Nonconforming Finite Elements for Incompressible Flow Problems

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Abstract

We present nonconforming finite elements suitable for an accurate approximation of convection dominated effects and of divergence–free functions and hence also well suited for the numerical solution of incompressible flow problems like the Stokes and Navier–Stokes equations. Particularly, we give a detailed error analysis for the convection dominated case and we thoroughly describe the construction of a modification of the Crouzeix–Raviart element. We present several numerical results which demonstrate the advantages of the new elements.

Keywords: Navier–Stokes equations, Stokes equations, convection–diffusion problems, streamline diffusion method, nonconforming finite elements, inf–sup condition, error estimates

Mathematics Subject Classification (1991): 65N30, 65N15

1 Introduction

In this paper we discuss the properties of some nonconforming finite elements when they are applied to the numerical solution of incompressible flow problems. The basic purpose of these investigations is to develop robust and accurate nonconforming methods for the numerical solution of the incompressible Navier–Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\,\Delta\,\boldsymbol{u} + (\nabla \boldsymbol{u})\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega\,, \tag{1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega\,, \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{u}_b \qquad \text{on } \partial\Omega\,. \tag{3}$$

These equations describe the velocity $\boldsymbol{u}$ and the pressure p in an incompressible viscous fluid contained in a domain Ω. We shall assume that Ω is a bounded domain in $\mathbb{R}^2$ and that its boundary $\partial\Omega$ is polygonal. For simplicity, we consider Dirichlet boundary conditions on $\partial\Omega$, but other types of boundary conditions are also possible. The parameter $\nu > 0$ denotes the kinematic viscosity and $\boldsymbol{f}$ is an outer volume force.

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The numerical solution of the Navier–Stokes equations (1)–(3) by means of the finite element method is connected with various difficulties and we shall concetrate on two of them. The first one is caused by the fact that in typical applications the viscosity ν is very small and therefore the convection term in (1) dominates the diffusion term. A standard model problem for investigating convection dominated effects is the scalar convection–diffusion equation

$$-\varepsilon\,\Delta\,u+\boldsymbol{b}\cdot\nabla u+c\,u=f\quad\text{in }\Omega,\qquad\qquad u=u_b\quad\text{on }\partial\Omega. \tag{4}$$

The reaction term $c\,u$ can arise due to a time discretization of (1). Again, $\varepsilon\in(0,1)$ is a constant but typically small parameter and we assume that $\boldsymbol{b}\in W^{1,\infty}(\Omega)^2$, $c\in L^\infty(\Omega)$, $f\in L^2(\Omega)$ and $u_b\in H^{3/2}(\partial\Omega)$. In addition we make the standard assumption

$$c-\frac{1}{2}\operatorname{div}\boldsymbol{b}\geq c_0\,, \tag{5}$$

where c_0 is a positive constant. This assumption guarantees that (4) admits a unique solution. Note that (5) is satisfied if (4) corresponds to time–discretized Navier–Stokes equations (1)–(3).

It is well known that, for $\varepsilon\ll 1$, the standard Galerkin finite element method produces unphysical oscillations if the local mesh size is not small enough. In this case the discrete problem has to be stabilized and an efficient way to do this is to apply the streamline diffusion method [11], [21]. For conforming finite elements, this method has been thoroughly studied during the two last decades and is well understood today (cf. e.g. [9], [13], [20], [21], [27]). Particularly, when using piecewise polynomial approximations of degree k, one can prove that in the convection dominated case ($\varepsilon\leq h$) the error estimate

$$|||u-u_h|||\leq C\,h^{k+1/2}\,\|u\|_{k+1,\Omega}$$

holds with respect to the usual streamline diffusion norm.

The situation is much more complicated if nonconforming finite element approximations are used. Nonconforming finite elements are more suitable for a parallel implementation than conforming elements since they lead to a cheap local communication between processors (cf. e.g. [8], [12], [22]). Moreover, they are attractive in computational fluid dynamics since they easily fulfil the Babuška–Brezzi condition. Unfortunately, compared to conforming approximations much less is known about the convergence properties of streamline diffusion type methods for nonconforming finite element approximations. Indeed, when considering nonconforming approximation spaces, we loose the continuity property over inner element edges and the coercivity of the corresponding bilinear form depends on the type of discretization for the convective term. Moreover, the nonconformity leads to consistency errors which are often very difficult to estimate uniformly in ε. In [14], [15] these difficulties have been overcome by adding some special jump terms and thus modifying the standard streamline diffusion finite element method. Another possibility is to employ superconvergence properties on uniform meshes for particular finite elements, cf. [26]. However, in general, without using jump terms and on general meshes, we cannot guarantee the same convergence results as in the conforming case. This is not only a problem of the mathematical techniques used for deriving the error estimates since numerical results show that the standard nonconforming finite elements often lead to very inaccurate results in the convection dominated case.

We shall show that a successful application of nonconforming finite elements to convection dominated problems requires that the finite element spaces satisfy a patch test of a higher order than usual. We shall describe a class of such spaces defined on triangular meshes and we shall call the corresponding finite elements the P_n^{mod} elements. For clarity, we shall mostly confine ourselves to first order elements and we shall only give some hints how to proceed in case of higher order elements. Particularly, we shall thoroughly derive a modification of the Crouzeix–Raviart element, which we call the P_1^{mod} element, and we shall present both theoretical and numerical results demonstrating its optimal convergence behaviour, robustness and high accuracy. Let us also mention that the P_1^{mod} element satisfies the discrete Korn inequality (cf. [16]) which does not hold for the most nonconforming first order elements including the Crouzeix–Raviart element.

A second difficulty in solving the Navier–Stokes equations (1)–(3) numerically is caused by the incompressibility constraint (2). In view of this constraint, finite element spaces $\mathbf{V}_h$ and Q_h for approximating the velocity $\boldsymbol{u}$ and pressure p, respectively, cannot be chosen arbitrarily if one wants to obtain a stable discretization with respect to $h \to 0$ and no additional stabilization of the equation (2) is used (see e.g. [4], [10] for details). A sufficient requirement on the spaces $\mathbf{V}_h$, Q_h is the validity of the inf–sup condition

$$\sup_{\boldsymbol{v}_h \in \mathbf{V}_h \setminus \{\mathbf{0}\}} \frac{b_h(\boldsymbol{v}_h, q_h)}{|\boldsymbol{v}_h|_{1,h}} \geq \beta \, \|q_h\|_{0,\Omega} \qquad \forall \; q_h \in \mathrm{Q}_h \,, \tag{6}$$

where $\beta > 0$ is independent of the discretization parameter h,

$$b_h(\boldsymbol{v}_h, q_h) = -\sum_{K \in \mathcal{T}_h} \int_K q_h \operatorname{div} \boldsymbol{v}_h \, \mathrm{d}x \,, \qquad |\boldsymbol{v}_h|_{1,h} = \left(\sum_{K \in \mathcal{T}_h} |\boldsymbol{v}_h|_{1,K}^2 \right)^{1/2}$$

and $\mathcal{T}_h$ is a triangulation of Ω consisting of elements K used for constructing the spaces $\mathbf{V}_h$ and Q_h. The inf–sup condition (6) makes it possible to establish optimal error estimates for the discrete solution of (1)–(3), see e.g. [10]. Therefore, we shall also disuss for which spaces Q_h the inf–sup condition (6) is satisfied if the velocity space $\mathbf{V}_h$ is defined using the P_n^{mod} elements.

The plan of the paper is as follows. First, in Section 2, we introduce some notation needed in the subsequent sections. Then, in Section 3, we describe a nonconforming streamline diffusion finite element discretization of the convection–diffusion equation (4). The corresponding error analysis is presented in Section 4 and motivates the definition of the P_1^{mod} element described in Section 5. Section 6 is devoted to numerical results for the convection–diffusion equation (4) which demonstrate the good behaviour of discretizations employing the P_1^{mod} element. In Section 7 we mention how the P_1^{mod} element can be extended to higher order elements and, in Section 8, we discuss the validity of the inf–sup condition (6) for these elements. Finally, in Section 9, some theoretical and numerical results for the Stokes equations are presented.

2 Notation

We assume that we are given a family $\{\mathcal{T}_h\}$ of triangulations of the domain Ω parametrized by a positive parameter $h \to 0$ and having the following properties. Each triangulation $\mathcal{T}_h$ consists of a finite number of closed triangular elements K such that $h_K \equiv \mathrm{diam}(K) \le h$, $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$ and any two different elements K, $\widetilde{K} \in \mathcal{T}_h$ are either disjoint or possess either a common vertex or a common edge. In order to prevent the elements from degenerating when h tends to zero, we assume that the family of triangulations is regular, i.e., there exists a constant C independent of h such that

$$\frac{h_K}{\varrho_K} \le C \qquad \forall\, K \in \mathcal{T}_h,\ h > 0\,,$$

where ϱ_K is the maximum diameter of circles inscribed into K. For any element K, we shall denote by $\boldsymbol{n}_{\partial K}$ the unit outer normal vector to the boundary of K.

We denote by $\mathcal{E}_h$ the set of the edges E of $\mathcal{T}_h$. The set of inner edges will be denoted by $\mathcal{E}_h^i$ and the set of boundary edges by $\mathcal{E}_h^b$. Further, we denote by h_E the length of the edge E, by C_E the midpoint of E, by $x_{E,1}$, $x_{E,2}$ the end points of E and by $\lambda_{E,1}$, $\lambda_{E,2}$ the barycentric coordinates on E with respect to $x_{E,1}$, $x_{E,2}$, respectively. Thus, $\lambda_{E,1}$, $\lambda_{E,2}$ are linear functions on E and satisfy $\lambda_{E,i}(x_{E,j}) = \delta_{ij}$, $i, j = 1, 2$. The union of the elements adjacent to an edge E will be denoted by S_E (i.e., S_E consists of one or two elements). For any edge E, we choose a fixed unit tangent vector $\boldsymbol{t}_E$ and a fixed unit normal vector $\boldsymbol{n}_E$. If $E \subset \partial\Omega$, then $\boldsymbol{n}_E$ coincides with the outer normal vector to $\partial\Omega$. Consider any $E \in \mathcal{E}_h^i$ and let K, $\widetilde{K}$ be the two elements possessing the edge E denoted such that $\boldsymbol{n}_E$ points into $\widetilde{K}$. If v is a function belonging to the space

$$H^{1,h}(\Omega) = \{v \in L^2(\Omega)\,;\ v|_K \in H^1(K)\ \ \forall\, K \in \mathcal{T}_h\}\,,$$

then we define the jump of v across E by

$$[|v|]_E = (v|_K)|_E - (v|_{\widetilde{K}})|_E\,. \tag{7}$$

If $E \in \mathcal{E}_h^b$, then we set

$$[|v|]_E = v|_E \,,$$

which is the jump defined by (7) with v extended by zero outside Ω.

In the following sections, we shall need the spaces

$$\widetilde{\mathrm{V}}_h^{conf} = \{v_h \in C(\overline{\Omega})\,;\; v_h|_K \in P_1(K) \;\; \forall\, K \in \mathcal{T}_h\}\,, \qquad \mathrm{V}_h^{conf} = \widetilde{\mathrm{V}}_h^{conf} \cap H_0^1(\Omega)\,,$$
$$\mathrm{V}_h^{nc} = \{v_h \in L^2(\Omega)\,;\; v_h|_K \in P_1(K) \;\; \forall\, K \in \mathcal{T}_h\,, \;\; \int_E [|v_h|]_E \, \mathrm{d}\sigma = 0 \;\; \forall\, E \in \mathcal{E}_h\}\,,$$

and we shall denote by $i_h : H^2(\Omega) \to \widetilde{\mathrm{V}}_h^{conf}$ the Lagrange interpolation operator. It is well known that

$$|v - i_h v|_{m,K} \le C\, h_K^{2-m}\, |v|_{2,K} \qquad \forall\, v \in H^2(K),\, K \in \mathcal{T}_h,\, m = 0, 1\,. \tag{8}$$

Throughout the paper we use standard notation $L^p(\Omega)$, $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$, $C^k(\overline{\Omega})$, etc. for the usual function spaces, see e.g. [5]. In addition, we define a broken space $H^2(\Omega)$ by

$$H^{2,h}(\Omega) = \{v \in L^2(\Omega)\,;\; v|_K \in H^2(K) \;\; \forall\, K \in \mathcal{T}_h\}\,.$$

The norm and seminorm in the Sobolev space $W^{k,p}(\Omega)$ will be denoted by $\|\cdot\|_{k,p,\Omega}$ and $|\cdot|_{k,p,\Omega}$, respectively, and we set $\|\cdot\|_{k,\Omega} = \|\cdot\|_{k,2,\Omega}$ and $|\cdot|_{k,\Omega} = |\cdot|_{k,2,\Omega}$. The scalar product in the space $L^2(G)$ will be denoted by $(\cdot,\cdot)_G$ and we set $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$. Finally, we use the notation C, $\widetilde{C}$ to denote generic constants independent of h, ν and ε.

3 Discretization of the convection–diffusion equation (4)

The starting point of a finite element discretization is the weak formulation of the problem under consideration. Denoting by $\widetilde{u}_b \in H^2(\Omega)$ an extension of u_b, a natural weak formulation of the convection–diffusion equation (4) reads:

Find $u \in H^1(\Omega)$ such that $u - \widetilde{u}_b \in H_0^1(\Omega)$ and

$$a(u, v) = (f, v) \qquad \forall\, v \in H_0^1(\Omega)\,,$$

where

$$a(u, v) = \varepsilon\, (\nabla u, \nabla v) + (\boldsymbol{b} \cdot \nabla u, v) + (c\, u, v)\,.$$

This weak formulation has a unique solution.

We intend to approximate the space $H_0^1(\Omega)$ by a nonconforming first order finite element space V_h and at this stage we only assume that

$$\mathrm{V}_h^{conf} \subset \mathrm{V}_h \subset H^{2,h}(\Omega)\,. \tag{9}$$

The inclusion $\mathrm{V}_h^{conf} \subset \mathrm{V}_h$ assures first order approximation properties of V_h with respect to $|\cdot|_{1,h}$ when $h \to 0$.

A finite element discretization of (4) could be simply obtained by using the bilinear forms

$$a_h^d(u,v)=\varepsilon\sum_{K\in\mathcal{T}_h}(\nabla u,\nabla v)_K\,,\quad a_h^c(u,v)=\sum_{K\in\mathcal{T}_h}(\boldsymbol{b}\cdot\nabla u,v)_K\,,\qquad u,v\in H^{1,h}(\Omega)\,,$$

instead of the first two terms in $a(u,v)$ and by replacing the space $H_0^1(\Omega)$ in the weak formulation by the finite element space V_h. However, the bilinear form corresponding to the discrete problem generally would not be coercive and therefore, before passing from the weak formulation to the discrete problem, we first apply integration by parts to the convective term $(\boldsymbol{b}\cdot\nabla u,v)$ to obtain

$$(\boldsymbol{b}\cdot\nabla u,v)=\frac{1}{2}\left[(\boldsymbol{b}\cdot\nabla u,v)-(\boldsymbol{b}\cdot\nabla v,u)-(\operatorname{div}\boldsymbol{b},u\,v)\right],\qquad u\in H^1(\Omega),\ v\in H_0^1(\Omega)\,.$$

Thus, a discrete analogue of the second term in the bilinear form a also is

$$a_h^s(u,v)=\frac{1}{2}\sum_{K\in\mathcal{T}_h}\left[(\boldsymbol{b}\cdot\nabla u,v)_K-(\boldsymbol{b}\cdot\nabla v,u)_K-(\operatorname{div}\boldsymbol{b},u\,v)_K\right],\qquad u,v\in H^{1,h}(\Omega)\,.$$

This bilinear form is skew–symmetric if $\operatorname{div}\boldsymbol{b}=0$. That gives rise to the notation a_h^{skew} below. For $u\in H^{2,h}(\Omega)$ and $v\in H^{1,h}(\Omega)$, we define a streamline diffusion term by

$$a_h^{sd}(u,v)=\sum_{K\in\mathcal{T}_h}(-\varepsilon\,\Delta\,u+\boldsymbol{b}\cdot\nabla u+c\,u,\,\delta_K\,\boldsymbol{b}\cdot\nabla v)_K\,,$$

where $\delta_K\geq 0$ is a control parameter. Now, denoting

$$\begin{aligned} a_h^{skew}(u,v) &= a_h^d(u,v)+a_h^s(u,v)+(c\,u,v)+a_h^{sd}(u,v)\,,\\ l_h(v) &= (f,v)+\sum_{K\in\mathcal{T}_h}(f,\,\delta_K\,\boldsymbol{b}\cdot\nabla v)_K\,, \end{aligned}$$

the streamline diffusion finite element method investigated in this paper reads:

Find $u_h\in H^{2,h}(\Omega)$ such that $u_h-i_h\widetilde{u}_b\in\mathrm{V}_h$ and

$$a_h^{skew}(u_h,v_h)=l_h(v_h)\qquad\forall\ v_h\in\mathrm{V}_h\,. \tag{10}$$

In the remaining part of this section we shall investigate the coercivity of the bilinear form a_h^{skew} with respect to the streamline diffusion norm

$$|||v|||=\left(\sum_{K\in\mathcal{T}_h}\{\varepsilon\,|v|_{1,K}^2+c_0\,\|v\|_{0,K}^2+\delta_K\,\|\boldsymbol{b}\cdot\nabla v\|_{0,K}^2\}\right)^{1/2}.$$

Using standard arguments (cf. [5], Chapter III), we deduce that there exist constants μ_1, μ_2 independent of h such that

$$\|\Delta\,v_h\|_{0,K}\ \leq\ \mu_1\,h_K^{-1}\,|v_h|_{1,K}\qquad\forall\ v_h\in\mathrm{V}_h,\ K\in\mathcal{T}_h\,, \tag{11}$$

$$|v_h|_{1,K}\ \leq\ \mu_2\,h_K^{-1}\,\|v_h\|_{0,K}\qquad\forall\ v_h\in\mathrm{V}_h,\ K\in\mathcal{T}_h\,. \tag{12}$$

Assuming that the control parameter δ_K satisfies

$$0 \le \delta_K \le \min\left\{ \frac{c_0}{2\,\|c\|^2_{0,\infty,K}}, \frac{h_K^2}{2\,\varepsilon\,\mu_1^2} \right\}, \tag{13}$$

we can prove the following result which implies that the discrete problem (10) has a unique solution.

Theorem 1 *Under the assumption (13), the bilinear form a_h^{skew} is coercive, i.e.,*

$$a_h^{skew}(v_h, v_h) \ge \frac{1}{2}\, |||v_h|||^2 \qquad \forall\ v_h \in \mathrm{V}_h\,. \tag{14}$$

Proof. Using (5), we immediately obtain

$$a_h^{skew}(v_h, v_h) \ge |||v_h|||^2 + \sum_{K\in\mathcal{T}_h} (-\varepsilon\,\Delta\,v_h + c\,v_h,\ \delta_K\,\boldsymbol{b}\cdot\nabla v_h)_K\,.$$

Applying (11) and (13), we get

$$\begin{aligned}(-\varepsilon\,\Delta\,v_h + c\,v_h,\ \delta_K\,\boldsymbol{b}\cdot\nabla v_h)_K &\le (\varepsilon^{1/2}\,|v_h|_{1,K} + c_0^{1/2}\,\|v_h\|_{0,K})\,\frac{\delta_K^{1/2}}{\sqrt{2}}\,\|\boldsymbol{b}\cdot\nabla v_h\|_{0,K} \\ &\le \frac{1}{2}\,(\varepsilon\,|v_h|^2_{1,K} + c_0\,\|v_h\|^2_{0,K} + \delta_K\,\|\boldsymbol{b}\cdot\nabla v_h\|^2_{0,K})\,,\end{aligned}$$

which gives (14). ♣

Remark 1 We shall also discuss the convergence properties of the discrete problem (10) when a_h^{skew} is replaced by the convective bilinear form a_h^{conv} defined by

$$a_h^{conv}(u,v) = a_h^d(u,v) + a_h^c(u,v) + (c\,u, v) + a_h^{sd}(u,v)\,. \tag{15}$$

Note that a result similar to (14) does not hold for this bilinear form. Indeed,

$$a_h^{conv}(v_h, v_h) \ge \frac{1}{2}\,|||v_h|||^2 + \frac{1}{2}\sum_{E\in\mathcal{E}_h}\int_E (\boldsymbol{b}\cdot\boldsymbol{n}_E)\,[|v_h^2|]_E\,\mathrm{d}\sigma \qquad \forall\ v_h \in \mathrm{V}_h\,,$$

where the additional term is of order $O(\|v_h\|^2_{0,\Omega}/h)$ in general (cf. [26]).

4 Error analysis

If the weak solution of (4) satisfies $u \in H^2(\Omega)$, then it fulfils the equation (4) a. e. in Ω. Multiplying (4) by $v_h \in \mathrm{V}_h$ and integrating by parts, we infer that

$$a_h^{skew}(u, v_h) = l_h(v_h) + r_h^d(u, v_h) + r_h^s(u, v_h) \qquad \forall\ v_h \in \mathrm{V}_h\,, \tag{16}$$

where the consistency errors r_h^d, r_h^s, are given by

$$\begin{aligned} r_h^d(u,v_h) &= \varepsilon \sum_{K\in\mathcal{T}_h} \int_{\partial K} \frac{\partial u}{\partial \boldsymbol{n}_{\partial K}}\, v_h \,\mathrm{d}\sigma = \varepsilon \sum_{E\in\mathcal{E}_h} \int_E \frac{\partial u}{\partial \boldsymbol{n}_E}\, [|v_h|]_E \,\mathrm{d}\sigma\,,\\ r_h^s(u,v_h) &= -\frac{1}{2} \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\boldsymbol{b}\cdot\boldsymbol{n}_{\partial K})\, u\, v_h \,\mathrm{d}\sigma = -\frac{1}{2} \sum_{E\in\mathcal{E}_h} \int_E (\boldsymbol{b}\cdot\boldsymbol{n}_E)\, u\, [|v_h|]_E \,\mathrm{d}\sigma\,. \end{aligned}$$

For estimating the consistency errors, we shall use the following lemma.

Lemma 1 *For any edge $E \in \mathcal{E}_h$ and any integer $k \geq 0$, let $\mathcal{M}_E^k$ be the projection operator from $L^2(E)$ onto $P_k(E)$ defined by*

$$\int_E q\, \mathcal{M}_E^k\, v \,\mathrm{d}\sigma = \int_E q\, v \,\mathrm{d}\sigma \qquad \forall\, q \in P_k(E),\ v \in L^2(E)\,.$$

Then there exists a constant C independent of E and h such that

$$\left| \int_E \varphi\, (v - \mathcal{M}_E^k\, v) \,\mathrm{d}\sigma \right| \leq C\, h_E^{k+1}\, |\varphi|_{1,K}\, |v|_{k+1,K} \tag{17}$$

for all $K \in \mathcal{T}_h$, $E \subset \partial K$, $\varphi \in H^1(K)$ and $v \in H^{k+1}(K)$.

Proof. See [6], Lemma 3. ♣

Now we are in a position to prove a convergence result for the discrete problem (10).

Theorem 2 *Let the assumptions (9) and (13) be fulfilled and let the space V_h satisfy the patch test of order $k+1$, i.e.,*

$$\int_E [|v_h|]_E\, q \,\mathrm{d}\sigma = 0 \qquad \forall\, v_h \in \mathrm{V}_h,\ q \in P_k(E),\ E \in \mathcal{E}_h\,, \tag{18}$$

where $k \geq 0$ is a given integer. Let the weak solution of (4) belong to $H^m(\Omega)$, $m = \max\{2, k+1\}$, and let $\boldsymbol{b} \in W^{k+1,\infty}(\Omega)^2$. Then the discrete solution u_h satisfies

$$|||u - u_h||| \leq C\, h \left(\sum_{K\in\mathcal{T}_h} \gamma_K\, |u|_{2,K}^2 \right)^{1/2} + C\, h^k \left(\sum_{E\in\mathcal{E}_h} \gamma_E\, \|u\|_{m,S_E}^2 \right)^{1/2}, \tag{19}$$

where

$$\gamma_K = \varepsilon + h_K^2 + \delta_K + (\max\{\varepsilon, \delta_K\})^{-1}\, h_K^2\,, \qquad \gamma_E = \min\left\{ \frac{h_E^2}{\varepsilon}, 1 \right\}.$$

Proof. Denoting $w = i_h u - u$ and $w_h = i_h u - u_h$ (where i_h is the Lagrange interpolation operator), we have $w_h \in \mathrm{V}_h$ and it follows from (10) and (16) that

$$a_h^{skew}(w_h, v_h) = a_h^{skew}(w, v_h) + r_h^d(u, v_h) + r_h^s(u, v_h) \qquad \forall\, v_h \in \mathrm{V}_h\,.$$

Applying (14) and the triangular inequality, we infer that

$$\frac{1}{2}\,|||u-u_h||| \le \frac{1}{2}\,|||w||| + \sup_{v_h\in \mathrm{V}_h} \frac{a_h^d(w,v_h)+r_h^d(u,v_h)}{|||v_h|||}$$
$$+ \sup_{v_h\in \mathrm{V}_h} \frac{a_h^s(w,v_h)+r_h^s(u,v_h)}{|||v_h|||} + \sup_{v_h\in \mathrm{V}_h} \frac{(c\,w,v_h)}{|||v_h|||} + \sup_{v_h\in \mathrm{V}_h} \frac{a_h^{sd}(w,v_h)}{|||v_h|||}\,.$$

Since the estimation of the terms on the right–hand side of this inequality follows the lines of the proof of Theorem 1 in [15], we only mention the main steps. First, a simple application of (8) gives

$$|||w||| \le C\,h \left(\sum_{K\in\mathcal{T}_h} (\varepsilon + h_K^2 + \delta_K)\,|u|_{2,K}^2 \right)^{1/2},$$
$$(c\,w,v_h) \le C\,h \left(\sum_{K\in\mathcal{T}_h} h_K^2\,|u|_{2,K}^2 \right)^{1/2} |||v_h|||\,.$$

Using (13), we further get

$$a_h^{sd}(w,v_h) \le C\,h \left(\sum_{K\in\mathcal{T}_h} (\varepsilon + \delta_K)\,|u|_{2,K}^2 \right)^{1/2} |||v_h|||\,.$$

In view of (18), we have for any $E\in\mathcal{E}_h$

$$\int_E \frac{\partial u}{\partial \boldsymbol{n}_E}\,[|v_h|]_E\,\mathrm{d}\sigma = \int_E \left(\frac{\partial u}{\partial \boldsymbol{n}_E} - \mathcal{M}_E^0\,\frac{\partial u}{\partial \boldsymbol{n}_E} \right) [|v_h|]_E\,\mathrm{d}\sigma$$

and hence, applying (17), we deduce that

$$a_h^d(w,v_h) + r_h^d(u,v_h) \le C\,h\,\varepsilon^{1/2}\,|u|_{2,\Omega}\,|||v_h|||\,.$$

Integrating by parts, we obtain

$$a_h^s(w,v_h) = -\sum_{K\in\mathcal{T}_h} (\boldsymbol{b}\cdot\nabla v_h, w)_K - (\operatorname{div}\boldsymbol{b}, w\,v_h) + \frac{1}{2}\sum_{E\in\mathcal{E}_h} \int_E (\boldsymbol{b}\cdot\boldsymbol{n}_E)\,w\,[|v_h|]_E\,\mathrm{d}\sigma\,.$$

Since $(\boldsymbol{b}\cdot\nabla v_h, w)_K \le C\,h_K^2\,|u|_{2,K}\,\|\boldsymbol{b}\cdot\nabla v_h\|_{0,K}$, we have

$$(\boldsymbol{b}\cdot\nabla v_h, w)_K \le C\,h_K^2\,|u|_{2,K}\,(\max\{\varepsilon,\delta_K\})^{-1/2}\,(\varepsilon\,|v_h|_{1,K}^2 + \delta_K\,\|\boldsymbol{b}\cdot\nabla v_h\|_{0,K}^2)^{1/2}$$

and hence

$$\sum_{K\in\mathcal{T}_h} (\boldsymbol{b}\cdot\nabla v_h, w)_K \le C\,h \left(\sum_{K\in\mathcal{T}_h} (\max\{\varepsilon,\delta_K\})^{-1}\,h_K^2\,|u|_{2,K}^2 \right)^{1/2} |||v_h|||\,.$$

The term $(\operatorname{div} \boldsymbol{b}, w\, v_h)$ can be estimated in the same way as $(c\, w, v_h)$. To estimate the term with the jumps, we apply (18) and Lemma 1 and we obtain

$$\begin{aligned}\int_E (\boldsymbol{b}\cdot\boldsymbol{n}_E)\, w\, [|v_h|]_E \,\mathrm{d}\sigma &= \int_E [(\boldsymbol{b}\cdot\boldsymbol{n}_E)\, w - \mathcal{M}_E^k((\boldsymbol{b}\cdot\boldsymbol{n}_E)\, w)]\, [|v_h|]_E \,\mathrm{d}\sigma \\ &\le C\, h_E^{k+1}\, \|w\|_{k+1,S_E}\, |v_h|_{1,S_E} \le \widetilde{C}\, h_E^{k+1}\, \|u\|_{m,S_E}\, |v_h|_{1,S_E}\,,\end{aligned}$$

where the norms over S_E are considered to be defined elementwise. Using (12), we derive

$$\int_E (\boldsymbol{b}\cdot\boldsymbol{n}_E)\, w\, [|v_h|]_E \,\mathrm{d}\sigma \le C\, h_E^k\, \|u\|_{m,S_E}\, \gamma_E^{1/2}\, (\varepsilon\, |v_h|^2_{1,S_E} + c_0\, \|v_h\|^2_{0,S_E})^{1/2}\,,$$

which implies that

$$\sum_{E\in\mathcal{E}_h} \int_E (\boldsymbol{b}\cdot\boldsymbol{n}_E)\, w\, [|v_h|]_E \,\mathrm{d}\sigma \le C\, h^k \left(\sum_{E\in\mathcal{E}_h} \gamma_E\, \|u\|^2_{m,S_E}\right)^{1/2} |||v_h|||\,.$$

The consistency error $r_h^s(u, v_h)$ can be estimated in the same way. Summing up all the estimates, we obtain the theorem. ♣

Remark 2 The above estimate together with the condition (13) suggests to set

$$\delta_K = \begin{cases} \kappa_K\, h_K & \text{if } h_K > \varepsilon, \\ 0 & \text{if } h_K \le \varepsilon, \end{cases}$$

where κ_K is bounded independently of h and satisfies

$$0 < \kappa_0 \le \kappa_K \le \min\left\{ \frac{c_0}{2\, \|c\|^2_{0,\infty,K}\, h_K}, \frac{h_K}{2\, \varepsilon\, \mu_1^2} \right\}.$$

Then $(\max\{\varepsilon, \delta_K\})^{-1}\, h_K^2 \le (\min\{1, \kappa_0\})^{-1}\, h_K$ and hence $\gamma_K \le C\,(\varepsilon + h_K)$.

Let $\varepsilon \le h$ and let δ_K be defined as in Remark 2, which implies that $\gamma_K \le C\, h$. Since the sum over edges in (19) stems from the nonconformity only, we obtain for $\mathrm{V}_h = \mathrm{V}_h^{conf}$ the well–known optimal ε–uniform estimate

$$|||u - u_h||| \le C\, h^{3/2}\, |u|_{2,\Omega}\,.$$

For a general nonconforming space V_h satisfying the assumptions of Theorem 2, the estimate (19) leads to the ε–uniform estimate

$$|||u - u_h||| \le C\, h^{3/2}\, |u|_{2,\Omega} + C\, h^k \|u\|_{\max\{2,k+1\},\Omega}\,. \tag{20}$$

Thus, if we use the space $\mathrm{V}_h = \mathrm{V}_h^{nc}$, which satisfies (18) for $k = 0$ only, the ε–uniform convergence order is 0. Numerical experiments really confirm this pessimistic prediction, which suggests that it is generally a property of the method and not a consequence of an inaccurate estimation. On the other hand, the estimate (20) shows that, for obtaining the optimal convergence order 3/2, it is sufficient to require that V_h satisfies (18) with $k = 2$, i.e., the patch test of order 3. This is an unusual requirement for a nonconforming first order finite element space but we shall show in the next section that such a space having a structure convenient for practical computations really can be constructed.

5 Derivation of the P_1^{mod} element

The aim of this section is to derive a nonconforming first order finite element space satisfying the patch test of order 3. The space V_h^{nc} does not possess this property and the basic idea of constructing the desired space is to enrich the space V_h^{nc} by a suitable linearly independent finite element space B_h and then to restrict the enlarged space to its subspace of functions satisfying the patch test of order 3. Since the resulting space is obtained not only by enriching but by *modifying* the space V_h^{nc}, we shall denote it by V_h^{mod}. Thus, we have

$$\mathrm{V}_h^{mod} = \{v_h \in \mathrm{V}_h^{nc} \oplus \mathrm{B}_h\,;\ \int_E [|v_h|]_E\, q \,\mathrm{d}\sigma = 0 \quad \forall\, q \in P_2(E),\ E \in \mathcal{E}_h\}\,. \tag{21}$$

Note that the space V_h^{mod} defined by (21) always is at least a first order space since $\mathrm{V}_h^{mod} \supset \mathrm{V}_h^{conf}$. Particularly, for $\mathrm{B}_h = \{0\}$, we have $\mathrm{V}_h^{mod} = \mathrm{V}_h^{conf}$.

There are various possibilities how to define the space B_h. We shall relate its properties to the standard basis functions $\{\zeta_E\}_{E\in\mathcal{E}_h^i}$ of V_h^{nc}. The basis functions ζ_E are piecewise linear, equal 1 on E and vanish in the midpoints of all edges different from E. Thus, the support of any ζ_E consists of the two elements adjacent to E, i.e., $\operatorname{supp}\zeta_E = S_E$. We require that, for any inner edge $E \in \mathcal{E}_h^i$, there exists a function $b_E \in \mathrm{B}_h$ such that the function

$$\psi_E = \zeta_E + b_E \tag{22}$$

belongs to V_h^{mod}. In addition, we want that ψ_E preserves some basic properties of ζ_E. Therefore, we require that

$$\operatorname{supp} b_E \subset S_E\,, \qquad b_E|_{S_E} \in H^1(S_E)\,, \qquad b_E|_E = 0\,, \tag{23}$$

$$b_E|_{S_E} \text{ is odd along the edges } E' \subset \partial S_E. \tag{24}$$

To assure these properties, we introduce, for each element $K \in \mathcal{T}_h$, three nonconforming bubble functions $b_{K,E}$ assigned to the edges E of K and having their supports in K. We shall define these functions using a fixed function $\widehat{b}$ defined on the standard reference element $\widehat{K}$.

Let $\widehat{E}$ be one of the edges of the reference element $\widehat{K}$ and let $\widehat{x}_1$, $\widehat{x}_2$ be the end points of $\widehat{E}$. The properties (9), (23), (24) and the requirement that ψ_E defined by (22) belongs to V_h^{mod} lead to the following assumptions on the function $\widehat{b}$:

$$\widehat{b} \in H^2(\widehat{K})\,, \qquad \widehat{b}|_{\partial\widehat{K}\setminus\widehat{E}} = 0\,, \qquad \|\widehat{b}\|_{0,\widehat{E}} \neq 0\,, \tag{25}$$

$$\widehat{b}|_{\widehat{E}} \text{ is odd with respect to the midpoint of } \widehat{E}, \tag{26}$$

$$\gamma \equiv \frac{1}{|\widehat{E}|}\int_{\widehat{E}} \widehat{b}\,\widehat{\lambda}_1 \,\mathrm{d}\widehat{\sigma} > 0\,, \tag{27}$$

where $\widehat{\lambda}_1$ is the barycentric coordinate on $\widehat{K}$ with respect to $\widehat{x}_1$ (i.e., $\widehat{\lambda}_1|_E \in P_1(\widehat{E})$, $\widehat{\lambda}_1(\widehat{x}_1) = 1$, $\widehat{\lambda}_1(\widehat{x}_2) = 0$).

Let K be any element of the triangulation $\mathcal{T}_h$ and let $E \in \mathcal{E}_h$ be an edge of K. Then there exists a unique regular affine mapping $F_K : \widehat{K} \to K$ such that $F_K(\widehat{K}) = K$, $F_K(\widehat{x}_1) = x_{E,1}$ and $F_K(\widehat{x}_2) = x_{E,2}$ (cf. e.g. [5]). Denoting

$$b_{K,E} = \begin{cases} \widehat{b} \circ F_K^{-1} & \text{in } K, \\ 0 & \text{in } \Omega \setminus K, \end{cases}$$

we obtain a nonconforming bubble function with support in K. The function $b_{K,E}|_K$ has properties analogous to (25)–(27) and satisfies

$$\int_E b_{K,E}|_K \, \lambda_{E,1} \, \mathrm{d}\sigma = \gamma \, h_E \, .$$

Note that

$$(b_{K,E}|_K)|_E = (b_{\widetilde{K},E}|_{\widetilde{K}})|_E \qquad \forall \; E \in \mathcal{E}_h^i \, , \tag{28}$$

where K, $\widetilde{K}$ are the two elements possessing the edge E.

Now, let us set

$$\mathrm{B}_h = \mathrm{span}\{b_{K,E}\}_{K \in \mathcal{T}_h,\, E \in \mathcal{E}_h,\, E \subset \partial K} \, . \tag{29}$$

Then, for any edge $E \in \mathcal{E}_h^i$, there exists a uniquely determined function $b_E \in \mathrm{B}_h$ such that (23) holds and the function ψ_E defined by (22) satisfies the patch test of order 2. To establish a formula for ψ_E, we denote by K, $\widetilde{K}$ the two elements adjacent to the edge E, by E, E_1, E_2 the edges of K and by E, E_3, E_4 the edges of $\widetilde{K}$. Then

$$\psi_E = \zeta_E + \beta_{E,E_1} \, b_{K,E_1} + \beta_{E,E_2} \, b_{K,E_2} + \beta_{E,E_3} \, b_{\widetilde{K},E_3} + \beta_{E,E_4} \, b_{\widetilde{K},E_4} \, , \tag{30}$$

where the coefficients β_{E,E_i} are determined by the requirement that ψ_E satisfies the patch test on E_i, $i = 1, \dots, 4$. In particular, we get

$$\beta_{E,E_i} = - \frac{\int_{E_i} [|\zeta_E|]_{E_i} \, \lambda_{E_i,1} \, \mathrm{d}\sigma}{\int_{E_i} [|b_{K,E_i}|]_{E_i} \, \lambda_{E_i,1} \, \mathrm{d}\sigma} = \pm \frac{1}{6\,\gamma} \, , \qquad i = 1, 2, \tag{31}$$

and β_{E,E_3}, β_{E,E_4} are defined analogously. The sign in (31) depends on the choice of $x_{E_i,1}$. A careful analysis shows that ψ_E satisfies not only the patch test of order 2 but also the patch test of order 3. First, from (26), it follows that, for any $v_h \in \mathrm{V}_h^{nc} \oplus \mathrm{B}_h$ and any $E' \in \mathcal{E}_h$, the jump $[|v_h|]_{E'}$ is odd with respect to the midpoint of E'. Therefore,

$$\int_{E'} [|v_h|]_{E'} \, \mathrm{d}\sigma = \int_{E'} [|v_h|]_{E'} \, \lambda_{E',1} \, \lambda_{E',2} \, \mathrm{d}\sigma = 0 \qquad \forall \; v_h \in \mathrm{V}_h^{nc} \oplus \mathrm{B}_h, \; E' \in \mathcal{E}_h \, .$$

Further, (31) guarantees that

$$\int_{E'} [|\psi_E|]_{E'} \, \lambda_{E',1} \, \mathrm{d}\sigma = 0 \qquad \forall \; E' \in \mathcal{E}_h \, .$$

Since $P_2(E') = \mathrm{span}\{1,\, \lambda_{E',1},\, \lambda_{E',1}\lambda_{E',2}\}$, we deduce that really $\psi_E \in \mathrm{V}_h^{mod}$.

The functions $\{\psi_E\}_{E\in\mathcal{E}_h^i}$ are linearly independent and can be used as basis functions in V_h^{mod}. To obtain a complete basis of the space V_h^{mod}, we further introduce the functions

$$\chi_E = b_{K,E} + b_{\widetilde{K},E}\,, \qquad E \in \mathcal{E}_h^i\,, \tag{32}$$

where K, $\widetilde{K}$ are the two elements adjacent to the edge E. Then $\mathrm{supp}\,\chi_E \subset S_E$ and, in view of (28), we have $\chi_E \in H_0^1(\Omega)$. Thus, the functions χ_E generate a conforming subspace of V_h^{mod} whereas the functions ψ_E are purely nonconforming functions since they have jumps across the edges making up the boundary of S_E.

It is easy to verify that the functions $\{\psi_E, \chi_E\}_{E\in\mathcal{E}_h^i}$ are linearly independent and that

$$\mathrm{V}_h^{mod} = \mathrm{span}\{\psi_E, \chi_E\}_{E\in\mathcal{E}_h^i}\,.$$

Thus, we see that the space V_h^{mod} defined by (21) and (29) is an edge–oriented nonconforming finite element space. The supports of the basis functions ψ_E, χ_E are contained in the supports of the basis functions ζ_E of V_h^{nc} and hence the space V_h^{mod} can be implemented using the same data structures as the space V_h^{nc}. In addition, the functions from B_h are piecewise continuous and therefore, the space V_h^{mod} consists of piecewise continuous functions which are continuous in the midpoints of inner edges and vanish in the midpoints of boundary edges. This is a further feature common with the space V_h^{nc}.

As we know from the previous section, the properties of the space V_h^{mod} guarantee that the solution of the discrete problem (10) with $\mathrm{V}_h = \mathrm{V}_h^{mod}$ satisfies

$$|||u - u_h||| \le C\, h^{\min\{k,3/2\}}\, \|u\|_{k+1,\Omega}\,, \qquad k = 1, 2\,, \tag{33}$$

where C is independent of ε. Thus, for $u \in H^3(\Omega)$, we get the optimal ε–uniform convergence order 3/2. This is a dramatic improvement in comparison with the discrete problem (10) with $\mathrm{V}_h = \mathrm{V}_h^{nc}$ for which an ε–uniform convergence cannot be proved and numerical results are often of a very poor quality. Using the discretization (10) with a_h^{skew} replaced by a_h^{conv} defined in (15), for which no theoretical results are available, the numerical results are better than for a_h^{skew} but still much worse in the convection dominated case than for discretizations using the space V_h^{mod} (cf. Section 6).

The price we pay for the ε–uniform estimate (33) is that $\dim \mathrm{V}_h^{mod} = 2 \dim \mathrm{V}_h^{nc}$ and that, consequently, the stiffness matrix corresponding to V_h^{mod} is generally four times larger than the one corresponding to V_h^{nc}. Thus, the size of the discrete problem (10) with $\mathrm{V}_h = \mathrm{V}_h^{mod}$ defined on a triangulation $\mathcal{T}_h$ is comparable with the size of a discrete problem with $\mathrm{V}_h = \mathrm{V}_h^{nc}$ defined on the refined triangulation $\mathcal{T}_{h/2}$. Therefore, at the first sight, the space V_h^{mod} seems to be more expensive than the space V_h^{nc}. However, taking into account that the discretization using the space V_h^{mod} is more accurate, particularly in the convection dominated regime, we realize that using the space V_h^{mod} is in fact cheaper than using the space V_h^{nc}.

A particular example of the P_1^{mod} element can be obtained by setting

$$\widehat{b} = \widehat{\lambda}_1^2\,\widehat{\lambda}_2 - \widehat{\lambda}_1\,\widehat{\lambda}_2^2\,, \tag{34}$$

where $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ are the barycentric coordinates on the reference element $\widehat{K}$ with respect to $\widehat{x}_1$, $\widehat{x}_2$, respectively. To express the formulas (30), (32) for the basis functions ψ_E and χ_E in terms of barycentric coordinates, we denote by K and $\widetilde{K}$ the two elements adjacent to an edge $E \in \mathcal{E}_h^i$ and by λ_1, λ_2 and $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2$ the barycentric coordinates on K and $\widetilde{K}$ with respect to $x_{E,1}$, $x_{E,2}$, respectively. Further, we respectively denote by λ_3 and $\widetilde{\lambda}_3$ the remaining barycentric coordinates on K and $\widetilde{K}$. Then

$$\psi_E = \begin{cases} 1 - 2\,\lambda_3 - 10\,(\lambda_1^2\,\lambda_3 - \lambda_1\,\lambda_3^2) - 10\,(\lambda_2^2\,\lambda_3 - \lambda_2\,\lambda_3^2) & \text{in } K, \\ 1 - 2\,\widetilde{\lambda}_3 - 10\,(\widetilde{\lambda}_1^2\,\widetilde{\lambda}_3 - \widetilde{\lambda}_1\,\widetilde{\lambda}_3^2) - 10\,(\widetilde{\lambda}_2^2\,\widetilde{\lambda}_3 - \widetilde{\lambda}_2\,\widetilde{\lambda}_3^2) & \text{in } \widetilde{K} \setminus E, \\ 0 & \text{in } \Omega \setminus \{K \cup \widetilde{K}\}, \end{cases}$$

and

$$\chi_E = \begin{cases} \lambda_1^2\,\lambda_2 - \lambda_1\,\lambda_2^2 & \text{in } K, \\ \widetilde{\lambda}_1^2\,\widetilde{\lambda}_2 - \widetilde{\lambda}_1\,\widetilde{\lambda}_2^2 & \text{in } \widetilde{K} \setminus E, \\ 0 & \text{in } \Omega \setminus \{K \cup \widetilde{K}\}. \end{cases}$$

6 Numerical results for a convection–diffusion equation

In this section we present numerical results obtained for the convection–diffusion equation (4) with the following data:

Example 1 Let $\Omega = (0,1)^2$, $\boldsymbol{b} = (3,2)$, $c = 2$ and $u_b = 0$. For a given $\varepsilon > 0$, the right–hand side f of the equation (4) is chosen such that

$$u(x,y) = 100\,x^2\,(1-x)^2\,y\,(1-y)\,(1-2\,y)$$

is the exact solution of (4).

We considered either the discretization (10) or a discretization obtained from (10) by replacing a_h^{skew} by a_h^{conv} defined in (15) and we used the P_1^{nc} element ($\mathrm{V}_h = \mathrm{V}_h^{nc}$) and the P_1^{mod} element ($\mathrm{V}_h = \mathrm{V}_h^{mod}$) defined at the end of Section 5. The parameter δ_K was defined as in Remark 2 with $\kappa_K = 1$. The discrete problems were defined on a sequence of four unstructured triangulations (Level 1 – Level 4). The coarsest triangulation (Level 1) is depicted in Fig. 1 and was obtained using the anisotropic mesh adaptation technique of [7]. The next triangulation (Level 2) was obtained by mirroring the triangulation of Level 1 with respect to the upper edge and than by mirroring this new mesh of the rectangle $[0,1] \times [0,2]$ with respect to the its right edge. This gives a mesh of the square $[0,2] \times [0,2]$ which was finally transformed onto the square $[0,1] \times [0,1]$. The resulting mesh can be seen in Fig. 2. Triangulations of Level 3 and Level 4 were obtained analogously.

The computed results are presented in Tables 1–3. We considered $\varepsilon = h^4$ to investigate whether the methods are ε–uniform. The errors of the discrete solutions were measured in the norms $\|\cdot\|_{0,\Omega}$, $|\cdot|_{1,h}$, $|||\cdot|||$ and in the discrete L^∞ norm $\|\cdot\|_{0,\infty,h}$

Table 1: Example 1, errors for a_h^{skew} with the P_1^{nc} element and $\varepsilon = h^4$.

h	ε	$\|\cdot\|_{0,\Omega}$	$\|\cdot\|_{1,h}$	$\|\|\|\cdot\|\|\|$	$\|\cdot\|_{0,\infty,h}$
7.34−2	2.91−5	3.67−1	2.98+1	6.44−1	1.92+0
3.67−2	1.81−6	3.95−1	6.53+1	7.00−1	1.97+0
1.83−2	1.13−7	3.44−1	1.15+2	6.29−1	2.03+0
9.17−3	7.09−9	2.91−1	1.95+2	5.76−1	2.24+0
conv. order		0.24	−0.76	0.13	−0.14

Table 2: Example 1, errors for a_h^{conv} with the P_1^{nc} element and $\varepsilon = h^4$.

h	ε	$\|\cdot\|_{0,\Omega}$	$\|\cdot\|_{1,h}$	$\|\|\|\cdot\|\|\|$	$\|\cdot\|_{0,\infty,h}$
7.34−2	2.91−5	3.93−2	2.71+0	1.23−1	4.68−1
3.67−2	1.81−6	2.35−2	3.25+0	5.10−2	2.28−1
1.83−2	1.13−7	1.22−2	3.22+0	2.21−2	1.68−1
9.17−3	7.09−9	6.09−3	3.17+0	9.89−3	7.99−2
conv. order		1.00	0.03	1.16	1.07

which is defined as the maximum of the errors in the midpoints of edges. For the P_1^{mod} element, we only give the errors of the piecewise linear part u_h^{lin} of u_h since it can be proved that u_h^{lin} converges to the weak solution with the same convergence orders as u_h (cf. [19]). The convergence orders were always computed using values obtained for the two finest triangulations.

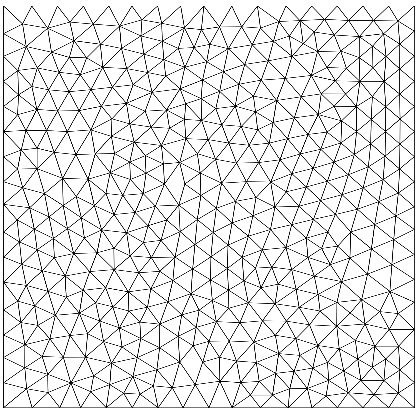

Figure 1: Triangulation of Level 1.

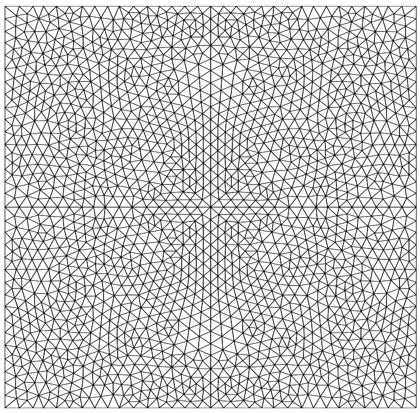

Figure 2: Triangulation of Level 2.

Table 3: Example 1, errors for a_h^{skew} with the P_1^{mod} element and $\varepsilon = h^4$.

h	ε	$\|\cdot\|_{0,\Omega}$	$\vert\cdot\vert_{1,h}$	$\vert\vert\vert\cdot\vert\vert\vert$	$\|\cdot\|_{0,\infty,h}$
7.34−2	2.91−5	2.62−3	2.19−1	1.18−1	1.47−2
3.67−2	1.81−6	6.78−4	1.13−1	4.21−2	3.54−3
1.83−2	1.13−7	1.90−4	5.73−2	1.50−2	1.31−3
9.17−3	7.09−9	5.01−5	2.87−2	5.31−3	3.52−4
conv. order		1.93	1.00	1.50	1.89

The results in Table 1 show a very bad convergence behaviour for the discretization (10) with the P_1^{nc} element which is in agreement with our theoretical results in Section 4. The method a_h^{conv}/P_1^{nc}, for which no theoretical results are available, is better but nevertheless it does no lead to optimal ε–uniform convergence rates on unstructured meshes, although it gives optimal rates on Friedrichs–Keller triangulations (cf. [19]). In contrast to this, the results in Table 3 indicate that the solutions of the discretization a_h^{skew}/P_1^{mod} converge with optimal ε–uniform rates also on unstructured meshes and we see that the accuracy of the discrete solutions obtained using the P_1^{mod} element is much higher than in case of discretizations employing the P_1^{nc} element. For further numerical experiments concerning the three methods considered here we refer to [19].

7 Extension of the P_1^{mod} element to higher order finite elements

It is easy to see that, for higher order nonconforming finite elements, we can proceed in a similar way as in Section 4 and that we can also draw similar conclusions. This implies that it is also desirable to construct higher order nonconforming finite element spaces satisfying a patch test of a higher order than usual. For this, we can employ the fact that, imposing further conditions on the function $\widehat{b}$ defining the P_1^{mod} element, we can enhance the order of the patch test satisfied by the space V_h^{mod}.

Let us investigate which conditions on $\widehat{b}$ are necessary to have a patch test of a given order. It suffices to investigate the validity of the patch test for the basis functions $\{\psi_E\}_{E\in\mathcal{E}_h^i}$ defined by (30) and (31) since the remaining basis functions $\{\chi_E\}_{E\in\mathcal{E}_h^i}$ belong to $H_0^1(\Omega)$. Consider any edge $E \in \mathcal{E}_h^i$ and let E_i, $i \in \{1,\dots,4\}$, be one of the edges used in (30). Since ψ_E satisfies the patch test of order 3, we have for any $q \in L^2(E_i)$

$$\begin{aligned}\int_{E_i} [|\psi_E|]_{E_i}\, q\,\mathrm{d}\sigma &= \int_{E_i} [|\psi_E|]_{E_i}\,(q - \mathcal{M}_{E_i}^1\, q)\,\mathrm{d}\sigma = \int_{E_i} [|\zeta_E + \beta_{E,E_i}\, b_{K,E_i}|]_{E_i}\,(q - \mathcal{M}_{E_i}^1\, q)\,\mathrm{d}\sigma \\ &= \int_{E_i} [|\beta_{E,E_i}\, b_{K,E_i}|]_{E_i}\,(q - \mathcal{M}_{E_i}^1\, q)\,\mathrm{d}\sigma\,.\end{aligned}$$

Since

$$\int_{E_i} \{b_{K,E_i}|_K + 6\,\gamma\,(1-2\,\lambda_{E_i,1})\}\; v\,\mathrm{d}\sigma = 0 \qquad \forall\ v \in P_1(E_i)\,,$$

we have

$$\int_{E_i} b_{K,E_i}|_K\, \mathcal{M}^1_{E_i}\, q\,\mathrm{d}\sigma = -6\,\gamma \int_{E_i} (1-2\,\lambda_{E_i,1})\,\mathcal{M}^1_{E_i}\, q\,\mathrm{d}\sigma = -6\,\gamma \int_{E_i} (1-2\,\lambda_{E_i,1})\, q\,\mathrm{d}\sigma$$

and hence

$$\int_{E_i} [|\psi_E|]_{E_i}\, q\,\mathrm{d}\sigma = \pm\beta_{E,E_i} \int_{E_i} \{b_{K,E_i}|_K + 6\,\gamma\,(1-2\,\lambda_{E_i,1})\}\; q\,\mathrm{d}\sigma\,,$$

where the sign depends on whether the normal vector $\boldsymbol{n}_{E_i}$ is outward or inward with respect to K. Consequently, if the function $\widehat{b}$ satisfies

$$\int_{\widehat{E}} \left\{\widehat{b} + 6\,\gamma\,(1-2\,\widehat{\lambda}_1)\right\} q\,\mathrm{d}\widehat{\sigma} = 0 \qquad \forall\ q \in P_k(\widehat{E})\,, \tag{35}$$

we have

$$\int_E [|v_h|]_E\, q\,\mathrm{d}\sigma = 0 \qquad \forall\ v_h \in \mathrm{V}_h^{mod},\ q \in P_k(E),\ E \in \mathcal{E}_h\,,$$

i.e., the space V_h^{mod} satisfies the patch test of order $k+1$. For any integer k we can find a function $\widehat{b}$ satisfying (25)–(27) and (35). As we know, the function $\widehat{b}$ defined by (34) satisfies these conditions with $k=2$. Another example is the function

$$\widehat{b} = (28\,\widehat{\lambda}_1\,\widehat{\lambda}_2 - 126\,\widehat{\lambda}_1^2\,\widehat{\lambda}_2^2)(\widehat{\lambda}_1 - \widehat{\lambda}_2)\,, \tag{36}$$

for which the mentioned conditions hold with $k=4$.

Now, enriching a P_1^{mod} space satisfying the patch test of a given order $k+1$ by suitable continuous piecewise quadratic functions associated with edges, we obtain a nonconforming space satisfying again the patch test of order $k+1$ and containing all continuous piecewise quadratic functions, i.e., a space of second order of accuracy. We shall call the corresponding finite element the P_2^{mod} element. Similarly, enriching the P_2^{mod} space by suitable continuous piecewise cubic functions, we get a nonconforming space containing all continuous piecewise cubic functions and defining the P_3^{mod} element. In this way we can proceed further and construct a family of P_n^{mod} elements. Since, for any of these elements, we can enforce an arbitrarily high order of the patch test, we can always recover the optimal convergence rates known from the conforming finite element method when we use these elements in the discrete problem (10). For more details concerning the P_n^{mod} elements, we refer to [18].

8 Validity of the inf–sup condition for the P_n^{mod} elements

In this section we discuss for which pressure spaces Q_h the P_n^{mod} element satisfies the inf–sup condition (6) with a positive constant β independent of h. First let us look in the literature whether some known results can be applied to the P_n^{mod} element. We can employ the fact that all continuous piecewise polynomial functions of degree n are contained in a P_n^{mod} space. This does not imply any inf–sup condition for the P_1^{mod} element, but for $n > 1$ we can use

$$\mathrm{Q}_h = \mathrm{Q}_h^{n-1,cont} \equiv \{q_h \in C(\overline{\Omega}) \cap L_0^2(\Omega)\,;\ q_h|_K \in P_{n-1}(K) \quad \forall\, K \in \mathcal{T}_h\}$$

and for $n > 3$ even

$$\mathrm{Q}_h = \mathrm{Q}_h^{n-1,disc} \equiv \{q_h \in L_0^2(\Omega)\,;\ q_h|_K \in P_{n-1}(K) \quad \forall\, K \in \mathcal{T}_h\}\,.$$

This follows from the theory of Taylor–Hood elements (cf. [1], [28], [25], [3]) and the results of [23] which hold under additional assumptions on the triangulations.

However, most of these results can be improved. Let us start with the P_1^{mod} element which was obtained by modifying the nonconforming piecewise linear Crouzeix–Raviart element. Here it is not surprising that the inf–sup condition holds with the space $\mathrm{Q}_h^{0,disc}$, which can be proved using analogous techniques as in [6]. However, using similar techniques as in [1] and [28], one can also show that the inf–sup condition holds with the space $\mathrm{Q}_h^{1,cont}$. Finally, applying the macroelement technique (cf. [2] and [24]) using macroelements consisting of elements sharing one vertex, we can even derive (cf. [17]) that the inf–sup condition holds for the space $\mathrm{Q}_h^{1,disc}$. The last two results were proved under the assumption that

(37) *any element $K \in \mathcal{T}_h$ has at least one vertex in Ω*

and that

$$\int_{\widehat{K}} \widehat{b}\,\mathrm{d}\widehat{x} = 0\,, \tag{38}$$

which is satisfied for the examples of $\widehat{b}$ given in (34) and (36). Of course, all the results available for the P_1^{mod} element also hold for any P_n^{mod} element. But for the P_n^{mod} elements with $n > 1$ one can prove further results. Particularly, the inf–sup condition holds for the P_2^{mod} element with the space $\mathrm{Q}_h^{2,cont}$ and for the P_3^{mod} element with the space $\mathrm{Q}_h^{3,cont}$. The proves are again based on the macroelement technique and use the assumptions (37) and (38). The situation with discontinuous pressure spaces it not clear, but it was shown that on the mentioned macroelement one cannot use the P_2^{mod} element with discontinuous piecewise quadratic pressures.

Let us remark that an optimal convergence rate with respect to $h \to 0$ is guaranteed when we use the P_n^{mod} element together with a pressure space Q_h of piecewise polynomial degree $n-1$. However, it may often pay to use pressure spaces of a higher degree than $n-1$ since a better pressure space can help to increase the accuracy of the discrete solution if the viscosity ν is small. This will be clearly seen from the numerical results presented in the next section.

9 Application of the P_1^{mod} element to the numerical solution of the Stokes equations

In this section we concentrate on effects connected with the incompressibility constraint in the Navier–Stokes equations and therefore, we simplify the problem (1)–(3) to the stationary Stokes equations

$$-\nu\,\Delta\,\boldsymbol{u} + \nabla p = \boldsymbol{f}\,, \qquad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega\,, \qquad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega\,. \tag{39}$$

Assuming that $\boldsymbol{f} \in L^2(\Omega)^2$ and denoting

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x\,, \quad b(\boldsymbol{v},p) = -\int_\Omega p\operatorname{div}\boldsymbol{v}\,\mathrm{d}x\,, \quad (\boldsymbol{f},\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x\,,$$

the usual weak formulation of (39) reads: Find $\boldsymbol{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega)$ such that

$$\nu\,a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) - b(\boldsymbol{u},q) = (\boldsymbol{f},\boldsymbol{v}) \qquad \forall\ \boldsymbol{v} \in H_0^1(\Omega)^2,\ q \in L_0^2(\Omega)\,.$$

It can be shown that this problem has a unique solution (cf. [10, p. 80, Theorem 5.1]).

We will approximate the space $H_0^1(\Omega)^2$ in the weak formulation by a first order nonconforming finite element space $\mathbf{V}_h$ satisfying the patch test of the order $m \geq 1$. The space $L_0^2(\Omega)$ will be approximated by a general finite element space $\mathrm{Q}_h \subset L_0^2(\Omega)$ possessing the approximation property

$$\lim_{h\to 0}\ \inf_{q_h\in \mathrm{Q}_h} \|q - q_h\|_{0,\Omega} = 0 \qquad \forall\ q \in L_0^2(\Omega)\,.$$

Since the functions from $\mathbf{V}_h$ are only piecewise in $H^1(\Omega)$, we have to replace the bilinear form a by its 'piecewise' counterpart

$$a_h(\boldsymbol{u},\boldsymbol{v}) = \sum_{K\in\mathcal{T}_h} \int_K \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x$$

and the bilinear form b by the bilinear form b_h defined in Section 1. Then a finite element discretization of the Stokes equations (39) reads: Find $\boldsymbol{u}_h \in \mathbf{V}_h$ and $p_h \in \mathrm{Q}_h$ such that

$$\nu\,a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) + b_h(\boldsymbol{v}_h,p_h) - b_h(\boldsymbol{u}_h,q_h) = (\boldsymbol{f},\boldsymbol{v}_h) \qquad \forall\ \boldsymbol{v}_h \in \mathbf{V}_h,\ q_h \in \mathrm{Q}_h\,. \tag{40}$$

Using the techniques of [6] and [10], we obtain the following result:

Theorem 3 *Let the spaces $\mathbf{V}_h$ and Q_h satisfy the inf–sup condition (6) with $\beta > 0$ independent of h. Then the problem (40) has a unique solution $\boldsymbol{u}_h$, p_h and we have*

$$\lim_{h\to 0}\{|\boldsymbol{u}-\boldsymbol{u}_h|_{1,h} + \|p-p_h\|_{0,\Omega}\} = 0\,,$$

where $\boldsymbol{u}$, p is the weak solution of (39). Moreover, if $\boldsymbol{u} \in H^2(\Omega)^2$ and $p \in H^k(\Omega)$, $k \in \{1,\ldots,m\}$, then

$$|\boldsymbol{u}-\boldsymbol{u}_h|_{1,h} \leq C\,h\,|\boldsymbol{u}|_{2,\Omega} + \frac{C}{\nu}\,h^k|p|_{k,\Omega} + \frac{\sqrt{2}}{\nu}\inf_{q_h\in\mathrm{Q}_h}\|p-q_h\|_{0,\Omega}\,,$$

$$\|p-p_h\|_{0,\Omega} \leq \nu\,C\,h\,|\boldsymbol{u}|_{2,\Omega} + C\,h^k|p|_{k,\Omega} + C\inf_{q_h\in\mathrm{Q}_h}\|p-q_h\|_{0,\Omega}\,.$$

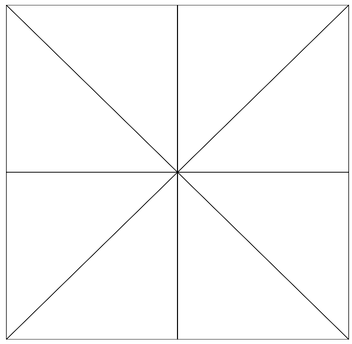

Figure 3: Coarse triangulation.

If we use $\mathbf{V}_h = [\mathrm{V}_h^{nc}]^2$ or $\mathbf{V}_h = [\mathrm{V}_h^{mod}]^2$, where the space V_h^{mod} is defined using the P_1^{mod} element, then the inf–sup condition mentioned in Theorem 3 holds for $\mathrm{Q}_h = \mathrm{Q}_h^{0,disc}$. It is well known (cf. e.g. [10, pp. 102 and 126]) that

$$\inf_{q_h \in \mathrm{Q}_h^{0,disc}} \|p - q_h\|_{0,\Omega} \leq C\, h\, |p|_{1,\Omega} \qquad \forall\, p \in H^1(\Omega) \cap L_0^2(\Omega)$$

and hence the choice $\mathrm{Q}_h = \mathrm{Q}_h^{0,disc}$ is optimal with respect to the convergence order of the discrete solution. However, since the parameter ν is small in typical applications, the pressure parts of the estimates of Theorem 3 will often dominate the velocity parts and the choice $\mathrm{Q}_h = \mathrm{Q}_h^{0,disc}$ will cause that the relation of these parts will not change significantly for $h \to 0$. Therefore, if $\mathbf{V}_h = [\mathrm{V}_h^{mod}]^2$, it may also be reasonable to require that

$$\inf_{q_h \in \mathrm{Q}_h} \|p - q_h\|_{0,\Omega} \leq C\, h^k\, |p|_{k,\Omega} \qquad \forall\, p \in H^k(\Omega) \cap L_0^2(\Omega),\ k = 1, 2\,.$$

This is satisfied if $\mathrm{Q}_h \supset \mathrm{Q}_h^{1,cont}$, which includes the cases $\mathrm{Q}_h = \mathrm{V}_h^{nc} \cap L_0^2(\Omega)$ and $\mathrm{Q}_h = \mathrm{Q}_h^{1,disc}$. In all these cases, as we know from the previous section, the inf–sup condition (6) holds provided the assumptions (37) and (38) are satisfied. Hence, if $\boldsymbol{u} \in H^2(\Omega)^2$ and $p \in H^2(\Omega)$, we have

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h} \leq C\, h\, |\boldsymbol{u}|_{2,\Omega} + \frac{C}{\nu}\, h^2\, |p|_{2,\Omega}\,, \qquad \|p - p_h\|_{0,\Omega} \leq \nu\, C\, h\, |\boldsymbol{u}|_{2,\Omega} + C\, h^2 |p|_{2,\Omega}\,.$$

Numerical experiments show that, in this way, we can often significantly improve the accuracy of the discrete solution in comparison to the choice $\mathrm{Q}_h = \mathrm{Q}_h^{0,disc}$. Let us demonstrate it on the following example.

Example 2 Let $\Omega = (0,1)^2$ and $\psi(x,y) = x^2\,(1-x)^2\,y^2\,(1-y)^2$. For a given $\nu > 0$, the right–hand side $\boldsymbol{f}$ of the problem (39) is chosen such that the functions

$$\boldsymbol{u}(x,y) = \left(\frac{\partial \psi(x,y)}{\partial y}, -\frac{\partial \psi(x,y)}{\partial x} \right), \qquad p(x,y) = x^3 + y^3 - \frac{1}{2}$$

are the exact solution of (39), see Fig. 4.

We considered the discrete problem (40) on a triangulation obtain by five uniform refinements of the coarse triangulation shown in Fig. 3 so that the resulting mesh consists of 65×65 points. The discrete spaces were defined using the P_1^{nc} element ($\mathbf{V}_h = [\mathrm{V}_h^{nc}]^2$), the P_1^{mod} element described at the end of Section 5 ($\mathbf{V}_h = [\mathrm{V}_h^{mod}]^2$), the P_0 element ($\mathrm{Q}_h = \mathrm{Q}_h^{0,disc}$) and the P_1^{disc} element ($\mathrm{Q}_h = \mathrm{Q}_h^{1,disc}$). The discrete problem was solved for $\nu = 1$, 0.01 and 0.001 and the errors of the discrete solutions are given in Tables 4–6. We see that the velocity error is considerably growing for decreasing ν if piecewise constant pressures are used whereas the discretization with piecewise linear pressure gives velocity errors almost independent of ν in the considered range of ν. Moreover, the pressure approximation is several orders better for piecewise linear pressures than for piecewise constant pressures. The discrete velocity fields corresponding to $\nu = 0.01$ are depicted in Fig. 5–7 and we see that, for the Crouzeix–Raviart element P_1^{nc}, the discrete velocity is completely wrong. This is caused by the fact that the influence of small ν is not compensated by a sufficiently good approximation properties of the discrete spaces as we mentioned above. For the P_1^{mod} element with piecewise constant pressure the discrete velocity is better but still far away from the exact solution. The improvement is caused by the fact that the P_1^{mod} element leads to a smaller consistency error since it satisfies the patch test of a higher order than the Crouzeix–Raviart element. Finally, for the P_1^{mod} element with discontinuous piecewise linear pressure, the space Q_h is sufficiently good to compensate the negative influence of ν and we see that the discrete velocity is in a very good agreement with the exact solution.

Table 4: Example 2, errors $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}$.

ν	P_1^{nc}/P_0	P_1^{mod}/P_0	P_1^{mod}/P_1^{disc}
1	7.23−5	1.86−5	1.78−5
0.01	7.19−3	1.27−3	1.78−5
0.001	7.19−2	1.27−2	1.79−5

Table 5: Example 2, errors $|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h}$.

ν	P_1^{nc}/P_0	P_1^{mod}/P_0	P_1^{mod}/P_1^{disc}
1	9.57−3	2.51−3	2.05−3
0.01	9.36−1	1.72−1	2.05−3
0.001	9.36+0	1.72+0	2.12−3

Table 6: Example 2, errors $\|p - p_h\|_{0,\Omega}$.

ν	P_1^{nc}/P_0	P_1^{mod}/P_0	P_1^{mod}/P_1^{disc}
1	7.70−3	7.78−3	3.19−5
0.01	7.67−3	7.53−3	3.38−7
0.001	7.67−3	7.53−3	1.65−7

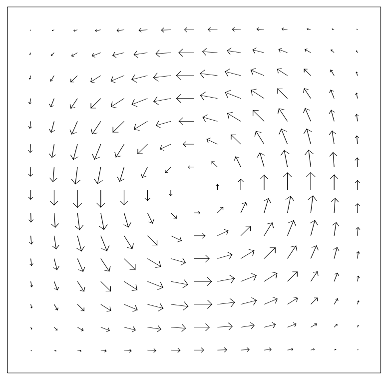

Figure 4: Exact solution $\boldsymbol{u}$.

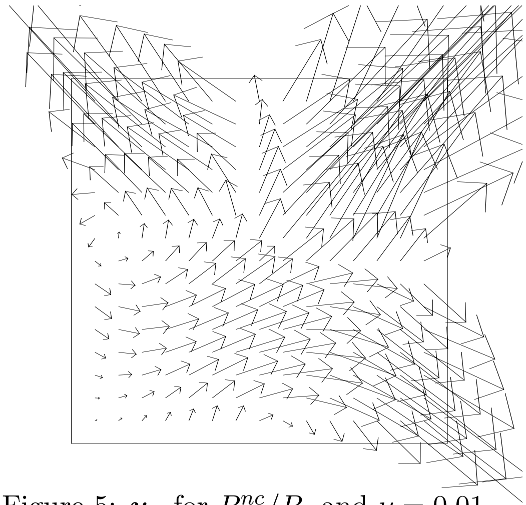

Figure 5: $\boldsymbol{u}_h$ for P_1^{nc}/P_0 and $\nu = 0.01$.

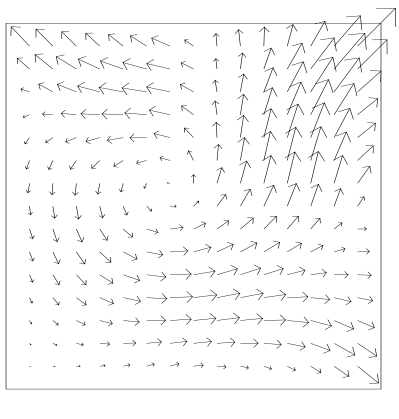

Figure 6: $\boldsymbol{u}_h$ for P_1^{mod}/P_0 and $\nu = 0.01$.

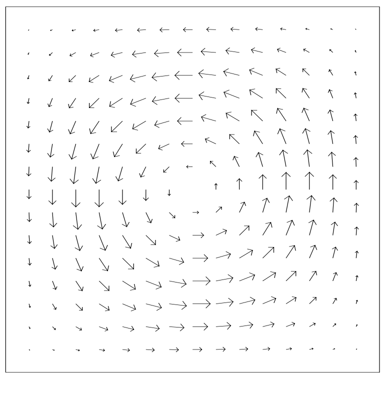

Figure 7: $\boldsymbol{u}_h$ for P_1^{mod}/P_1^{disc} and $\nu = 0.01$.

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