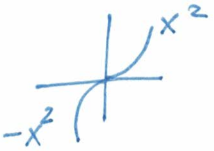
$\boxed{7.\ \text{hodina}}$

Spočtěte derivace (z definice)

1. $f(x) = x|x|$



$$f'(\underline{0}) = \lim_{h\to 0} \frac{(0+h)|0+h| - f(0)}{h} = \lim_{h\to 0} |h| \overset{VOLSF}{=}$$

$$\underset{=}{a)}\ \left|\lim_{h\to 0} h\right| = |0|$$

$$\underset{=}{b)}\ \lim_{h\to 0^+} h = 0 \quad \left(= \lim_{h\to 0} h = 0, \text{ pokud } \exists\right)$$

$$\lim_{h\to 0^-} -h = -0 = 0 \quad \left(= \lim_{h\to 0} -h = -0, \text{ pokud } \exists\right)$$

$$c)\ f'_+(0) = \lim_{h\to 0^+} \frac{(0+h)|0+h| - f(0)}{h} = \lim_{h\to 0^+} h \overset{[\text{neb } h>0}{=} \lim_{h\to 0} h = 0 \quad \leftarrow \text{if } \exists\,]$$

$$f'_-(0) = \lim_{h\to 0^-} \frac{(0+h)(-h) - 0}{h} = \lim_{h\to 0^+} (-h) = -\lim_{h\to 0} h = 0$$

$\nwarrow$ if $\exists$) $\longrightarrow$ další strana

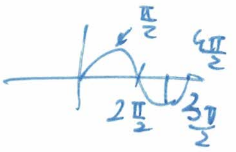
*) Viz také s. 1 $\boxed{\%}$ (vůbec o spojitosti této fce).

2. $$f(x) = \begin{cases} x \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f'(0) = \lim_{h\to 0} \frac{h \sin\frac{1}{h} - f(0)}{h} = \lim_{h\to 0} \sin\frac{1}{h}$$

Neexistuje $h = \frac{2}{\pi(2n+1)}$ $\sin\left(\frac{\pi(2n+1)}{2}\right) = (-1)^n$

$h = \frac{2}{\pi 2n}$ $\sin\left(\frac{\pi 2n}{2}\right) = 0$



$\alpha > 1$

3. $$f(x) = \begin{cases} x^\alpha \sin\frac{1}{x} &, x \neq 0 \\ 0 &, x = 0 \end{cases}$$

$$f'(0) = \lim_{h\to 0} \frac{h^\alpha \sin\frac{1}{h}}{h} = \lim_{h\to 0} h^{\alpha-1} \sin\frac{1}{h}$$

Lemma o policistech:

$$0 \leq \left|h^{\alpha-1} \sin\frac{1}{h}\right| \leq |h^{\alpha-1} \cdot 1|$$

$$\begin{array}{ccc} \downarrow & & \downarrow \ \alpha > 1 \\ 0 & & 0 \end{array}$$

$$\Rightarrow \lim_{h\to 0} \left|h^{\alpha-1} \sin\frac{1}{h}\right| \ \underline{\underline{\exists \overset{\&}{=} 0}} \Longrightarrow$$

$$\overset{\alpha>1}{} \lim_{h\to 0} \left(h^{\alpha-1} \sin\frac{1}{h}\right) \overset{\exists}{=} 0$$

Je vůbec $f(x)=x \sin \frac{1}{x}$ spojitá ?

$f(x)$ pro $x \neq 0$ je složením spojitých

Pro $x = 0$. Jak spočítat limitu ?

$$\begin{matrix} 0 \leq & \left| x \sin \frac{1}{x} \right| & \leq |x| \\ \downarrow & & \downarrow \; x \to 0 \\ 0 & & 0 \end{matrix}$$

Dle lemma/věty o policistech je $\lim_{x \to 0} \left| x \sin \frac{1}{x} \right| = 0$.

Dle věty $\lim_{x \to x_0} |f| = 0 \Rightarrow \lim_{x \to x_0} f = 0$ je $\lim_{x \to 0} \left(x \sin \frac{1}{x} \right) = 0$

limit. $\exists \wedge f(0) = 0$. Takže f je spoj. i v 0.
a je rovna 0

$\lim_{x \to x_0} |f| = c \Rightarrow \lim_{x \to x_0} f = c$ neplatí . $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

$|f| = 1$ $\lim_{x \to 0} |f|$ $\exists$, ale $\lim_{x \to 0} f(x) \nexists$ $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$.

Pozn.: $\sin \frac{1}{x}$ nemá limitu pro $x \to 0$: Polož $x = \frac{2}{(2k+1)\pi}$

$\sin \frac{1}{x} = \sin \frac{(2k+1)\pi}{2} = -(-1)^k$. Takže $\exists$ limity vysporují jako u Dir. fce : Vždy najdu δ (nebo k), že $(-\delta, \delta)$ $\sin \frac{1}{x}$ bude mít hodnotu + i − jedna.

4. Derivace sudé je lichá. Buď $x_0 \in \mathbb{R}$

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=\lim_{h\to 0}\frac{f(-x_0-h)-f(-x_0)}{h}=$$

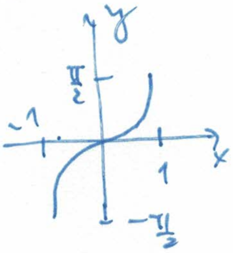
$\tilde{h}=-h$

$$=\lim_{\tilde{h}\to 0}\frac{f(-x_0+\tilde{h})-f(-x_0)}{-\tilde{h}}\overset{!}{=}-\lim_{\tilde{h}\to 0}\frac{f(-x_0+\tilde{h})-f(-x_0)}{\tilde{h}}=-f'(-x$$

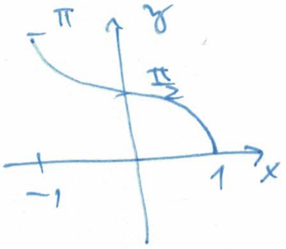
Tj. $f'(x_0)=-f'(-x_0)$ ⇒ lichá.

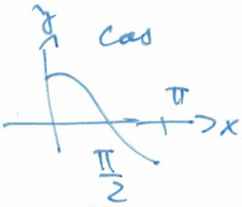
Některé Elementární funkce (méně známé).

1. arcsin $= \sin|_{[-\frac{\pi}{2},\frac{\pi}{2}]}^{-1}$

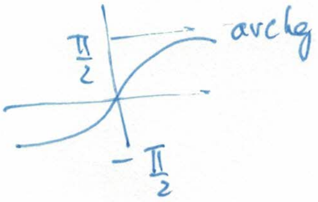


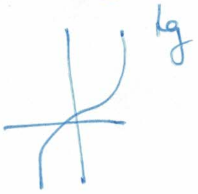
2. arccos $= \cos|_{[0,\pi]}^{-1}$



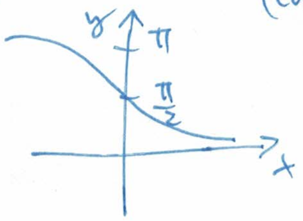


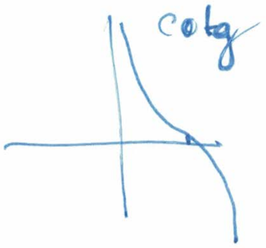
3. arctan $:= \tan|_{[-\frac{\pi}{2},\frac{\pi}{2}]}^{-1}$



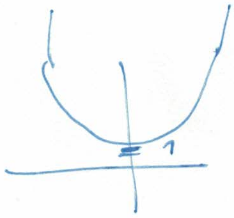


4. arccot (arccotg) $:=$ cotan (cotg) $|_{[0,\pi]}^{-1}$



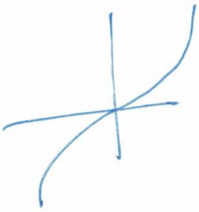


5. $\cosh x := \frac{e^x+e^{-x}}{2}$ (sudá)



„rychleji než parabola"

6. $\sinh x := \frac{e^x-e^{-x}}{2}$ (lichá)



Derivace elementárních funkcí

Viz na mých stránkách.

Vytisknout /ztáhnout si.

Doporučuji a^x i $\log_a x$ $\xrightarrow{\prime}$ si umět odvodit.

1) Derivace :

Nechť $\exists f', g'$. Pak

a) $(f+g)' = f' + g'$

b) $(cf)' = cf'$

c)! $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

d) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ $\forall x$, kde $g(x) \neq 0$

Def: $f'(a) = \lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$

e) $(f \circ g)'(x) = f'(g(x))g'(x)$

f) $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

1
1 1
1 2 1
1 3 3 1
1 4 6 6 1

Př.: $(x^4)'(x_0) = \lim\limits_{h \to 0} \frac{(x_0+h)^4 - x_0^4}{h} = \lim\limits_{h \to 0} \frac{x_0^4 + 4x_0^3h + 6x_0^2h^2 + 4x_0h^3 + h^4 - x_0^4}{h} = \lim\limits_{h \to 0} \left(4x_0^3 + 6x_0^2h + 4x_0h^2 + h^3\right) = 4x_0^3.$

$\boxed{(x^n)' = nx^{n-1}}$ obecně.

Př.: $\arcsin'(x) = \frac{1}{\sin'(\arcsin(x))} = \frac{1}{\cos(\arcsin x)} =$

$f = \sin$
$f^{-1} =$

$$= \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}} \overset{\uparrow}{=} \frac{1}{\sqrt{1 - \sin(\arcsin x)\sin(\arcsin x)}} =$$

konvence
$\sin^2 x \neq \sin x^2$
nýbrž $\sin x \sin x$, tj. $(\sin x)^2$.

$$= \frac{1}{\sqrt{1-x^2}} \text{ na } (-1,1)$$

O derivaci v ± 1 můžem mluvit jen jako o jednostranné.

1. $$f(x)=\frac{\overset{f}{2x}}{\underset{g}{1-x^2}} \qquad f'(x)=\frac{2(1-x^2)-2x(-2x)}{(1-x^2)^2}=\frac{2x^2+2}{(1-x^2)^2}$$

$x \neq \pm 1$

2. $$f(x)=\sqrt[3]{\frac{1+x}{1-x^2}}=\left(\frac{1+x}{1-x^2}\right)^{\frac{1}{3}}$$

$x \neq \pm 1$ $\qquad g(x)=\frac{1+x}{1-x^2}$

$$f'(x)=\frac{1}{3}\left(\frac{1+x}{1-x^2}\right)^{-\frac{2}{3}}\cdot\left[\frac{1+x}{1-x^2}\right]'=$$

$$=\frac{1}{3}\left(\frac{1+x}{1-x^2}\right)^{-\frac{2}{3}}\frac{(1-x^2)+2x(1+x)}{(1-x^2)^2}=$$

$$=\frac{1}{3}\left(\frac{1+x}{1-x^2}\right)^{-2/3}\frac{1+2x+x^2}{(1-x^2)^2}=\frac{1}{3}\frac{(1+x)^{4/3}}{(1-x^2)^{4/3}}=\frac{1}{3}\frac{(\cancel{1+x})^{\frac{4}{3}}}{(1-x)^{\frac{4}{3}}(\cancel{1+x})^{4/3}}$$

$$=\frac{1}{3}(1-x)^{-4/3}$$ Nebo mnohem snáze:

$$f(x)=\left(\frac{1+x}{(1+x)(1-x)}\right)^{1/3}=(1-x)^{-1/3}\Rightarrow f'(x)=-\frac{1}{3}(1-x)^{-4/3}\cdot(-1)=\frac{1}{3}(1-x)^{4/3}!$$

3. $$f(x)=\overset{f}{\sin}(\underbrace{\sin(\sin x)}_{g(x)})$$

$$f'(x)=\cos(\underbrace{\sin(\sin(x))}_{g(x)})\cdot(\overset{\overset{f}{\shortparallel}}{\underset{\uparrow\,\tilde{f}}{\sin}}\circ\overset{\overset{g}{\shortparallel}}{\underset{\uparrow\,\tilde{g}}{\sin}})'(x)=$$

Mimodem to vůbec není derivace $\sin^2 x$

$\boxed{x\in\mathbb{R}}$

$$=\underset{\tilde{f}'(\tilde{g})}{\cos(\sin(\sin x))\cos(\sin x)}\underset{\tilde{g}'}{\overset{\shortparallel}{\cos x}}.$$

opět: převádím na e^x [ny: nemám probl. s def. oborem]

4. $$f(x)=2^{\operatorname{tg}\frac{1}{x}}\overset{!}{=}e^{(\ln 2)\operatorname{tg}\frac{1}{x}} \qquad x\neq(2k+1)\frac{\pi}{2},\ x\neq 0$$

$$f'(x)=e^{\ln 2\operatorname{tg}\frac{1}{x}}\left(\ln 2\operatorname{tg}\frac{1}{x}\right)'=2^{\operatorname{tg}\frac{1}{x}}\ln 2\frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\left(\frac{1}{x}\right)'=$$

$$=2^{\operatorname{tg}\frac{1}{x}}\ln 2\frac{x^2}{1+x^2}\left(-\frac{1}{x^2}\right) \qquad x\neq 0,\ \underbrace{x\neq(2k+1)\frac{\pi}{2}\ \forall k\in\mathbb{Z}}$$

$$=2^{\operatorname{tg}\frac{1}{x}}\ln 2\frac{1}{1+x^2}$$ stále ale udržím "$\underline{x\neq 0}$ &

5. $f(x)=x^{a^a}+a^{x^a}+a^{a^x}$ (bez def. oboru; tendowa)

$f(x) = x^{(a^a)} + a^{(x^a)} + a^{(a^x)}$ „zřejmá" konvence.

Jinak např. $(x^a)^a = x^{aa} = x^{(a^2)} \ldots$

$$f'(x) = a^a x^{a^a-1} + \left(e^{\ln a \, (x^a)}\right)' + \left(e^{a^x \ln a}\right)' =$$

$$= a^a x^{a^a-1} + a^{x^a} \ln a \cdot a x^{a-1} + a^{a^x} \left(a^x \ln a\right)' =$$

$$= a^a x^{a^a-1} + a^{x^a} \ln a \cdot a \cdot x^{a-1} + a^{a^x} \ln a \left(e^{\ln a \, x}\right)' =$$

$$= a^a x^{a^a-1} + a^{x^a} a \ln a \, x^{a-1} + a^{a^x} \ln^2 a \; a^x .$$

Takže pozor: $(a^{x^a})' \neq x^a a^{x^a - 1}$.

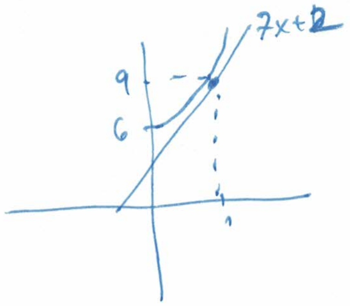
6. Spočtěte rovnici tečny k $f(x)=x^3+2x^2+6$ v $x=1$

$f'(x)=3x^2+4x$, $f'(1)=7$

$y = 7x + b$

$y(1) = 7+b = f(1) = 9 \Rightarrow$

$b=2$



$y = 7x+2$

! Dělení polynomů se zbytkem: $S(x) = Q(x)\,P(x) + R(x)$

$\forall S$ a $\forall Q \neq 0$ $\exists P$ $\exists R$ $\deg R < \deg Q$, aby $\uparrow$

Př.: $x^4 + 2x^2 + x + 2 : (x^2+1) = x^2+1$

$$\begin{array}{l} x^4 + 2x^2 + x + 2 \\ -(x^4 + x^2) \\ \hline \quad x^2 + x + 2 \\ \quad -(x^2 + 1) \\ \hline \qquad x + 1 \end{array}$$

Tj. $\underbrace{\dfrac{x^4+2x^2+x+2}{\underbrace{x^2+1}_{Q}}}_{} = \underbrace{x^2+1}_{P(x)} + \dfrac{\overbrace{x+1}^{R(x)}}{\underbrace{x^2+1}_{Q(x)}}$

$= S$

$$x^5+3x+6 : (x+1) = x^4 - x^3 + x^2 - x + 4 + \frac{2}{x+1} .$$

$$\begin{array}{l}
x^5+3x+6 \\
-(x^5+x^4) \\
\hline
-x^4+3x+6 \\
-(-x^4-x^3) \\
\hline
x^3+3x+6 \\
-(x^3+x^2) \\
\hline
-x^2+3x+6 \\
-(-x^2-x) \\
\hline
4x+6 \\
-(4x+4) \\
\hline
+2
\end{array}$$

$\deg 2 < \deg (x+1)$

Primitivní funkce

Pro f hledáme F, aby $F'=f$. F není jednoznačná: $(F+c)'=f$

Př.: Pro $f(x) = x$ hledáme F, aby $F' = x$.

Víme $(x^2)'=2x$. Zkusme užít $F(x) = \frac{1}{2}x^2$. $F'(x) =$

$= \frac{1}{2}\cdot 2x = x$.

Píšeme $\int x\,dx = \frac{1}{2}x^2 + C$

Př.: $f(x) = \frac{1}{2}\sin x$ | Nejdřív zkusme $\cos x$; $(\cos x)' = -$

$-\sin x$ | Tak $(-\cos x)' = -1\cdot(-\sin x) = \sin x$

Ještě přidáme $\frac{1}{2} \rightsquigarrow F(x) = \frac{-\cancel{1}\cos x}{2}$

$F'(x) = -\frac{1}{2}\cdot(-\sin x) = \frac{1}{2}\sin x$. Píšeme tedy

$\int \frac{1}{2}\sin x\,dx = -\frac{\cos x}{2} + C$.