\tfrac{3+\sqrt{3}}{6}, \tfrac{1}{2})\,, \qquad (\tfrac{1}{2}, \tfrac{3-\sqrt{3}}{6})\,, \qquad (\tfrac{1}{2}, \tfrac{3+\sqrt{3}}{6})$$

in the reference element $\widehat{T}$. That assures that

$$\int_T g \,\mathrm{d}x = \frac{|T|}{2}\,[g(A_T^1) + g(A_T^2)] \quad \text{for} \quad g(x) = (\xi_0 + \xi_1\, x_1)\, q(x)\,, \tag{17}$$

$$\int_T h \,\mathrm{d}x = \frac{|T|}{2}\,[h(A_T^3) + h(A_T^4)] \quad \text{for} \quad h(x) = (\eta_0 + \eta_1\, x_2)\, q(x)\,, \tag{18}$$

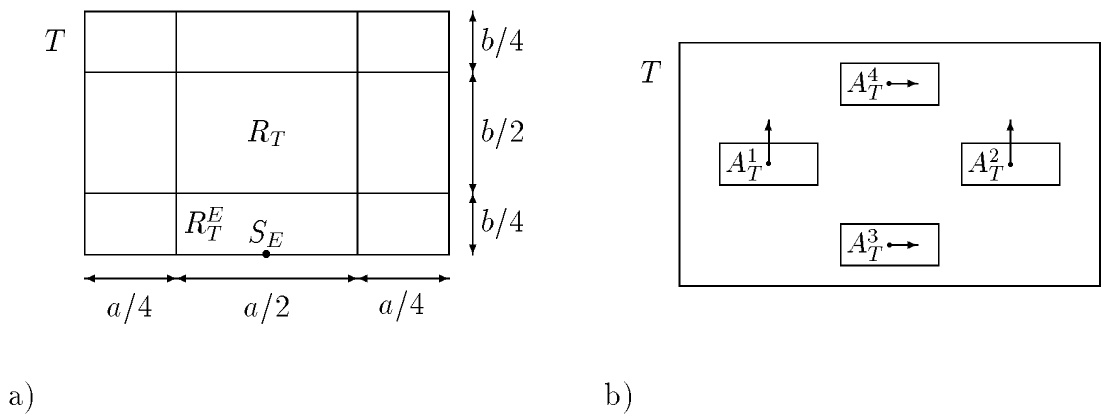

Figure 1: Subdivisions of an element T

where $q \in Q_1(T)$ and ξ_0, ξ_1, η_0, η_1 are arbitrary real numbers. The dimensions of the rectangles R_T^i are $\frac{3-\sqrt{3}}{6}\, a \times \frac{3-\sqrt{3}}{6}\, b$.

For any $T \in \mathcal{T}_h$, we define functions φ_T^1, φ_T^2, ψ_T^1, ψ_T^2, ψ_T^3, $\psi_T^4 \in H_0^1(\Omega)$ by

$$\begin{aligned}
&\varphi_T^1|_T = \widehat{\varphi}_1 \circ F^{-1} \quad \text{with} \quad F(\widehat{T}) = T\,, && \varphi_T^1|_{\Omega\setminus T} = 0\,, \\
&\varphi_T^2|_{R_T} = \widehat{\varphi}_1 \circ F^{-1} \quad \text{with} \quad F(\widehat{T}) = R_T\,, && \varphi_T^2|_{\Omega\setminus R_T} = 0\,, \\
&\psi_T^i|_{R_T^i} = \widehat{\varphi}_1 \circ F^{-1} \quad \text{with} \quad F(\widehat{T}) = R_T^i\,, && \psi_T^i|_{\Omega\setminus R_T^i} = 0\,, \qquad i = 1,2,3,4\,,
\end{aligned}$$

where F always is a regular affine mapping. Note that

$$\int_\Omega \psi_T^i \,\mathrm{d}x = \int_\Omega \psi_T^1 \,\mathrm{d}x\,, \qquad |\psi_T^i|_{1,\Omega} = |\psi_T^1|_{1,\Omega} \qquad \forall\, T \in \mathcal{T}_h,\ i = 1,2,3,4\,. \tag{19}$$

For edges $E \in \mathcal{E}_h$, we define functions φ_E^1, $\varphi_E^2 \in H_0^1(\Omega)$ satisfying $\varphi_E^1|_{\Omega\setminus P_E} = 0$, $\varphi_E^2|_{\Omega\setminus R_E} = 0$ and

$$\begin{aligned}
&\varphi_E^1|_T = \widehat{\varphi}_2 \circ F^{-1} && \text{with} \quad F(\widehat{T}) = T && \forall\, T \subset P_E && \text{if } E \not\subset \partial\Omega, \\
&\varphi_E^2|_{R_T^E} = \widehat{\varphi}_2 \circ F^{-1} && \text{with} \quad F(\widehat{T}) = R_T^E && \forall\, T \subset P_E && \text{if } E \not\subset \partial\Omega, \\
&\varphi_E^2|_{R_T^E} = \widehat{\varphi}_3 \circ F^{-1} && \text{with} \quad F(\widehat{T}) = R_T^E && \forall\, T \subset P_E && \text{if } E \subset \partial\Omega,
\end{aligned}$$

where F always is a reagular affine mapping satisfying $F(\frac{1}{2}, 0) = S_E$.

It follows from (16) that, for any $T \in \mathcal{T}_h$ and any $E \in \mathcal{E}_h$ (in case of φ_E^1 only for $E \not\subset \partial\Omega$), we have

$$\begin{aligned}
&\int_T (x - C_T)\, \varphi_T^i(x) \,\mathrm{d}x = 0\,, \quad \boldsymbol{n}_E \cdot \int_T (x - S_E)\, \varphi_E^i(x) \,\mathrm{d}x = 0\,, \qquad i = 1,2\,, \\
&\int_T (x - A_T^i)\, \psi_T^i(x) \,\mathrm{d}x = 0\,, \qquad i = 1,2,3,4\,.
\end{aligned}$$

Clearly, all the functions also satisfy (10) and (11). Thus, setting

$$\begin{aligned}
&\mathrm{W}_h^1 = \mathrm{span}\{\varphi_E^1\, \boldsymbol{t}_E\}_{E\not\subset\partial\Omega}\,, \quad \mathrm{W}_h^2 = \mathrm{span}\{\varphi_T^1\, \boldsymbol{e}^1\}_{T\in\mathcal{T}_h}\,, \quad \mathrm{W}_h^3 = \mathrm{span}\{\varphi_T^1\, \boldsymbol{e}^2\}_{T\in\mathcal{T}_h}\,,\\
&\mathrm{W}_h^4 = \mathrm{span}\{\varphi_E^2\, \boldsymbol{t}_E\}_{E\in\mathcal{E}_h}\,, \quad \mathrm{W}_h^5 = \mathrm{span}\{\varphi_T^2\, \boldsymbol{e}^1\}_{T\in\mathcal{T}_h}\,, \quad \mathrm{W}_h^6 = \mathrm{span}\{\varphi_T^2\, \boldsymbol{e}^2\}_{T\in\mathcal{T}_h}\,,\\
&\mathrm{W}_h^7 = \mathrm{span}\{\psi_T^1\, \boldsymbol{e}^2,\, \psi_T^2\, \boldsymbol{e}^2,\, \psi_T^3\, \boldsymbol{e}^1,\, \psi_T^4\, \boldsymbol{e}^1\}_{T\in\mathcal{T}_h}\,,
\end{aligned}$$

where $\boldsymbol{e}^1 = (1,0)$ and $\boldsymbol{e}^2 = (0,1)$, we can define V_h^2 e.g. as one of the following spaces

$$\mathrm{V}_h^{2,1} = \mathrm{W}_h^1 \oplus \mathrm{W}_h^2 \oplus \mathrm{W}_h^3\,, \quad \mathrm{V}_h^{2,2} = \mathrm{W}_h^4\,, \quad \mathrm{V}_h^{2,3} = \mathrm{W}_h^4 \oplus \mathrm{W}_h^5 \oplus \mathrm{W}_h^6\,, \quad \mathrm{V}_h^{2,4} = \mathrm{W}_h^7\,.$$

For all these spaces, the assumption (7) is satisfied (cf. Remark 1) so that the Babuška–Brezzi condition holds. Since $\mathrm{V}_h^{2,1}$ consists only of piecewise biquadratic functions, we particularly infer that the Babuška–Brezzi condition is satisfied for the pair V_h, Q_h with V_h consisting of piecewise biquadratic functions from $H_0^1(\Omega)^2$. Note that the functions from $\mathrm{W}_h^2 \oplus \mathrm{W}_h^3$ are needed for the validity of the Babuška–Brezzi condition for $\mathrm{V}_h^1 \oplus \mathrm{V}_h^{2,1}$, Q_h only on those elements which have two or three edges on $\partial\Omega$.

5 Stabilized methods for the Stokes problem

In this section, we investigate the relation between modified Galerkin methods with bubble functions and stabilized methods for the Stokes problem (1). Following [11], we consider a reduced discretisation of the Stokes equations given by: Find $\boldsymbol{u}_h^1 \in \mathrm{V}_h^1$, $\boldsymbol{u}_h^2 \in \mathrm{V}_h^2$ and $p_h \in \mathrm{Q}_h$ satisfying

$$\nu\, a(\boldsymbol{u}_h^1, \boldsymbol{v}_h^1) \qquad\qquad + b(\boldsymbol{v}_h^1, p_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h^1\rangle \qquad \forall\ \boldsymbol{v}_h^1 \in \mathrm{V}_h^1\,, \tag{20}$$

$$\nu\, a(\boldsymbol{u}_h^2, \boldsymbol{v}_h^2) + b(\boldsymbol{v}_h^2, p_h) = 0 \qquad \forall\ \boldsymbol{v}_h^2 \in \mathrm{V}_h^2\,, \tag{21}$$

$$b(\boldsymbol{u}_h^1, q_h) + b(\boldsymbol{u}_h^2, q_h) \qquad = 0 \qquad \forall\ q_h \in \mathrm{Q}_h\,, \tag{22}$$

the solution of which has, for $\boldsymbol{f} \in L^2(\Omega)^2$, the same convergence properties as the solution of the original problem (3).

We assume that the basis functions of the space V_h^2 satisfy $a(\varphi^i\, \boldsymbol{t}^i, \varphi^j\, \boldsymbol{t}^j) = 0$ for $i \neq j$, which holds e.g. for the spaces $\mathrm{V}_h^{2,2}$, $\mathrm{V}_h^{2,3}$ and $\mathrm{V}_h^{2,4}$. Using the basis representation $\boldsymbol{u}_h^2 = \sum_{j=1}^{N_h} \alpha^j\, \varphi^j\, \boldsymbol{t}^j$ and setting $\boldsymbol{v}_h^2 = \varphi^i\, \boldsymbol{t}^i$ in (21), we get

$$\alpha^i\, \nu\, a(\varphi^i\, \boldsymbol{t}^i, \varphi^i\, \boldsymbol{t}^i) + b(\varphi^i \boldsymbol{t}^i, p_h) = 0\,.$$

Eliminating $\boldsymbol{u}_h^2$ from (20)–(22) and applying Lemma 1, we obtain a stabilised Q_1/Q_1 discretisation in the form: Find $\boldsymbol{u}_h^1 \in \mathrm{V}_h^1$ and $p_h \in \mathrm{Q}_h$ satisfying

$$\nu\, a(\boldsymbol{u}_h^1, \boldsymbol{v}_h^1) + b(\boldsymbol{v}_h^1, p_h) \;=\; \langle \boldsymbol{f}, \boldsymbol{v}_h^1\rangle \qquad \forall\ \boldsymbol{v}_h^1 \in \mathrm{V}_h^1\,, \tag{23}$$

$$b(\boldsymbol{u}_h^1, q_h) - c_h(p_h, q_h) \;=\; 0 \qquad \forall\ q_h \in \mathrm{Q}_h\,, \tag{24}$$

where the stabilising term is given by

$$c_h(p_h, q_h) = \sum_{i=1}^{N_h} \frac{\partial p_h}{\partial \boldsymbol{t}^i}(A^i)\, \frac{\partial q_h}{\partial \boldsymbol{t}^i}(A^i)\, \frac{\left|\int_\Omega \varphi^i\, \mathrm{d}x\right|^2}{\nu\, |\varphi^i|_{1,\Omega}^2}\,.$$

For the spaces $\mathrm{V}_h^{2,2}$, $\mathrm{V}_h^{2,3}$ and $\mathrm{V}_h^{2,4}$ defined in the preceding section, the stabilizing term can be respectively written as

$$c_h^2(p_h, q_h) = \sum_{E \in \mathcal{E}_h} \frac{\delta_E\, h_E^4}{\nu} \frac{\partial p_h}{\partial \boldsymbol{t}_E}(S_E) \frac{\partial q_h}{\partial \boldsymbol{t}_E}(S_E)\,, \tag{25}$$

$$c_h^3(p_h, q_h) = \sum_{E \in \mathcal{E}_h} \frac{\delta_E\, h_E^4}{\nu} \frac{\partial p_h}{\partial \boldsymbol{t}_E}(S_E) \frac{\partial q_h}{\partial \boldsymbol{t}_E}(S_E) + \sum_{T \in \mathcal{T}_h} \frac{\delta_T\, h_T^4}{\nu} \nabla p_h(S_T) \cdot \nabla q_h(S_T)\,, \tag{26}$$

$$c_h^4(p_h, q_h) = \sum_{T \in \mathcal{T}_h} \frac{\gamma_T\, h_T^2}{\nu} \int_T \nabla p_h \cdot \nabla q_h \,\mathrm{d}x\,, \tag{27}$$

where h_E denotes the length of the edge E and the parameters

$$\delta_E = \frac{|\int_\Omega \varphi_E^2 \,\mathrm{d}x|^2}{h_E^4\, |\varphi_E^2|_{1,\Omega}^2}\,, \qquad \delta_T = \frac{|\int_\Omega \varphi_T^2 \,\mathrm{d}x|^2}{h_T^4\, |\varphi_T^2|_{1,\Omega}^2}\,, \qquad \gamma_T = \frac{2\,|\int_\Omega \psi_T^1 \,\mathrm{d}x|^2}{h_T^2\, |T|\, |\psi_T^1|_{1,\Omega}^2}$$

are bounded from below and from above by positive constants independent of h. The relation (27) folows from (17)–(19) and from the fact that $\partial p_h / \partial x_k|_T$ is a linear function which does not depend on x_k, $k = 1, 2$. In the sums of (26), it is sufficient to consider only those terms which assure that the assumption (7) is satisfied. For instance, if $\nabla p_h(S_T) \cdot \nabla q_h(S_T)$ is present for a given T, we need only one edge $E \subset T$ in the first sum of (26). Note also that the matrix corresponding to (25) has only five entries in each row. The matrix corresponding to (26) has nine entries per row in general. The stabilizations (25) and (26) are new whereas in (27) we recovered a stabilization originally introduced and studied by Brezzi and Pitkäranta [6].

Finally, using (20)–(22) with $\langle \boldsymbol{f}, \boldsymbol{v}_h^2 \rangle$ instead of the right–hand side of (21) (a consistent reduced discretization), we obtain a stabilized problem which differs from (23)–(24) by the right–hand side of (24) which is, for $\boldsymbol{f} \in L^2(\Omega)^2$, given by

$$-\sum_{i=1}^{N_h} \frac{\partial q_h}{\partial \boldsymbol{t}^i}(A^i) \int_\Omega \boldsymbol{f} \cdot \boldsymbol{t}^i\, \varphi^i \,\mathrm{d}x\, \frac{\int_\Omega \varphi^i \,\mathrm{d}x}{\nu\, |\varphi^i|_{1,\Omega}^2}\,.$$

For $\mathrm{V}_h^2 = \mathrm{V}_h^{2,4}$ and a piecewise bilinear function $\boldsymbol{f}$, we infer using (17)–(19) that the modified continuity equation reads

$$b(\boldsymbol{u}_h^1, q_h) + \sum_{T \in \mathcal{T}_h} \frac{\gamma_T\, h_T^2}{\nu} \int_T (\boldsymbol{f} - \nabla p_h) \cdot \nabla q_h \,\mathrm{d}x = 0\,. \tag{28}$$

This stabilization is identitical with the Petrov–Galerkin formulation of the Stokes equations introduced in [10]. Increasing the number of the bubble functions, we can obtain (28) also for $\boldsymbol{f}$ being a piecewise higher degree polynomial.

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