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9) $I=\int \frac{\sin x \cos x}{1+\sin^3 x}\,dx \overset{1.\,VoS}{=} \left| \begin{array}{l} y=\sin x \\ dy=\cos x\,dx \end{array} \right| = \int \frac{y \cancel{\cos x}}{1+y^3}\cdot\frac{dy}{\cancel{\cos x}} =$

$$=\int \frac{y}{1+y^3}\,dy$$

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$$\frac{y}{1+y^3}=\frac{A}{y+1}+\frac{By+C}{y^2-y+1}=\frac{-\frac{1}{3}}{y+1}+\frac{\frac{1}{3}y+\frac{1}{3}}{y^2-y+1}$$

$$y=Ay^2-Ay+A+By^2+Cy+By+C$$

$A+B=0 \;\rightarrow\; A=-B \;\Rightarrow\; A=-\frac{1}{3}$

$A+C=0 \;\rightarrow\; C=-A=B \;\Rightarrow\; C=\frac{1}{3}$

$-A+B+C=1 \;\Rightarrow\; 3B=1 \;\Rightarrow\; B=\frac{1}{3}$

(A je výhodné metodou zakrývání)

$$\int\frac{y}{1+y^3}\,dy=\int -\frac{1}{3(y+1)}\,dy+\int\frac{y+1}{3(y^2-y+1)}\,dy=-\frac{1}{3}\int\frac{1}{(y+1)}\,dy+\frac{1}{3}\int\frac{y+1}{y^2-y+1}\,dy$$

$$=-\frac{1}{3}\int\frac{1}{(y+1)}\,dy+\frac{1}{3}\int\frac{2y-1}{2(y^2-y+1)}+\frac{3}{2(y^2-y+1)}\,dy=$$

$$=-\frac{1}{3}\int\frac{1}{(y+1)}\,dy+\frac{1}{6}\int\frac{2y-1}{y^2-y+1}\,dy+\frac{1}{2}\int\frac{1}{y^2-y+1}\,dy$$

ŘEŠÍME:
$$\int \frac{1}{y^2-y+1}\,dy = \int \frac{1}{(y-\frac{1}{2})^2+\frac{3}{4}}\,dy = \left| \begin{array}{l} t = \frac{2y-1}{\sqrt{3}} \\ dt = \frac{2}{\sqrt{3}}\,dy \end{array} \right| =$$

$$= \int \frac{\sqrt{3}}{2\left(\frac{3t^2}{4}+\frac{3}{4}\right)}\,dt = \frac{2}{\sqrt{3}}\int \frac{1}{t^2+1}\,dt = \frac{2}{\sqrt{3}}\,\mathrm{arctg}\,t$$

ZPĚTNÁ SUBSTITUCE:
$$\Rightarrow \frac{2\,\mathrm{arctg}\left(\frac{2y-1}{\sqrt{3}}\right)}{\sqrt{3}} + C'$$

$$\int \frac{1}{y+1}\,dy = \ln|y+1| + C'' \qquad \int \frac{2y-1}{y^2-y+1}\,dy = \ln|y^2-y+1| + C'''$$

MŮŽEME PSÁT CELÉ ŘEŠENÍ:
$$-\frac{1}{3}\int \frac{1}{y+1}\,dy + \frac{1}{6}\int \frac{2y-1}{y^2-y+1}\,dy + \frac{1}{2}\int \frac{1}{y^2-y+1}\,dy =$$

$$= -\frac{1}{3}\ln|y+1| + \frac{1}{6}\ln|y^2-y+1| + \frac{\mathrm{arctg}\left(\frac{2y-1}{\sqrt{3}}\right)}{\sqrt{3}} + C$$

PO ZPĚTNÉ SUBSTITUCI:
$$I = -\frac{1}{3}\ln|\sin x + 1| + \frac{1}{6}\ln|\sin^2 x - \sin x + 1| + \frac{\mathrm{arctg}\left(\frac{2\sin x - 1}{\sqrt{3}}\right)}{\sqrt{3}} + C$$

ZPĚTNĚ DOSADÍME DO SUBSTITUCE:

$$t = \sqrt{x^2+x+1} - x \quad \longrightarrow$$

$$\underline{\underline{\frac{1}{2}\left(\sqrt{x^2+x+1} - x\right) - 2\ln\left|\sqrt{x^2+x+1} - x + 1\right| + \frac{1}{2}\ln\left|2\left(\sqrt{x^2+x+1} - x\right) - 1\right| +}}$$

$$\underline{\underline{+ \frac{3}{2}\left(\frac{1}{4\left(\sqrt{x^2+x+1} - x\right) - 2}\right) + C}}$$

$$\parallel$$

$$\frac{3}{4}\left(\frac{1}{2\left(\sqrt{x^2+x+1} - x\right) - 1}\right)$$

$$\int e^{3x} \cos x \, dx = \left| \begin{array}{ll} f = \cos x & g = \frac{1}{3}e^{3x} \\ f' = -\sin x & g' = e^{3x} \end{array} \right| = \frac{1}{3} e^{3x} \cos x + \frac{1}{3}\int \sin x \, e^{3x} \, dx =$$

$$= \left| \begin{array}{ll} f = \sin x & g = \frac{1}{3}e^{3x} \\ f' = \cos x & g' = e^{3x} \end{array} \right| = \frac{1}{3} e^{3x} \cos x + \frac{1}{3}\left(\frac{1}{3} e^{3x} \sin x - \frac{1}{3}\int e^{3x} \cos x \, dx\right)$$

$$\Rightarrow \frac{10}{9}\int e^{3x} \cos x \, dx = \frac{1}{3} e^{3x} \cos x + \frac{1}{9} e^{3x} \sin x$$

$$\int e^{3x} \cos x \, dx = \underline{\underline{\frac{1}{10} e^{3x} \left(3\cos x + \sin x\right) + C}}$$

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pěkné a přehledné

5) $\displaystyle\int \frac{x+\sqrt{x^2+x+1}}{1+x+\sqrt{x^2+x+1}}\,dx$

Euler; $a>0$: $\sqrt{x^2+x+1} = t+x$ ✓

$$\cancel{x^2}+x+1 = \cancel{x^2}+2tx+t^2$$

$$x(1-2t) = t^2-1$$

$$x = \frac{t^2-1}{1-2t} \quad ✓$$

$$t+x = t+\frac{t^2-1}{1-2t} = \frac{-2t^2+t+t^2-1}{1-2t} = \frac{-t^2+t-1}{1-2t}$$

$$dx = \frac{(t^2-1)'(1-2t)-(t^2-1)(1-2t)'}{(1-2t)^2} = \frac{2t(1-2t)-(t^2-1)(-2)}{(1-2t)^2} = \frac{-2t^2+2t-2}{(1-2t)^2}\,dt \quad ✓$$

$$\int \frac{x+\sqrt{x^2+x+1}}{1+x+\sqrt{x^2+x+1}}\,dx = \int \frac{\frac{t^2-1}{1-2t}+\frac{-t^2+t-1}{1-2t}}{1+\frac{t^2-1}{1-2t}+\frac{-t^2+t-1}{1-2t}}\cdot\frac{-2t^2+2t-2}{(1-2t)^2}\,dt =$$

$$= \int \frac{\frac{t-2}{\cancel{1-2t}}}{\frac{-t-1}{\cancel{1-2t}}}\cdot\frac{-2t^2+2t-2}{(1-2t)^2}\,dt = \int \frac{t-2}{-t-1}\cdot\frac{-2(t^2-t+1)}{(1-2t)^2}\,dt = \int \frac{2t^3-6t^2+6t-4}{4t^3-3t+1}\,dt \quad ✓$$

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$$\frac{2t^3-6t^2+6t-4}{4t^3-3t+1} = \frac12 + \frac{-6t^2+\frac{15}{2}t-\frac92}{(t+1)(2t-1)^2} = \frac12+\frac{A}{t+1}+\frac{B}{2t-1}+\frac{C}{(2t-1)^2}$$

(A a C výhodné zakrýváním, ale nebezpečí chyb ...)

$$-6 = 4A+2B \longrightarrow A=\frac{-6-2B}{4} \Rightarrow A=\underline{-2} \quad ✓$$

$$\frac{15}{2} = -4A+B+C \longrightarrow 30 = 24+18B-12 \Rightarrow B=\underline{1} \quad ✓$$

$$-\frac92 = A-B+C \longrightarrow C = -\frac92+\frac{6+2B}{4}+B \Rightarrow C=\frac{-12+6}{4} = \underline{\underline{-\frac32}} \quad ✓$$

$$\int \frac{2t^3-6t^2+6t-4}{4t^3-3t+1}\,dt = \int\frac12\,dt+\int\frac{-2}{t+1}\,dt+\int\frac{1}{2t-1}\,dt+\int -\frac{3}{2(2t-1)^2}\,dt =$$

$$= \frac12 t - 2\ln|t+1| + \frac12\ln|2t-1| - \frac32\left(-\frac{1}{4t-2}\right) + C$$