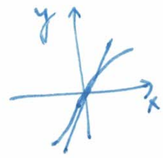
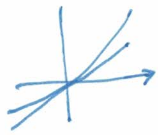
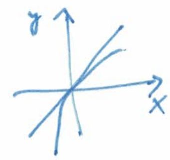
6. hodina

$\lim\limits_{x\to0}\frac{\sin x}{x}=1$ $\qquad$ $\lim\limits_{x\to0}\frac{e^x-1}{x}=1$ $\qquad$ $\lim\limits_{x\to0}\frac{\ln(1+x)}{x}\left(=\lim\limits_{y\to0^+}\frac{\ln y}{y-1}\right)=1$

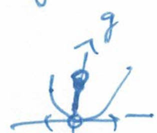
  

$$\lim_{x\to a} f(x)^{g(x)} = e^{\lim\limits_{x\to a} g(x)\ln f(x)}$$

(1) $\lim\limits_{x\to0}\frac{\sqrt{1-\cos x^2}}{1-\cos x} = \lim\limits_{x\to0}\sqrt{\frac{1-\cos x^2}{x^4}}\;\frac{x^2}{1-\cos x} \overset{\text{VOAL + VOLSF}}{=} \sqrt{\frac12}\;\frac12$, kde jsme

použili $\lim\limits_{x\to0}\frac{1-\cos x}{x^2} = \lim\frac{(1-\cos x)(1+\cos x)}{(1+\cos x)\,x^2} = \lim\limits_{x\to0}\frac{\sin^2 x}{x^2}\;\frac{1}{1+\cos x} = \frac12$

VOLSF - podmínky $f(y)=\frac{1-\cos y}{y^2}$ $\quad g(x)=x^2$: 1. $\lim\limits_{x\to0} g(x)=0$
2. $\lim\limits_{y\to0} f(y)=\frac12$ viz výše

+3. $g(x)\neq0$ na $P_\delta(0)$ (dokonce $\forall\delta>0$):  . (Lze použít i 2. VOLSF, víme-li, že f je spojitá ... f ale takto def. není ani v $y=0$ definovaná, musel bych dodefinovat ... snadné $f(0):=\frac12$.)

Dále VOLSF ještě na $f(y)=\sqrt y$, $y\ge0$, $g(x)=\frac{1-\cos x^2}{x^4}$:

1. $\lim\limits_{x\to0} g(x)=\frac12$, 2. $\lim\limits_{y\to\frac12} f(y)=\sqrt{\frac12}$, 3. $g(x)\neq\frac12$? ... Raději 2.

VOLSF : $\lim\limits_{x\to0} g(x)=\frac12$, $\underline{f \text{ spojitá}}$ $\Rightarrow$ $\lim\limits_{x\to0}(f\circ g)(x) = f\Big(\underset{\lim\limits_{x\to0} g}{\overset{\parallel}{\tfrac12}}\Big)$.

• $\lim\frac{1-\cos x}{x^2}=\frac12$ do paměti či libitě :-) odvodit.

jak $\frac{\sin x}{x}$ či $\frac{1-\cos x}{x^2}$?

(2) $\lim\limits_{x\to0}\frac{\operatorname{tg}x-\sin x}{x^3} = \lim\limits_{x\to0}\boxed{\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}\cdot\frac{\cos x}{\cos x}} = \lim\limits_{x\to0}\frac{\sin x-\sin x\cos x}{x^3} =$

$$= \lim_{x\to 0} \frac{\sin x}{x}\,\frac{(1-\cos x)}{x^2} \overset{\text{VOAL}}{=} 1\cdot\frac{1}{2} = \frac{1}{2} \quad \text{(opět dal. limita).}$$

$$\text{③}\ \lim_{x\to 0}\frac{\sin nx}{\sin mx} = \lim_{x\to 0}\frac{\sin(\overbrace{nx}^{=g(x)})}{\underbrace{nx}_{=g(x)}}\,\frac{mx}{\sin mx}\,\frac{n}{m} \overset{\substack{\text{VOAL}\\ \text{VOLSF}}}{=} \left|\begin{array}{l}\boxed{y_1 = nx}\\ y_2 = mx\end{array}\right| = \frac{n}{m}\lim_{y_1\to 0}\frac{\sin y_1}{y_1}\cdot\lim_{y_2\to 0}\frac{y_2}{\sin y_2} = \frac{n}{m} \qquad \forall m,n\in\mathbb{N}$$

g 0

$$\text{④}\ \lim_{x\to\pi}\frac{\sin nx}{\sin mx}\ \left|\begin{array}{l} y = x-\pi\\ \text{VOLSF ok:}\\ y \text{ je injektivní}\end{array}\right| = \lim_{y\to 0}\frac{\sin(n(y+\pi))}{\sin(m(y+\pi))} =$$

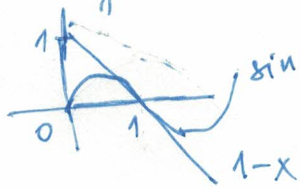
$$= \left|\begin{array}{l}\sin(ny+n\pi) = \sin ny\cos n\pi + \sin n\pi\cos ny = \\ \qquad = \sin ny\,(-1)^n\end{array}\right| =$$

$$= \lim_{y\to 0}\frac{\sin ny\,(-1)^n}{\sin my\,(-1)^m} = (-1)^{n-m}\,\frac{n}{m}$$

dle předch. limity

$$\text{⑤}\ \lim_{x\to 1}\frac{\sin \pi x}{1-x} = \left| y = x-1 \right| = \lim_{y\to 0}\frac{\sin(\pi y+\pi)}{-y} = \lim_{y\to 0}\frac{-\sin\pi y}{-\pi y}\,\pi =$$

$$= \left|\begin{array}{l} z = \pi y\\ \text{předp.}\\ \text{VOLSF}(1)\end{array}\right| = \lim_{z\to 0}\frac{\sin z}{z}\,\pi = \pi$$



$$\text{⑥}\ \lim_{x\to 0}\frac{\ln(1+\overbrace{xe^x}^{\approx 1})}{\underbrace{\ln(x+\sqrt{1+x^2})}_{\approx 1}} = \lim_{x\to 0}\frac{\ln(1+xe^x)}{xe^x}\,\frac{xe^x}{\ln(x+\sqrt{1+x^2})}\,\frac{(x+\sqrt{1+x^2}-1)}{(x+\sqrt{1+x^2}-1)}$$

$$\ln(x+\sqrt{1+x^2}) = \ln(1+(x+\sqrt{1+x^2}-1))$$

$$\overset{\text{VOLSF}}{=} \lim_{x\to 0}\left(\frac{\ln(1+xe^x)}{xe^x}\right)\frac{\lim\limits_{x\to 0}(x+\sqrt{1+x^2}-1)}{\ln(x+\sqrt{1+x^2})}\cdot\lim_{x\to 0}\frac{xe^x}{x+\sqrt{1+x^2}-1} = 1\cdot 1.$$

3

$\cdot \lim\limits_{x\to 0} \dfrac{x e^x}{x+\sqrt{1+x^2}-1} = \lim\limits_{x\to 0} \dfrac{\dfrac{xe^x - x + x}{x}\cdot x}{x+\sqrt{1+x^2}-1} =$

kde / jak $\dfrac{e^x-1}{x}$

opet: pokud limity napravo existují

$= \lim\limits_{x\to 0} \dfrac{\left[x\left(\dfrac{e^x-1}{x}\right)+\dfrac{x}{\cancel{x}}\right]\cdot x}{x+\sqrt{1+x^2}-1} \overset{!}{=} \lim\limits_{x\to 0}\left[x\left(\dfrac{e^x-1}{x}\right)+\dfrac{x}{\cancel{x}}\right] \lim\limits_{x\to 0}$

$\dfrac{x}{x+\sqrt{1+x^2}-1}\cdot\dfrac{x+\sqrt{1+x^2}+1}{x+\sqrt{1+x^2}+1} = \cancel{0}\cdot \lim\limits_{x\to 0}\dfrac{x^2+x\sqrt{1+x^2}+x}{x+1+x^2+2x\sqrt{1+x^2}-1} =$

$= \cancel{0}\cdot\lim\limits_{x\to 0}\dfrac{x^2+x\sqrt{1+x^2}+x}{x^2+x^2+2x\sqrt{1+x^2}} = 0\cdot\lim\limits_{x\to 0}\dfrac{x+\sqrt{1+x^2}+1}{\cancel{x}+x+2\sqrt{1+x^2}} = \cancel{0}\cdot\dfrac{2}{2} = \cancel{0}.$

(7) $\lim\limits_{x\to 0}\dfrac{\ln(\cos ax)}{\ln(\cos bx)}$, $b\neq 0$, $\cos ax \approx 1$

$= \lim\limits_{x\to 0}\dfrac{\ln(\cos ax - 1 + 1)}{\ln(\cos bx - 1 + 1)} \overset{!}{=} \lim\limits_{x\to 0}\dfrac{\ln(\cos ax - 1 + 1)}{\cos ax - 1}$

$\cdot \lim\limits_{x\to 0}\dfrac{\cos bx - 1}{\ln(\cos bx - 1 + 1)}\cdot\lim\limits_{x\to 0}\dfrac{\cos ax - 1}{\cos bx - 1} = 1\cdot 1\cdot \boxed{\lim\limits_{x\to 0}\dfrac{\cos ax - 1}{(ax)^2}}$

VOLSF

$\lim\limits_{x\to 0}\dfrac{(bx)^2}{\cos bx - 1}\cdot\lim\limits_{x\to 0}\dfrac{(ax)^2}{(bx)^2} = -\dfrac{1}{2}\cdot(-2)\dfrac{a^2}{b^2} = \dfrac{a^2}{b^2}$ ($b\neq 0$ ok)

(8) $\lim\limits_{x\to 0^+}(1+x)^{\frac{1}{x}} = \lim\limits_{x\to 0^+} e^{\ln(1+x)\frac{1}{x}} = e^{\lim\limits_{x\to 0^+}\frac{\ln(1+x)}{x}} = e^1 = e$

(9) $\lim\limits_{x\to 0}\left(\dfrac{1+\operatorname{tg} x}{1+\sin x}\right)^{\frac{1}{\sin^3 x}} \Rightarrow \lim\limits_{x\to 0}\dfrac{1}{\sin^3 x}\ln\left(\dfrac{1+\operatorname{tg} x}{1+\sin x}\right).$

$$\lim_{x\to 0}\left[\ln \underbrace{\frac{1+\operatorname{tg} x}{1+\sin x}}_{\to 1}\ \frac{1}{\sin^3 x}\right] = \lim_{x\to 0} \frac{\ln\left(\frac{1+\operatorname{tg} x}{1+\sin x}\right)}{\frac{1+\operatorname{tg} x}{1+\sin x}-1}\left(\frac{1+\operatorname{tg} x}{1+\sin x}-1\right)\frac{1}{\sin^3 x}$$

$$\overset{!}{=} \lim_{x\to 0}\left(\frac{1+\operatorname{tg} x}{1+\sin x}-1\right)\frac{1}{\sin^3 x} = \lim_{x\to 0}\left(\frac{1+\operatorname{tg} x-1-\sin x}{1+\sin x}\ \frac{1}{\sin^3 x}\right) =$$

$$= \lim_{x\to 0}\frac{\operatorname{tg} x-\sin x}{1+\sin x}\ \frac{1}{\sin^3 x} \overset{\pm\text{bylo...}}{=} \lim_{x\to 0}\left(\frac{\operatorname{tg} x-\sin x}{x^3}\ \frac{x^3}{\sin^3 x}\ \frac{1}{1+\sin x}\right)$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^3 \frac{1}{1} = \frac{1}{2} \Rightarrow \lim \dots = e^{1/2}$$

(10) $$\lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x}{1-\cos x} = \lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x}{x^2}\ \frac{x^2}{1-\cos x} \overset{!}{=}$$

$$= 2\lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x}{x^2} = 2\lim_{x\to 0}\frac{1-\cos x+\cos x(1-\cos 2x\cos 3x)}{x^2} =$$

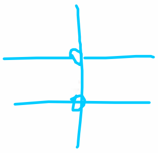
$$\overset{!}{=} 2\left[\lim_{x\to 0}\frac{1-\cos x}{x^2} + \cos 0\lim_{x\to 0}\frac{1-\cos 2x\cos 3x}{x^2}\right] = 2\left[\frac{1}{2}+\lim_{x\to 0}\frac{1-\cos 2x+\cos 2x(1-\cos 3x)}{x^2}\right]$$

VOL SF $y=2x$

$$\overset{!}{=} 2\left[\frac{1}{2}+\lim_{x\to 0}\frac{1-\cos 2x}{4x^2}\,4 + \cos 0\lim_{x\to 0}\frac{1-\cos 3x}{9x^2}\cdot 9\right] =$$

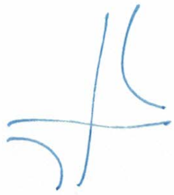
$$= 2\left(\frac{1}{2}+\frac{1}{2}\cdot 4+\frac{1}{2}\cdot 9\right) = 2\,\frac{1+4+9}{2} = 14$$

Pozn.: Podmíněnost rovností:



$$\lim_{x\to 0}\frac{x}{x} \overset{\text{Lemma}}{=} \lim_{x\to 0}\underline{\left(\frac{x}{x}+\frac{1}{x}-\frac{1}{x}\right)} \overset{\text{VOAL}}{=} \lim_{x\to 0}\left(\frac{x}{x}+\frac{1}{x}\right)-\lim_{x\to 0}\frac{1}{x} = ?$$

neexistuje !!



$\lim_{x\to 0}\frac{1}{x}$ neex $\left(\lim_{x\to 0}\left(\frac{x}{x}+\frac{1}{x}\right) \text{ taky}\right)$

Odtud neplyne, že $\lim_{x\to 0}\frac{x}{x}$ neexistuje. VOAL říká: Pokud limity napravo $\exists$, existuje lim. nalevo.

Měli bychom psát $\lim_{x\to 0}\left(\frac{x}{x}+\frac{1}{x}-\frac{1}{x}\right) \overset{(!)}{\underset{\text{VOAL}}{=}} \lim_{x\to 0}\left(\frac{x}{x}+\frac{1}{x}\right)+\lim_{x\to 0}\frac{1}{x}$.

Pozn.: Dále: neplatnost část. dosazení:

$$\lim_{x\to 0^+}\underline{(1+x)^{\frac{1}{x}}} = \lim_{x\to 0^+} 1^{\frac{1}{x}} \overset{\text{lemma}}{=} \lim_{x\to 0^+} 1 = 1.$$

$1^{\text{cokoliv}} = 1$

Chyba je zde: na to nemám větu

Zkusme

$$\lim_{x\to 0^+} e^{\frac{1}{x}\ln(1+x)} = e^{\lim_{x\to 0}\frac{\ln(1+x)}{x}} = e^1.$$

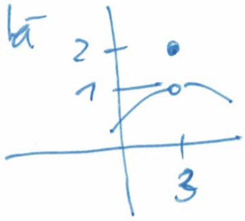
- VOLSF nejde: $f(y) := (1+y)^{\frac{1}{y}}$, $g(x) = 1+x$ ... nejde
- $(1+x)^{\frac{1}{x}} = e^{\ln(1+x)\frac{1}{x}}$: $\boxed{f(y)=e^y,\ g(x)=\frac{\ln(1+x)}{x}}$ To to

skládání jde

Nespojitost – spojitost

(A) Spojitost f v a $\Rightarrow$ f v a nutně def $\wedge$ $\boxed{\lim_{x\to a} f(x)} = f(a)$, mj. má lim

(B) Má lim $\not\Rightarrow$ spojitá



$\lim_{x\to 3}$ ex ... 1 $\lim_{x\to 3} f(x) \neq f(3)$

$f(3)$ ... 2

nespojitá (непрерывная, not continuous, unstetig)

Neexistence limity v a: 0.) $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$ nebo jedn. neexistuje

1. z def: $\forall A\ \exists \varepsilon>0\ \forall \delta>0\ \nexists x \in P_\delta(a)$ $f(x) \notin U_\varepsilon(A)$

(limita není A: $\exists \varepsilon>0\ \forall \delta>0\ \exists x \in P_\delta(a)\ f(x) \notin U_\varepsilon(A)$)

$[U_\varepsilon(A) := (A-\varepsilon, A+\varepsilon) = \{x \mid |x-A| < \varepsilon\}]$

Př.: $Dir(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$ není spojitá v žádném bodě (= v $\forall$ bodě je nespojitá) a nikde nemá limitu.

$x_0 \in \mathbb{R}$ a pro spor $\lim_{x\to x_0} Dir(x) = L$.

a) $L<0$ $|f(x)-L| < \varepsilon$? $|f(x)-L| \geq |L|$

Zvol $\varepsilon = |L|$ ... $\delta>0$ lib. $x \in P_\delta(x_0)$ lib.

$|f(x)-L| \begin{cases} |1-L| > |L|, \text{ tj. ne } \underbrace{|1-L| < |L|}_{x\in\mathbb{Q}}! \\ |L| \nless |L|, \ x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$

b) $L>1$ $|f(x)-L| \geq |L-1|$, obdobně zvol $\varepsilon = |L-1|$.

c) $L \in [0,1]$; $\varepsilon := \frac{1}{2}\max\{L, 1-L\}$ lib. $\delta>0$ $x \in P_\delta(x_0)$

1. $\varepsilon = \frac{1}{2}L$ $|f(x)-L| = L \nless \frac{1}{2}L$, x irac

2. $\varepsilon = \frac{1-L}{2}$ $|f(x)-L| = 1-L \nless \frac{1-L}{2}$ x rac

Používáme $\forall \delta>0$ $(a-\delta, a+\delta)$ obsahuje racionální i iracionální čísla

Ilustrace (snazší fakt): Není $\lim_{x \to 0} Dir(x) = 1$.

Dk.: $\varepsilon = \frac{1}{2}$. $|Dir(x) - 1| < \frac{1}{2}$ na $(-\delta, \delta)$?

x ~~i~~rac z $(-\delta, \delta)$: $|1-1| < \frac{1}{2}$ OK

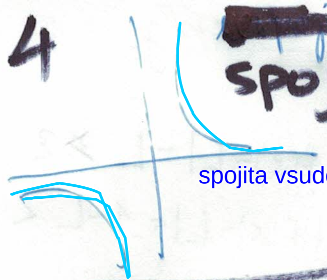
Zvol ale x irac z $(-\delta, \delta)$: $|0-1| \not< \frac{1}{2}$ Není OK.

Stačí tedy vzít lib. irac.

[Co máme dokázat? $\exists \underset{= \frac{1}{2}}{\varepsilon > 0}\ \forall \delta\ \exists x_1\ (x_1 \in (-\delta, \delta)$ irac$)$
$|Dir(x) - 1| \not< \frac{1}{2}$]

Vždy musím vědět:

4



[redacted]: $\frac{1}{x}$ spoj.

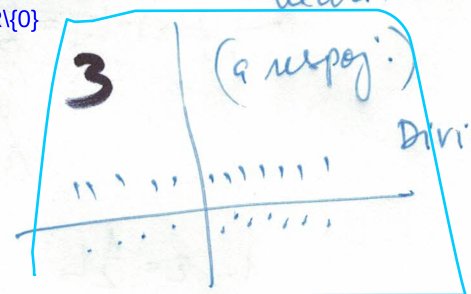
spojita vsude na def. oboru, ktery je R\{0}

1

nespoj. a nemá limi

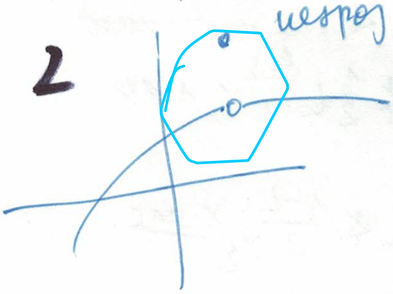
nemá limitu (a nespoj.)

3



Dirichlet

2



nespoj. a má lim.

Spojitost

1. $f(x)=\dfrac{1-\cos x}{x^2}$, $x\in\mathbb{R}\setminus\{0\}$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}, \quad f(0):=\frac{1}{2}$$

f je spojitá: $\underline{\lim_{x\to 0} f(x)} = \frac{1}{2} = \underline{f(0)}$.

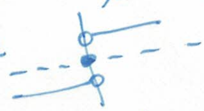
2. a) $f(x)=e^{-\frac{1}{x}}$, $D_f=\mathbb{R}\setminus\{0\}$. f je spojitá na svém def. oboru: $\frac{1}{x}$ spoj. a exp. spoj.

$\frac{1}{x}$  spojitá!

Lze ji spoj. dodefinovat v nule? $\left(\lim_{x\to 0^+} e^{-\frac{1}{x}} = e^{-\lim_{x\to 0^+}\frac{1}{x}} = 0\right.$

$\lim_{x\to 0^-} e^{-\frac{1}{x}} = +\infty$ ← to ještě ale neumíme)

b) $f(x)=\operatorname{sgn}\cos\frac{1}{x}$, $\frac{1}{x}$ spojitá na $\mathbb{R}\setminus\{0\}$, cos také.

sgn ale ne . Takže nevíme. Zkusíme dok. nespojitost.

$$\cos\frac{1}{x}=0 \iff \frac{1}{x}=\frac{\pi}{2}+k\pi \iff x=\frac{1}{\frac{\pi}{2}+k\pi}=\frac{2}{\pi+2k\pi}=\frac{2}{\pi}\frac{1}{2k+1}$$

$x=\frac{2}{\pi}$ $\quad \cos\frac{\pi}{2}=0$ $\quad \cos>0$ na $(\frac{\pi}{2}-\delta,\frac{\pi}{2})$ $\quad \operatorname{sgn}\cos = 1$

$\cos<0$ na $(\frac{\pi}{2},\frac{\pi}{2}+\delta)$ $\quad \operatorname{sgn}\cos = -1$

Není spoj. v $\frac{2}{\pi}$. Obdobně v $\frac{2}{\pi}\frac{1}{2k+1}$

3. $f(x)=\operatorname{sgn} x$, $g(x)=x(1-x^2)$

$f(g(x))=?$

	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
g	+	−	+	−
sgn g	1	−1	1	−1

není spojitá.

$g(f(x))=?$

	$(-\infty,0)$	0	$(0,\infty)$
sgn	−1	0	1
$g(f(x))=$	0	0	0

$\Longrightarrow$ je spojitá.