Basics of Banach algebras & Gelfand map

Definition: $(A, \|\ \|)$ is called Banach algebra, if a A is an assoc. $\mathbb{R}/\mathbb{C}$-algebra, $\|\ \| : A \to \mathbb{R}_{\geq 0}$ is submultiplicative norm ($\|ab\| \leq \|a\|\|b\|$, $a, b \in A$) and $(A, \|\ \|)$ is complete.

Remark: 1) Associative R-algebra A (R is a ring) is a ring and an R-module with the compatibility conditions:

$$r(ab) = (ra)b = a(rb), \quad a, b \in A, r \in R$$

2) multiplication: $\|a_n b_n - a_0 b_0\| = \|a_n b_n - a_n b_0 + a_n b_0 - a_0 b_0\|$
$\leq \|\underbrace{a_n}_{\text{bounded (conv} \Rightarrow \text{bound.)}}(\underbrace{b_n - b_0}_{\to 0})\| + \|\underbrace{(a_n - a_0)}_{\to 0} b_0\|$. Continuous.

Definition: A Banach alg. $\Delta_A := \{ m : A \to \mathbb{C} \mid m$ homom of alg. A and $\mathbb{C}$, $m \neq 0$, m cont.$\}$ - structure/state space. $\forall m, n \in \Delta_A$: $(m+n)(a) := m(a) + n(a)$, $(\lambda m)(a) := \lambda m(a)$, $\lambda \in \mathbb{R}/\mathbb{C}$. $(m \cdot n)(a) = m(a) n(a)$, $a \in A$,

Rem.: a) Δ_A not a vectorspace: $m + (-m) = 0 \notin \Delta_A$ (if $A \neq 0$, that is supposed). b) Homom of $\mathbb{C}/\mathbb{R}$-algs: $m(a+b) = m(a) + m(b)$, $m(\lambda a) = \lambda m(a)$, $m(ab) = m(a) m(b)$

Augmentation: A Ban. alg (over $\mathbb{C}$); $\tilde{A} := A \times \mathbb{C} \simeq A \oplus \mathbb{C}$
$(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$, $a, b \in A$, $\alpha, \beta \in \mathbb{C}$.
$1 := (0, 1)$ is a/the unit $\quad 1(a, \alpha) = (a, \alpha)1 = (a, \alpha)$.
$\|(a, \alpha)\|^{\sim} := \|a\| + |\alpha|$ ← in $\mathbb{C}$

$(\tilde{A}, \|\ \|^{\sim})$ augm. of A. If A has unit $\Rightarrow$ it is replaced by $1 = (0,1)$ in $\tilde{A}$. If not the aug. $\tilde{A}$ has one new unit.

Remark: If A has unit, we call it unital.

Remark: $1a (= a1) = a$ & $1'a (= a1') = a$ $\forall a \in A$; Espec.

$$1 \underset{\substack{1' \text{ left} \\ \text{unit}}}{=} 1'1 = 11' \underset{\substack{1 \text{ left} \\ \text{unit}}}{=} 1' \Rightarrow 1 = 1' \,!$$

Lemma (Neumann series): $(A, \|\cdot\|)$ unital Banach alg, $a \in A$, $\|a\| < 1$. Then 1) $1 - a \in A^{\times}$ ($:=$ invertible elements of A), $(1-a)^{-1} = \sum_{n=0}^{+\infty} a^n$

2) $A^{\times}$ open

Pf.: 1) • $s_n := \sum_{m=0}^{n} a^m$; $\|s_{n+p} - s_n\| = \|\sum_{k=n+1}^{n+p} a^k\| \underset{\text{triang}}{\leq} \sum_{n+1}^{n+p} \|a^k\| \underset{\text{submult}}{\leq}$ ($p \geq 1$)

$$\leq \sum_{n+1}^{n+p} \|a\|^k \leq \|a\|^{n+1} \sum_{k=0}^{p-1} \|a\|^k \leq \|a\|^{n+1} \sum_{k=0}^{\infty} \|a\|^k \leq$$

$$\underbrace{\leq \|a\|^{n+1} \frac{1}{1 - \|a\|}}_{(*)} \longrightarrow 0, \; n \to \infty \Rightarrow s_n \text{ Cauchy}$$

($\forall \varepsilon \, \exists n_0 \, \forall p \dots$, n_0 chosen indep. on p). A complete $\Rightarrow$ s_n converges. (The convergence is also absolute. trivial)

• $(1-a) \sum_{n=0}^{m} a^n - 1 = a + \dots + a^m - a - \dots - a^{m+1} =$

$= -a^{m+1} \Rightarrow \|(1-a) \sum_{n=0}^{m} a^n - 1\| \leq \|a\|^{m+1} \longrightarrow 0 \Rightarrow$

$(1-a) \sum_{n=0}^{\infty} a^n = 1$ ✓ ($\Rightarrow$ $1-a$ invertible)

2) • $B_1(1) = \{y \in A \mid \|y - 1\| < 1\}$ open. $\forall a \in B_1(1)$ $(\dots)$

$a^{-1} = (1 - \underset{\|<1}{(1-a)})^{-1} \left(= \sum (1-a)^n\right)$ exists $\underset{\text{by } 1)}{\Rightarrow}$ $B_1(1) \subseteq A^{\times}$

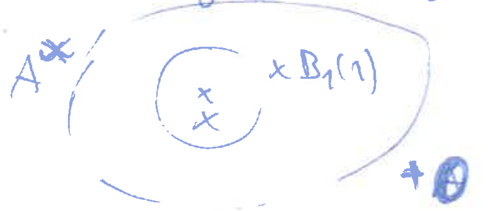
• $L_x : y \mapsto xy$; $L_x|_{B_1(1)} : B_1(1) \to xB_1(1) = \{xy \mid \|y-1\|<1,\ y\in A\}$

For $x\in A^\times$ $L_{x^{-1}}|_{xB_1(1)} : xB_1(1) \to B_1(1)$. Both continuous and right & left inverse of each other.

Esp. $xB_1(1)$ homeom $B_1(1)$ open $\forall x\in A^\times$:

• ($\forall x \in xB_1(1)$.) $\forall x\in A^\times$: $xB_1(1)$ open. Are elements of $xB_1(1)$ invertible?: For $z = xy$, take $y^{-1}x^{-1}$!

Thus $xB_1(1)\subseteq A^\times$



□

Remark: $(A^\times, 1, \cdot)$ is a topol. group. General fact:

"$\dim A = \infty \Rightarrow$ A not loc. cpt (f'tial analysis, (normed!) e.g., Trèves: Distr., Kernels and tensor products)"

If $\dim A = \infty \Rightarrow A^\times$ is not locally cpt.

From now, we consider complex Banach algebras only, i.e., $R=\mathbb{C}$.

Definition: $\forall a\in A$, unit. A assoc. alg. over $(\mathbb{R}/)\mathbb{C}$. $\sigma_A := \{\lambda\in\mathbb{C} \mid a-\lambda 1$ is not invertible (in A!)$\}$ — spectrum

$Res_A(a) = \mathbb{C}\setminus\sigma_A(a)$ — resolvent (set) of a.
$= \{\lambda \mid a-\lambda 1$ invertible$\}$

! Remark: $f: U \underset{\text{open}}{\subseteq} \mathbb{C} \to V$ (normed space, e.g., Banach, Banach alg etc.) is called holomorphic if $\forall z\in U$

$\exists \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ $(=: f'(z))$ (, where $\frac{1}{h} = \frac{\overline{h}}{\sqrt{h_1^2+h_2^2}}$ ← problem). $h\in\mathbb{C}$

(cpt. spec.) f holom $\Leftrightarrow \forall\varphi$ $\varphi\circ f$ holom.

Lemma: A unital Banach alg $\Rightarrow \forall a\in A$ $\sigma_A(a)$ is cpt.

Pf: 1) $F_a: \lambda\in\mathbb{C}\mapsto a-\lambda 1\in A$ continuous $\forall a$
$\Rightarrow F_a^{-1}(A^\times)$ open, i.e., $Res_A(a)$ open, i.e.,
↑ open (lemma above)

<u>Remark</u>: Continuity of inverse:

$b \to a$: $\quad |a^{-1}-b^{-1}| = |b^{-1}\underline{(b-a)}a^{-1}| \overset{(*)}{=} |(b^{-1}-a^{-1})(b-a)a^{-1} + a^{-1}(b-a)a^{-1}|$

$\leq |b^{-1}-a^{-1}|\,|b-a|\,|a^{-1}| + |b-a|\,|a^{-1}|^2$

$$|a^{-1}-b^{-1}|\underbrace{(1-\underset{\text{small}}{|b-a|\,|a^{-1}|})}_{\geq \frac{1}{2}\ (\text{„big"})} \leq \underset{\text{small}}{|b-a|}\,\underset{\text{~~bounded~~ fixed}}{|a^{-1}|^2}$$

$\Rightarrow |a^{-1}-b^{-1}|$ small.

<u>Remark</u>: $(*)$ „basic trick"

<u>Remark</u>: A unital Banach alg $\Rightarrow$ $(A^\times, 1, \cdot)$ is a topological group.

In general, not locally compact.

$\sigma_A(a)$ closed.

2) $\lambda \in \mathbb{C}$ · $|\lambda| > \|a\|$: $\|\lambda^{-1}a\| < 1 \overset{\text{le Neumann ser.}}{\Rightarrow}$ $1-\lambda^{-1}a$ invertible

$\Rightarrow$ $\lambda 1 - a = \lambda(1-\lambda^{-1}a)$ $(\lambda \neq 0 \Leftarrow |\lambda| > \|a\|!)$ $(\Rightarrow \lambda \neq 0)$

invertible as well $\Rightarrow$ λ not in spectr. $\Rightarrow \sigma_A(a) \subseteq B_{\|a\|}(0) \subseteq \mathbb{C}$

bounded.

<u>Thm.</u>: A unital Banach algebra (over cplx). Then $\forall a \in A$ $\boxed{\sigma_A(a) \neq \emptyset.}$

Proof: ↯ $\exists a \in A$ $\sigma_A(a) = \emptyset$. Be $\alpha : A \to \mathbb{C}$ cont. ftial and

def. $\boxed{f_\alpha(\lambda) := \alpha\left(\frac{1}{a-\lambda}\right)}$. f_α is holom. $\left(\frac{1}{a-\lambda}\right.$

is holo & α: $A \to \mathbb{C}$ $\mathbb{C}$-linear.)

For $|\lambda| > 2\|a\|$: $|f_\alpha(\lambda)| = \left|\alpha\left(\frac{1}{a-\lambda}\right)\right| = \frac{1}{|\lambda|}\left|\alpha\left(1-\frac{a}{\lambda}\right)\right| =$

$\leq \frac{1}{|\lambda|}\left|\alpha\left(\sum^{+\infty}\left(\frac{a}{\lambda}\right)^n\right)\right| \leq \frac{1}{|\lambda|}\left|\alpha \sum^{+\infty}\left(\frac{a}{\lambda}\right)^n\right| \leq \frac{1}{|\lambda|}\|\alpha\|_{op}\left\|\sum_{n=0}^{+\infty}\left(\frac{a}{\lambda}\right)^n\right\|$

$\exists$ by Neum. $\quad$ α cont. $\quad$ $\boxed{\alpha \text{ cont} \Rightarrow \alpha \text{ bound}}$

$\leq \frac{\|\alpha\|_{op}}{|\lambda|}\sum_0^{+\infty}\left\|\frac{a}{\lambda}\right\|^n \leq \frac{\|\alpha\|_{op}}{|\lambda|}\sum_0^{+\infty}\left(\frac{1}{2}\right)^n = \frac{\|\alpha\|_{op}}{|\lambda|} \xrightarrow[\lambda\to\infty]{} 0 \overset{\text{Liouville in cplx anal.}}{\Longrightarrow}$

Δ-in. + submult $\quad$ $\|ka\| = |k|\|a\|$, $k \in \mathbb{C}$

$\Rightarrow f_\alpha(\lambda) = 0 \ \forall \lambda \in \mathbb{C} \ \forall \alpha \in A'$

$\Rightarrow \frac{1}{a-\lambda} = 0$ (by Hahn-Banach) ↯.

important that $\mathbb{C}$, not $\mathbb{R}$

<u>Rem.</u>: A is Banach $\mathbb{C}$-alg.

Remark: holomorphicity of $\frac{1}{a-\lambda}$.

$$\lim_{\mu\to\lambda} \frac{\frac{1}{a-\mu} - \frac{1}{a-\lambda}}{\mu-\lambda} = \lim_{\mu\to\lambda} \frac{1}{\mu-\lambda} \frac{(a-\lambda)-(a-\mu)}{(a-\lambda)(a-\mu)} =$$

$$\left(\frac{1}{a-\mu}\right)'_{\mu=\lambda} = \lim_{\mu\to\lambda} \frac{\mu-\lambda}{\mu-\lambda}\frac{1}{(a-\lambda)(a-\mu)} \overset{\text{continuity of inversion}}{=} \frac{1}{(a-\lambda)^2}.$$

Btw.: $$\boxed{\frac{1}{a-\mu} - \frac{1}{a-\lambda} = (\mu-\lambda)(a-\lambda)^{-1}(a-\mu)^{-1} \quad \text{resolvent equation}}$$

Remark: Hahn-Banach: $\exists \alpha_0 \; 0 \neq \alpha_0(+a^{-1}) = \underbrace{f_{\alpha_0}(0)}_{\text{a priori}} = 0$. ↯

(Argument in the thm $\sigma_A(a) \neq \emptyset$.)

Lemma: f holom. at z $\iff$ $(\varphi\circ f)$ holom $\forall \varphi$ cont.

Pf: $\Rightarrow$ $\lim \frac{\varphi(f(z+h)) - \varphi(f(z))}{h} \overset{\varphi \text{ cont.}}{=}$

$= \varphi\left(\lim_h \frac{f(z+h)-f(z)}{h}\right) = \varphi\circ f'(z).$

φ con

$\Leftarrow$ $(\varphi\circ f)$ holom. in z $\forall \varphi$ $\exists (\varphi\circ f)'(z) = ia$. We don't need it... STOP. (But it is true.)

Corollary (Gelfand-Mazur): A complex unital Banach algebra that is a division algebra: $A \cong \mathbb{C}$ (and the isom. is unique)

Proof: $a \in A \Rightarrow \exists \lambda \in \sigma_A(a) \Rightarrow \lambda 1 - a$ noninvertible

$\Rightarrow a - \lambda 1 = 0 \Rightarrow a = \lambda 1 \Rightarrow A \cong \mathbb{C}1$

A div.

$\|a\| = \|\lambda 1\| = |\lambda|$. $a \mapsto \lambda$ ... isometric isomorphism. $\square$

$a \mapsto \lambda$? $\quad ab \mapsto \lambda_a \lambda_b$ (Why unique?) see below

$a \mapsto \lambda_a$

$b \mapsto \lambda_b$

Ex.: $\mathbb{H}$, multipl.

$\mathbb{H}$ is not a $\mathbb{C}$-algebra (Why?) (Otherwise It would contradict Coroll. above.)

$i \circlearrowleft \; a + bi + cj + dk \in Z(\mathbb{H})$

$-b + ai - dj + ck = -b + ai + dj - ck \Rightarrow c = d = 0$

$(a+bi)j = aj + bk$ $\Rightarrow b = 0 \quad Z(\mathbb{H}) \subseteq \mathbb{R}$, but

$j(a+bi) = ja - bk$

$Z(\mathbb{H}) \supseteq \mathbb{C}$ if $\mathbb{H}$ were $\mathbb{C}$-algebra:

$1 = 1 + 0i + 0j + 0k$

$x \in \mathbb{C}$ $\quad x1 \in \mathbb{H}$

$a \in \mathbb{H}$: $\boxed{a(x1)} = (ax)1 = x(a \cdot 1) = xa(1 \cdot a) = \boxed{(x \cdot 1)a}$

compatibility in R-algs, R ring.

$\Rightarrow x1 \in Z(\mathbb{H})$

Remark: uniqueness of φ:

$\varphi_1 : \mathbb{C} \to A \qquad \varphi_2(c) = \varphi_2(1 \cdot c) = \varphi_2(1) \qquad c = 1c$

$\varphi_2 : \mathbb{C} \to A \qquad \varphi_1(c) = \varphi_1(1 \cdot c) = \varphi_1(1) \cdot c = 1c$

$\boxed{\varphi_1 = \varphi_2}$ $\qquad \varphi$ nonzero hom. of algebras.

$\varphi(1) = 0 \Rightarrow \varphi = 0$

$\varphi(11) = \varphi(1)^2 = \varphi(1) \qquad \varphi(1) = 1$

<u>Def.</u>: $\forall a\in A$ (unital) Banach algebra

$$r(a) := \sup_\lambda \underbrace{\{|\lambda| \mid \lambda\in\sigma_A(a)\}}_{\le \|B_{\|a\|}(0)\|_{\mathbb{C}} \le \mathbb{R}} \in \mathbb{R}^{\ge 0}$$ is called spectral radius of a. If not unital $a\in\tilde{A}$.

<u>Rem.</u>: Exists by lemma on cpt. of spectra.

<u>Thm. (sp. radius)</u>: $\forall a\in A$ $r(a) = \limsup_{n\to\infty}\|a^n\|^{\frac{1}{n}}$ (In part. $r(a)\le\|a\|$.)

<u>Proof</u>: 1) From $\sigma_A(a)$ bounded, we know: $\lambda\in\sigma_A(a)$: $|\lambda|\le\|a\|$
Thus $r(a)\le\|a\|$.

2) We prove: $r(a)\le \liminf_n \|a^n\|^{\frac1n} \overset{\text{trivial}}{\le} \limsup_n \|a^n\|^{\frac1n} \le r(a)$

($! \Rightarrow \exists \lim$ & $= \liminf = \lim\sup$)

a) $\forall\lambda\ \forall n$ $\lambda^n 1 - a^n = \underbrace{(\lambda 1-a)}_{u}\underbrace{\sum_{j=0}^{n-1}\lambda^j a^{n-1-j}}_{v}$. $uv = vu \Leftarrow u,v$ comm.

Supp. $\lambda\in\sigma(A) \Rightarrow \lambda^n - a^n \notin A^\times$: ↯ $\lambda^n - a^n \in A^\times$; $1 = w(\lambda^n-a^n) = (wv)u$, $1 = (\lambda^n-a^n)w = u(vw)$ $\Rightarrow u\in A^\times$ ↯

$\Rightarrow \lambda^n\in\sigma(a^n)$: $|\lambda|^n = |\lambda^n| \le \|a^n\|$ (proof of bound. of $\sigma(b)$, $b=a^n$) $\Rightarrow |\lambda|\le\|a^n\|^{\frac1n} \Rightarrow |\lambda|\le\liminf\|a^n\|^{\frac1n}$

Taking $\sup_{\lambda\in\sigma_A(a)}$: $r(a)\le\liminf_n\|a^n\|^{\frac1n}$ ✓

If $|\lambda|>\|a\|$: b) $(\lambda 1-a)^{-1} = \lambda^{-1}\left(1-\frac{a}{\lambda}\right)^{-1} = \sum a^n\frac{1}{\lambda^{n+1}}$ since $\left\|1-\frac{a}{\lambda}-1\right\| < 1$. ($\|a\|<|\lambda|$)

Thus $\varphi\left(\frac{1}{\lambda-a}\right)$ is holom. on $|\lambda|>r(a)$. $\Leftrightarrow$

$\Rightarrow$ is holom. for $\forall\varphi: A\to\mathbb{C}$ continuous $\Rightarrow$

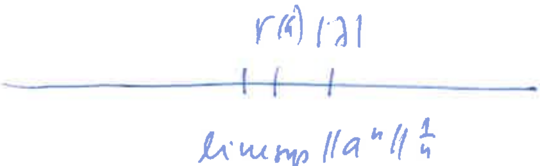
Again $\frac{1}{\lambda-a}$ is holom.

Thus $\varphi\left(\frac{a^n}{\lambda^{n+1}}\right)$ are bounded: $\varphi\left(\frac{a^n}{\lambda^{n+1}}\right)\le C_\varphi \Rightarrow \|a^n\|\le C|\lambda|^{n+1}$ (unif. boundedness principle)

$\|a^n\|^{\frac1n}\le C^{\frac1n}|\lambda||\lambda|^{\frac1n}$ $\limsup_n$

$\limsup\|a^n\|^{\frac1n}\le|\lambda|$ $\forall|\lambda|>r(a)$

$\Rightarrow \limsup\|a^n\|^{\frac1n}\le r(a)$. □



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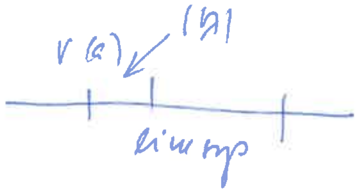

↗ norms on vec. space.

Lemma: $\forall m \in \Delta_A$: m is contin. & $\|m\|_{op} \le 1$. If A is unital $\Rightarrow \|m\|_{op} = 1$

Proof: • Only for $A \ni 1$.

$m(1^2) = m(1)^2 \Rightarrow$ a) $m(1) = 0 \Rightarrow m(a) = m(a1) = m(a)m(1) = 0 \ \forall a \Rightarrow m \notin \Delta_A$.

b) $m(1) = 1$

$\forall a \in A$: $a - m(a)1 \in A \setminus A^\times$: If $a - m(a)1 \in A^\times \Rightarrow \exists b \in A$

$1 = m(1) = m((a - m(a)1)b) = (m(a) - m(a))\, m(b) = 0$ ↯

$\Rightarrow m(a) \in \sigma_A(a) \Rightarrow |m(a)|_{\mathbb{C}} \le \|a\|$ (Lemma on cpt. spec)

$$\|m\|_{op} = \sup_{a \neq 0} \frac{|m(a)|_{\mathbb{C}}}{\|a\|_A} \le \sup \frac{\|a\|_A}{\|a\|_A} = 1. \quad \square$$

[Remark: • $m(1) = 1 \Rightarrow \|m\|_{op} = \sup_{\|a\|_A \le 1} |m(a)|_{\mathbb{C}} \ge \sup |m(1)|_{\mathbb{C}} = \sup 1 = 1$

$\|1\| = \|1 \cdot 1\| \le \|1\|\,\|1\|$

$\Rightarrow$ a) $\|1\| = 0$

b) $\|1\| \le 1$]

Banach-Alaoglu thm: $(V, \|\cdot\|)$ normed space. Then $\bar{B}_1' := \{f \in V' \mid \|f\|_{op} \le 1\} \subseteq V'$ is a cpt. Hausdorff with respect to the weak-$*$-topology.

Proof: $\bar{B}_1 = \{z \in \mathbb{C} \mid |z| \le 1\}$, $f \in \bar{B}_1'$: $\underbrace{|f(v)|_{\mathbb{C}}}_{\|f\|_{op}\|v\|_V \le \|v\|_V} \le \|v\|_V \ \forall v \in V \Rightarrow$

[Needed for thm on Gelfand map ∃f]

$f(v) \in \bar{B}_{\|v\|_V} \subseteq \mathbb{C}$, cpt.

$\Xi: \bar{B}_1' \to \bigtimes_{v \in V} (\|v\| \bar{B}_1)$, $f \mapsto (f(v))_{v \in V}$, $f \in \bar{B}'$

Tychonov (ff): $\bigtimes (\|v\| \bar{B}_1)$ cpt. Hausdorff

[Ξ injective — obvious.]

$\forall v \in V$: $f_\alpha(v) \xrightarrow[\times \|v\| \bar{B}_1]{} f(v) \Leftrightarrow f_\alpha \xrightarrow[\alpha]{\text{def}} f$ in the weak $*$-topol.

$(f_\alpha)_\alpha \to f \in \bigtimes_{v \in V} \|v\| \bar{B}_1 \Leftrightarrow f_\alpha(v) \to f(v) \ \forall v$

$$\|f\|_{op} = \sup_{\|v\| \le 1} |f(v)|_{\mathbb{C}} = \sup_{\|v\| \le 1} \Big|\underbrace{\lim_\alpha (f_\alpha(v))}_{\exists}\Big| \le \|f_\alpha\|_{op}\|v\| \le \|v\| \le 1$$

$\square$

Definition (Gelfand map): $\forall a \in A$, $\hat{a} : \Delta_A \to \mathbb{C}$, $\hat{a}(m) := m(a)$ (8)

$\hat{} : A \to \text{Ftions}(\Delta_A)$, $\hat{}(a) := \hat{a}$ is the Gelfand map

(Gelfand transform).

Thm. (on the Gelfand map): Let A be Banach algebra. Then

1. Δ_A is locally cpt Hausdorff in the weak $*$-topol.
2. A unital $\Rightarrow \Delta_A$ cpt. in the weak $*$-topol.
3. $\forall a \in A$ $\hat{a} \in \mathcal{C}(\Delta_A)$ and vanishes at infinity

   $\hat{} : A \to \mathcal{C}_0(A)$ is a homom. of algs.
4. $\forall a \in A$ $\|\hat{a}\|_{\mathcal{C}_0(\Delta_A)} \leq \|a\|$, and so $\hat{}$ is continuous.

Proof: Only $1 \in A$: $2) \Rightarrow 1)$ triv. We prove 2.

2) $m_n \in \Delta_A$, $m_n \xrightarrow{w} f$, $f \in A'$

$m_0 \neq m_1 \Rightarrow \exists a_0$ $m_0(a_0) \neq m_1(a_0)$

- $f(ab) = \left(\lim_{\substack{\alpha \\ \text{in } w\text{-}*}} m_\alpha\right)(ab) = \lim_{\substack{n \\ \text{in } \mathbb{C}}} (m_\alpha(ab)) =$

  $= \lim_{\substack{\alpha \\ \text{in } \mathbb{C}}} [m_\alpha(a) m_\alpha(b)] = \lim_{\substack{\alpha \\ \text{in } \mathbb{C}}} m_\alpha(a) \lim_{\substack{n \\ \text{in } \mathbb{C}}} m_\alpha(b)$

  $= \left(\lim_{\substack{\alpha \\ w\text{-}*}} m_\alpha\right)(a) \left(\lim_{\substack{\alpha \\ w*}} m_\alpha\right)(b) = f(a) f(b).$

- $f(\lambda a + \mu b) = \lambda f(a) + \mu f(b)$ analogical

- $f(1) = \lim m_n(1) = \dots = 1 \Rightarrow f \neq 0 \Rightarrow f \in \Delta_A$.

$\Rightarrow \Delta_A$ closed, Δ_A closed in cpt. $\Rightarrow \Delta_A$ cpt.

↑ by Banach-Alaoglu

3) vanishing in infinity $\emptyset$

Continuity: $m_\alpha \to m$

$\lim_\alpha \hat{a}(m_\alpha) = \lim_\alpha m_\alpha(a) = (\lim m_\alpha)(a) = m(a) = \hat{a}(m) =$

$= \hat{a}(\lim_\alpha m_\alpha) \Rightarrow \hat{a}$ cont.

homom. of algs.: $\widehat{ab}(m) = m(ab) = m(a) m(b) = \hat{a}(m) \hat{b}(m) =$

$= \hat{a}\hat{b}(m)$ $\forall m$

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4) $\|\hat{a}\|_{\mathcal{C}_0(\Delta_A)} = \sup_{\substack{m \in \Delta_A (\subseteq A^*) \\ \|m\|_{op} \le 1}} |\hat{a}(m)|_{\mathbb{C}}$ is definition.

Lemma on str. space

$|\hat{a}(m)|_{\mathbb{C}} \le |m(a)|_{\mathbb{C}} \le \|m\|_{op} \|a\|_A \le 1 \cdot \|a\|_A$ ✓

$\Rightarrow \|\hat{a}\|_{\mathcal{C}_0(\Delta_A)} \le \sup_{\|m\|_{op} \le 1} \|a\|_A = \|a\|_A$. Thus (trivially):

$\wedge : A \to \mathcal{C}_0(\Delta_A)$ is cont. (suff. $\wedge$ is bounded):

$\|\wedge\|_{op} = \sup_{\|a\|_A \le 1} \|\hat{a}\|_{\mathcal{C}_0(\Delta_A)} \le \sup_{\|a\|_A \le 1} (\|a\|_A) = 1$ $\square$

Def: A Banach alg.

A Banach $*$-alg, if $* : A \to A$ 1) antihomomom. $*(ab) = (*b)(*a)$

2) involution

and 3) isometry of $(A, \|\ \|)$, i.e., $\|a\| = \|*a\|$

Rem: $**a = a$, (invol.) $*(ab) = *b * a$ & $*(a + \lambda b) = \overset{*}{a} + \bar{\lambda} * b$ $*$-antihomom.

$\|^*a\| = \|a\|$, where $a^* := *a$

Def: A C^*-algebra: Banach $*$-alg & $\|a a^*\| = \|a\|^2$ (C^*-id.)

Ex.: 1) $(B(H), \cdot, \|\ \|_{op}, \overset{\text{adjoint}}{*})$, 2) $K(H)$

2) $(C_0(X), \cdot, \|\ \|, *)$, $f^*(x) := \overline{f(x)}$

} C^*-algs

Banach $*$-alg: { 4) $(L^1(G), *, \|\ \|_1, *)$ G loc. cpt.

$f^*(g) := \Delta_G^{-1}(g) \overline{f(g^{-1})}$, e.g. left μ.

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Examples:

1. X cpt. $A = C(X)$ $\quad \|f\| = \sup_{x\in X} |f(x)|_{\mathbb{C}} < +\infty$

$$\|fg\| = \sup_{x\in X} |f(x)g(x)| \le \sup_{x\in X} |f(x)||g(x)| \le \sup_{x\in X}|f(x)| \sup_{x\in X}|g(x)|$$

$\|f\|\,\|g\|.$

1 unital

$(f+g)(x) := f(x)+g(x)$, $\quad (\lambda f)(x) = \lambda f(x) \quad \forall x \in X.$

2. X loc. cpt. $A = C_0(X) = \{ f : X \to \mathbb{C} \mid f$ cont. $+$ vanishes at $+\infty \}$

$\forall \varepsilon > 0 \ \exists\, C$ cpt $\forall x \in X \setminus C \ |f(x)| < \varepsilon.$

$\|f\| = \sup_{x\in X} |f(x)| < +\infty \qquad \varepsilon = 1$

In classes,
these exmpls were done at the beginning of the chapter.

$b := \sup_{x\in C} \left| f|_C(x) \right| < +\infty$ cont. comp.

$\max\{1, b\}.$

→ Nonunital if X not cpt.

3. $(L^1(G), *, \| \ \|_1, +, 0)$, $*$ convolution, G any locally cpt. group

je Banachova algebra

$\|f * g\|_1 \le \|f\|_1 \|g\|_1$

Moreover, Banach $*$-algebra

involution

$f^*(x) := \Delta_G^{-1}(g)\, \overline{f(g^{-1})}$ $\quad \mu_G$ left Haar $\quad \|f\| = \int_G |f|\, d\mu_G$

$\|a\|^2 \le \|aa^*\| \le \|a\|\|a^*\| \Rightarrow \|a\| = \|a^*\|$

Proof of these facts (computational): each notes or Echterhoff, Deitmar.

we mean cont. linear Hilb. sp. maps

4. $(\mathrm{End}(H), \cdot, 1)$: $\boxed{\|AB\|_{op} \le \|A\|_{op} \|B\|_{op}}$

Hilbert

1) $\|A^*\|_{op} = \|A\|_{op}$

$(AB)^* = B^*A^*$ antihomom.

2) $\|A^*A\|_{op} = \|A\|_{op}^2$

follows from $\| \ \|_{op} = \sup$

Corollary of sp. radius thm.: If A is a C^*-algebra $\Rightarrow$

$r(a) = \|a\|$, $\forall a$ normal (normal $\equiv aa^* = a^*a$).

Proof.: 1) $a \overset{\text{self. adj.}}{=} a^* \Rightarrow \|a^2\| = \|aa^*\| \overset{C^*\text{-id.}}{=} \|a\|^2 \Longrightarrow \|a^{2^n}\| = \|a\|^{2^n}$

$$\|a\| = \lim_n \|a^{2^n}\|^{\frac{1}{2^n}} = r(A)$$

sp. radius thm. (+proof of it with the ex. of the limit)

2) a normal $\Rightarrow aa^*$ symmetric

$$\|a\|^2 \overset{C^*\text{-id}}{=} \|aa^*\| \overset{1)}{=} r(A) \overset{\text{sp. rad.}}{=} \lim_n \|(aa^*)^n\|^{\frac{1}{n}} \leq$$

$$\leq \lim_n \left(\|a\|^n \cdot \|a^*\|^n\right)^{\frac{1}{n}} \leq \lim_n \left(\|a^n\|^{\frac{1}{n}} \|a^{*n}\|^{\frac{1}{n}}\right) = r(A)^2 \underset{\text{bound. of } \sigma}{\leq} \|a\|^2. \quad \square$$

Lemma on reality: If it is a C^*-alg. $\Rightarrow \forall m \in \Delta_A$, $\forall a \in A$: $m(a^*) = \overline{m(a)}$.

Proof: $a = \mathrm{Re}(a) + i\,\mathrm{Im}(a)$, $\mathrm{Re}(a) = \frac{1}{2}(a+a^*)$ symm., $\mathrm{Im}(a) = \frac{1}{2i}(a-a^*)$ symmetric

m is $\mathbb{C}$-lin $\Rightarrow$ suff. take $a = a^*$.

(assume $1 \in A$) We prove $m(a) \in \mathbb{R}$. Suppose $m(a) = x + iy$.

$t \in \mathbb{R}$: Take $a_t = a + it$. $m(a_t) = m(a + it) = x + iy + it =$

$x + i(y+t) \Rightarrow |m(a_t)|_{\mathbb{C}}^2 = |x + i(y+t)|_{\mathbb{C}}^2 =$

$x^2 + (y+t)^2$ But $\|m\| \leq 1$. So $|m(a_t)|^2 \leq$

$\leq \|a_t a_t^*\| = \|(a+it)(a^*-it)\| = \|aa^* + t^2 + 0it\|$ ($a = a^*$)

$= \|a^2 + t^2\| \leq \|a\|^2 + t^2 \leq \|a\|^2 + t^2$

Sum-up $x^2 + y^2 + 2yt + t^2 \leq \|a\|^2 + t^2$

$x^2 + y^2 + 2yt \leq \|a\|^2 \ \forall t \Rightarrow y = 0.$

$\square$

Definition: By a left/right/both sided ideal I in an (assoc.) algebra A, we mean $I \subseteq A$ s.t. $a_1+a_2 \in I \Leftarrow a_1, a_2 \in I$ and: $a \in I$ and $a' \in A \Rightarrow a'a \in I / aa' \in I / a'aa'' \in I$, $a'' \in A$

i.e., an ideal in the ring $(A, +, \cdot)$.

Proper ideal $I \equiv I \subsetneq A$ (as usual)

Maximal ideal $I \subseteq A \equiv$ maximal among the proper ideals

maximal as usual $\equiv$ doesn't exist strictly bigger (wrt. to $\subseteq$)

Lemma 1: A unital assoc. alg. Any proper ideal is contained in a maximal ideal.

Proof: $\mathcal{P} := \{ M \mid M \text{ ideal}, M \supseteq I, M \text{ proper} \}$ for a proper ideal I. Zorn lemma assumptions

1) $I \in \mathcal{P} \Rightarrow \mathcal{P} \neq \emptyset$

2) $\mathcal{P}$ ordered part. by $\subseteq$

• $C \subseteq \mathcal{P}$ chain $\Rightarrow C' := \bigcup C$, (a) $\forall M \in C$ $M \overset{(\le)}{\subseteq} \bigcup C = C'$ (upper bound), (b) C' ideal: $c_1, c_2 \in \bigcup C \Rightarrow c_1, c_2 \le C \Rightarrow c_1 \in M_1, c_2 \in M_2$

C well ordered $\Rightarrow M_1 \subseteq M_2$ or $M_2 \subseteq M_1 \Rightarrow c_1, c_2 \in M_i$ $(\exists i) \Rightarrow c_1 + c_2 \in M_i$ (since M_i ideal) $\Rightarrow c_1 + c_2 \in \bigcup C = C'$

similarly $c \in \bigcup C \Rightarrow c \in M_1$, $a \in A$ $ac \in M_1 \subseteq \bigcup C$, $ac \in \bigcup C$ / (c) contains I (triv.), (d) proper?

if $C' = A \Rightarrow 1 \in C' \Rightarrow$ again $1 \in M_i \Rightarrow M_i$ not proper ↯ ($\in \mathcal{P}$)

conclusion

$\therefore$ $\mathcal{P}$ contains maximal and cont. I element. It's proper & def of $\mathcal{P}$. It is thus a maximal proper ideal containing I. □

Lemma 2: Each maximal is of form $\ker m$ for a $\neq$ homomorphism $m: A \to \mathbb{C}$ (alg. homom.)

11''

Proof: 1) $m \neq 0$ $\ker m$ maximal for dim reasons

$$A \simeq \ker m \oplus \operatorname{Im} m' \simeq \mathbb{C}$$

2)

Lemma 3 A unital. a) I proper & $a \in I \Rightarrow a$ noninv.

b) If A unproven comm: a noninv. $\Rightarrow Aa$ proper.

Proof: a) $\{$ a inv. $\Rightarrow 1 = a^{-1} a \in I \Rightarrow b = b \cdot 1 \in I$

$\forall b \in A$.

b) $A = Aa \Rightarrow 1 \in Aa \Rightarrow 1 = a'a = aa'$

$\Rightarrow a$ inv. $\square$

Lemma 4: A comm. with unit. $\forall a \in A$:

$\sigma(a) = \operatorname{Im} \hat{a}$ $\left[\hat{a}: \Delta_A \to \mathbb{C},\ \hat{a} \in C(\Delta_A)\right]$.

Proof: $\operatorname{Im} \hat{a} = \{\hat{a}(m) \mid m \in \Delta_A\} = \{m(a) \mid m \in \Delta_A\}$

1) $m(a) \in \sigma(a)$?

$m(m(a)1 - a) = 0 \Rightarrow m(a)1 - a \in \operatorname{Ker} m \xrightarrow{\text{Lemma above a)}}$ proper

$m(a)1 - a$ noninv. $\Rightarrow m(a) \in \sigma(a)$

2) $\lambda \in \sigma(a) \Rightarrow a - \lambda 1$ non.inv. $\Rightarrow A(a-\lambda 1)$ proper $\Downarrow$ It is in

some maximal $\Rightarrow m(a - \lambda 1) = 0$

$m(a) = \lambda \Rightarrow \lambda \in \operatorname{Im} \hat{a}$. $\square$

Corollary: $r(a) = \|\hat{a}\|_{C(\Delta_A)}$. $\Leftarrow$ A comm. with unit.

Proof: $r(a) = \sup\{|\lambda| \mid \lambda \in \sigma(a)\} =$

$= \sup\{|\lambda| \mid \lambda \in \operatorname{Im}(\hat{a})\} = \|\hat{a}\|_{C_0(\Delta_A)}$

$\uparrow$
Lemma just above

Lemma2: ~~Thm:~~ $\mathcal{M} = \{ M \leq A \mid M \text{ maximal} \} \overset{\cong}{\longrightarrow} \Delta_A$ (set bijection)

$11''''$

Proof:

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[rr, bend right, "\varphi_M"'] & A/M \arrow[r, "i"] & \mathbb{C} \end{tikzcd}$$

$\varphi_M = i \circ \pi : A \to \mathbb{C}$ $\neq 0$ homom. algs.

$0 \neq \varphi$ homom.

$$A \to A/\ker\varphi \xrightarrow{i} \mathbb{C}$$

$$\varphi_{\ker\varphi} = i \circ \pi \qquad \varphi_{\ker\varphi}(\underset{\ker\varphi}{\overset{\cup}{a}}) = i \circ \pi(0) = 0$$

$i\,\pi(a) = 0$

$\pi(a) = 0 \Leftrightarrow a \in \ker\varphi \qquad \pi(a) = [a] = 0 \Leftrightarrow a \in \ker\varphi$

$M \mapsto \varphi_M$

$\varphi \mapsto \ker\varphi$

Remark: $\widehat{a^*}(m) = \overline{\hat{a}(m)} = \overline{\hat{a}}(m)$ def of $\overline{f}$ $\left[f : \Delta_A \to \mathbb{C},\ \overline{f}(m) := \overline{f(m)},\ \text{where } m \in \Delta_A \right]$

Cons. of Gelf. map. thm.

$\boxed{\widehat{a^*} = \overline{\hat{a}}}$. "$*$ ^ commute".

Thm. (Gelfand–Najmark): A comm. C^*-alg. Then

i/ĭ = pronounced as 'iot'

the Gelfand map $a \in A \mapsto \hat{a} \in \mathcal{C}_0(\Delta_A)$ is an isometric $*$-isomorphism: $A \xrightarrow{\cong} \mathcal{C}_0(\Delta_A)$; i.e.,

$\|\hat{a}\|_{\mathcal{C}_0(\Delta_A)} = \|a\|_A$ & $\widehat{a^*} = \hat{a}^* (= \overline{\hat{a}})$.

Recall

Proof: 1. $r(b) = \|b\|_A$ for b normal (cor. of sp. radius)

C^*-id. in $C_0(\Delta_A)$

2. $\|\hat{a}\|^2_{C_0(\Delta_A)} = \|\overline{\hat{a}}\,\hat{a}\|_{C_0(\Delta_A)} =$ Corollary just above $= r(a^*a) =$

$= \|\widehat{a^*}\,\hat{a}\|_{C_0(\Delta_A)} = \|\widehat{a^* a}\|_{C_0(\Delta_A)}$

Remark just above

Thm. on Gelf. map (3.): ^ is alg. homomorph.

Cor. of sp. rad.

$= \|a^* a\| = \|a\|^2_A$.

C^*-id

Gelf. map is isometry

$\left[\begin{array}{l} \text{Is } a^*a \text{ normal?} \\ (a^*a)(a^*a)^* = a^*a\,a^*a \\ (a^*a)^*(a^*a) = a^*a\,a^*a \\ a^*a \text{ is even self-adjoint.} \end{array} \right]$

$\left[b^*b = bb = bb^* \quad \text{s.a.} \Rightarrow \text{normal"} \right]$

3. Stone–Weierstraß

1) $m_1 \neq m_2\ \exists\ \hat{a}(m_1) \neq \hat{a}(m_2)$? If $\forall\ \hat{a}(m_1) = \hat{a}(m_2)$ $\forall a \Rightarrow$ $m_1(a) = m_2(a)$ $m_1 = m_2$ ↯

2) $\hat{a}(m) \neq 0\ \forall m\ \exists a$? $m \in \Delta_A \Rightarrow m \neq 0 \Rightarrow$ $\exists a : m(a) \neq 0$ $\hat{a}(m) \neq 0$

3) $\hat{a} \in \hat{A} \subseteq \mathcal{C}_0(\Delta_A)$ 13

$\hat{a}^* = \widehat{a^*}$ by reality lemma.

Thus $\hat{A} \subseteq \mathcal{C}_0(\Delta_A)$ is dense by the Stone--Weierstrass theorem.

It is also closed since $\hat{\ }$ is an isometry.

$\Rightarrow \hat{A} = \mathcal{C}_0(\Delta_A)$, thus $\hat{A} \simeq \mathcal{C}(X)$. $\square$

Pontrjagin duality & Poisson fmla

1) G abelian locally compact

2) $\hat{G} := \hat{G}_{f.d}$. Repr. of ablian : $\hat{G}_{\beta} = \hat{G}_{1\text{-dim}}$.

Take

Recall $\rho_1, \rho_2 : G \to Aut(\mathbb{C})$ & $\rho_1 \cong \rho_2 \Rightarrow \rho_1 = \rho_2$.

3) $\hat{G} \subseteq C(G, \mathbb{C})$. On $C(G, \mathbb{C})$ take compact-open topology $\Rightarrow \hat{G}$ is a locally compact Hausdoff space.

4) Recall: $\hat{G}$ is a group $(\chi_1 \cdot \chi_2)(g) := \chi_1(g)\chi_2(g)$

$[\chi_1 \cdot (\chi_2 \cdot \chi_3)](g) = \chi_1(g)(\chi_2(g)\chi_3(g)) = (\chi_1(g)\chi_2(g))(\chi_3(g)) = [(\chi_1 \cdot \chi_2) \cdot \chi_3](g) \quad \forall g \in G$

$\chi^{-1}(g) := \chi(g)^{-1}$, $e(g) := 1 \in \mathbb{C}$

5) Not dificult $\cdot$ & $^{-1}$ continuous $\Rightarrow \hat{G}$ is a locally compact group; also easy $\hat{G}$ is abelian (since $\mathbb{C}$'s).

6) $\hat{\hat{G}} := \widehat{(\hat{G})}_{f.d.} = \widehat{(\hat{G})}$ again a loc. cpt. abelian.

$a \in \hat{\hat{G}} \Rightarrow$ $\underset{\hat{G}}{+}$ $a : \hat{G} \to Aut(\mathbb{C})$ $a(\chi) = ?$

7) $\delta : G \to \hat{\hat{G}}$ $\quad \underbrace{\delta(g)}_{\in \hat{\hat{G}}}(\chi)$, $g \in G$, $\chi \in \hat{G}$

$\underbrace{\delta(g)}_{a}(\chi) := \chi(g)$.

Pontrjagin map.

8) $\delta(g)(\chi\mu) = (\chi\mu)(g) = \chi(g)\mu(g) = \delta(g)\chi \cdot \delta(g)\mu$

$\Rightarrow \delta(g)$ homom.

9) δ je homom.: $[\delta(gh)](\chi) = \chi(gh) = \chi(g)\chi(h)$
$= \delta(g)(\chi)\cdot\delta(h)(\chi) =$
$= [\delta(g)\cdot\delta(h)](\chi) \quad \forall\chi$

$\delta(gh) = \delta(g)\cdot\delta(h) \quad \forall g,h\in G$

10) $\delta(g)$ is cont. : $\chi_i \to \chi$; $\lim_i[\delta(g)(\chi_i)] =$
$= \lim_i[\chi_i(g)] = \chi(g)$.

<u>Pontrjagin thm</u>: G loc. cpt. abelian. Then $\delta: G\to\hat{\hat{G}}$ is an isomorphism (onto $\hat{\hat{G}}$) of topological groups.

<u>Proof</u>: Deitm. / Echt.

<u>Examples</u>: 1) $\hat{\hat{S^1}} \cong \hat{\mathbb{Z}} \cong S^1$
2) $\hat{\hat{\mathbb{Z}}} \cong \hat{S^1} \cong \mathbb{Z}$

3) $\hat{\mathbb{R}} = \mathbb{R} \Rightarrow \hat{\hat{\mathbb{R}}} = \mathbb{R}$
($\hat{G}\cong G$ autodual)
def.

4) Doesn't hold for non-comm:
$G = SU(2)$, $\hat{G} \cong \mathbb{N}\cup\{0\}$ not a group.
even if $K(\mathbb{N}\cup\{0\}) \cong \mathbb{Z}$, $\hat{\mathbb{Z}} \cong S^1 \not\cong SU(2)$.

G loc. cpt. abel $\Rightarrow$ take a Haar measure μ_G
11) $\hat{G}$ loc. cpt. abelian $\Rightarrow$ $\exists$ (both-sided) Haar measure.
Moreover the measure can be normalized ~~s.t.~~.
denote it $\mu_{\hat{G},\mathrm{Plan.}}$, s.t.

<u>Thm. (gen. Plancherel)</u>: G locally cpt. abelian $\Rightarrow$ $\not\exists$ $L^2(G,\mu_G) \xrightarrow{\sim} L^2(\hat{G},\mu_{\hat{G}}^{\mathrm{Plan}})$ is an isomorphism of TVS. Proof: Deitm. / Echter. $\emptyset$. $\square$

Remark: G cpt. ab. $\Rightarrow \hat{G}$ discrete abel.
G diskr. abel $\Rightarrow \hat{G}$ compact abel.

Poisson summation

1. $f:\mathbb{R}\to\mathbb{C}$ contin. $g(x) := \sum_{\ell\in\mathbb{Z}} f(x+\ell)$, $x\in\mathbb{R}$ ① Assume $\sum$ converges absolutely.

2. g is 1-periodic. Four. series of g: $g(x)=\sum_k c_k e^{2\pi i k x}$ (② suppose it exists in the point-wise way)

3. $\sum_\ell f(\ell) = g(0) = \sum_k c_k = \sum_{k\in\mathbb{Z}} \int_0^1 g(y) e^{-2\pi i k y}\,dy$ unif. conv. ③ assume!

$= \sum_k \int_0^1 \sum_{\ell\in\mathbb{Z}} f(y+\ell) e^{-2\pi i k y}\,dy =$

$= \sum_k \left(\sum_{\ell\in\mathbb{Z}} \int_0^1 f(y+\ell) e^{-2\pi i k y}\,dy \right) =$

$= \sum_k \int_{\mathbb{R}} f(y) e^{-2\pi i k y}\,dy = \sum_k \hat{f}(k)$!

4. $\sum_k f(k) = \sum_k \hat{f}(k)$ Poiss. summ.

Thm (formal statement): $f\in L^1_{bc}(\mathbb{R})$, f piecewise C^1 with exc. of finitely many points. $\varphi(x) := \begin{cases} f'(x) & \exists f' \\ 0 & \text{otherwise} \end{cases}$

$x^2 f$ & $x^2\varphi$ be bounded on $\mathbb{R}$. Then

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \hat{f}(k)$$

Proof: ϕ, as above, but assumpt. ① – ③ are proved $\square$

5. $B \subseteq A$ closed subgr. of loc. comp. abel. A

A/B abelian, topological, moreover locally cpt.

6. $B^\perp := \{\chi \in \hat{A} \mid \chi(b) = 1 \ \forall b \in B\}$; $\underline{f \in L^1(A)}$
$\subseteq \hat{A}$

$f^B : A/B \to \mathbb{C}$ def. $f^B(aB) := \int_{b \in B} f(ab)\, d\mu_B(b)$

<u>Thm</u> (group version of Poisson summ): $\widehat{f^B} = \hat{f}|_{B^\perp}$ &

$$\int_{b \in B} f(ab)\, d\mu_B(b) = \int_{\chi \in B^\perp} \hat{f}(\chi)\chi(a)\, d\mu_B(\chi) \quad \forall \text{ a.e. } \underline{a}$$

<u>Proof</u>: $\emptyset$, Deitm.-Echterhoff.

Application: $A = (\mathbb{R}, +)$, $A \cong \hat{A} \cong \mathbb{R}$

$B \subseteq \mathbb{Z}$, $\hat{B} = S^1$, $A/B = \mathbb{R}/\mathbb{Z} = S^1$

$\widehat{A/B} = \hat{S^1} = \mathbb{Z}$, $a = 0$

lhs $\int f(b)\, d\mu_{\mathbb{Z}}(b) = \sum_{b \in \mathbb{Z}} f(b)$

rhs $\chi : \mathbb{R} \to \mathrm{Aut}(\mathbb{C})$

$\chi(x)_\eta = \chi_y(x) = e^{-2\pi i x y}$

$\chi(m) = e^{-2\pi i m y} = 1 \quad \forall m \in \mathbb{Z}$

$\Rightarrow y \in \mathbb{Z}$

Thus $B^\perp \cong \mathbb{Z}$

$$\int_{\chi \in \mathbb{Z}} \hat{f}(\chi)\chi(0)\, d\mu_{\mathbb{Z}}(\chi) = \sum_{\chi \in \mathbb{Z}} \hat{f}(\chi).$$

The classical Poisson. fml.

$\square$

Application of classical Poiss. sum.

Def: $\Theta(t) := \sum\limits_{k\in\mathbb{Z}} e^{-\pi t k^2}$, $t>0$; so called θ-function.

$= 2\sum\limits_{k\in\mathbb{N}} e^{-\pi t k^2} + 1$

Remark: Convergence: $\dfrac{a_{k+1}}{a_k} = \dfrac{e^{-\pi t (k+1)^2}}{e^{-\pi t k^2}}$ $e^{-\pi t(2k+1)} \leq e^{-\pi t} < 1$

Known formulas: $(T_a f)(x) := f(T_{a^{-1}}(x))$
$= f(x-a)$ (recall)

$D^\alpha := \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$ $\qquad \mathcal{F}\circ D^\alpha = (\pm 2\pi i\xi)^\alpha \mathcal{F}$

$\mathcal{F}\circ T_a = e^{\pm 2\pi i a\xi}\mathcal{F}$

e.t.c.

Dilation: $a>0$ $\quad \delta_a(x) := ax$, $x\in\mathbb{R}$
$(\delta_a f)(x) := f(\delta_{a^{-1}}x) = f(a^{-1}x)$.

Lemma: $\mathcal{F}\circ\delta_a = a^n\,\delta_{\frac{1}{a}}\circ\mathcal{F}$.

Proof: $[(\mathcal{F}\circ\delta_a)(f)](\xi) = [\mathcal{F}[f(a^{-1}\cdot)]](\xi) =$

$= \int f(a^{-1}x)\, e^{-2\pi i x\xi}\, dx = \left| \begin{matrix} y = a^{-1}x \\ dy = a^{-n}dx \end{matrix} \right|$

$= \int f(y)\, e^{-2\pi i y a\xi}\, a^n\, dy =$

$= a^n \int f(y)\, e^{-2\pi i a y\xi}\, dy = a^n \hat{f}(a\xi) =$

$= a^n (\delta_{\frac{1}{a}}\hat{f})(\xi)$. $\square$

Thm.: $\Theta\left(\frac{1}{t}\right) = \sqrt{t}\,\Theta(t) \quad \forall t>0$ (so called hard invariance)

Proof: a) $f_t(x) := e^{-\pi t x^2}$, $t>0$, i.e., $f_t = \delta_{\frac{1}{\sqrt{t}}} f_1$

$\left(f_1(x) = e^{-\pi x^2}, \text{ old friend}\right)$

$$\mathcal{F} f_t = \mathcal{F} \delta_{\frac{1}{\sqrt{t}}} f_1 \overset{\text{Lemma.}}{=} \frac{1}{\sqrt{t}} \delta_{\sqrt{t}} \mathcal{F} f_1 \overset{\text{exc. E.e. Schuster}}{=}$$

$$= \frac{1}{\sqrt{t}} \delta_{\sqrt{t}} f_1 = \frac{1}{\sqrt{t}} e^{-\pi\left(\frac{x}{\sqrt{t}}\right)^2} = \frac{1}{\sqrt{t}} e^{-\frac{\pi x^2}{t}}.$$

b) Poisson fmla: $\sum \mathcal{F} f_t(k) = \sum f_t(k)$

$$\frac{1}{\sqrt{t}} \sum_k e^{-\frac{\pi k^2}{t}} = \sum_k e^{-\pi k^2 t} \qquad \square$$

Remark: Comm. properties of $\mathcal{F}$ seem to be important.