Cvičení 3

Řešíme ODR

- $Ly = \delta$ ~~a~~ v $\mathcal{S}'(\mathbb{R})$ nebo $\mathcal{D}'(\mathbb{R})$
- $Ly = \sum_{k=0}^{n} a_k(x) D^k y$ v $\mathcal{D}'(\mathcal{S}')$

Potom $Ly^+ = 0$ na $[0,\infty)$ + podmínky: $D^\ell(y^+ - y^-)|_0 = 0$ $\ell = 0, 1, \dots, n-2$

$Ly^- = 0$ na $(-\infty, 0]$

~~$D^{n-1}(y^+ - y^-)|_0 = \frac{1}{a_n(0)}$~~

$D^{n-1}(y^+)(0_+) - D^{n-1}(y^-)(0_-) = \frac{1}{a_n(0)}$

$\Rightarrow$ y^+ $x \geq 0$ řeší naši úlohu
y^- $x < 0$

Příklady

$\boxed{1}$ $y' + ay = \delta$ $a \in \mathbb{R}$

1) $a > 0$
$y(x) = Ce^{-ax}$ řeší $\Rightarrow$

v $\mathcal{D}'(\mathbb{R})$ $y(x) = \begin{cases} C_1 e^{-ax} & x > 0 \\ C_2 e^{-ax} & x < 0 \end{cases}$ a $C_1 - C_2 = 1$

v $\mathcal{S}'(\mathbb{R})$ $y(x) = \begin{cases} e^{-ax} & x > 0 \\ 0 & x < 0 \end{cases}$

2) $a < 0$ v $\mathcal{S}'(\mathbb{R})$ je změna

$y(x) = \begin{cases} 0 & x > 0 \\ e^{-ax} & x < 0 \end{cases}$

3) $a = 0$

v $\mathcal{D}'(\mathbb{R})$ $y(x) = \begin{cases} C_1 & x > 0 \\ C_2 & x < 0 \end{cases}$ $C_1 - C_2 = 1$

v $\mathcal{S}'(\mathbb{R})$ totéž

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$$y'' + a^2 y = \delta \qquad a \geq 0$$

1) $a > 0$

$y^+(x) = C_1^+ \cos(ax) + C_2^+ \sin(ax)$ platí v $\mathcal{S}'(\mathbb{R})$ i $\mathcal{D}'(\mathbb{R})$ $\Rightarrow$ nebude rozdíl

$y^-(x) = C_1^- \cos(ax) + C_2^- \sin(ax)$

$y^+(0_+) = y^-(0_-) \Rightarrow C_1^+ = C_1^-$

$(y^+)'(0_+) - (y^-)'(0_-) = 1 \Rightarrow a \cdot C_2^+ - a C_2^- = 1$

$$\Rightarrow \boxed{\begin{aligned} y^+(x) &= C_1 \cos(ax) + C_2 \sin ax \qquad x \geq 0 \\ y^-(x) &= C_1 \cos(ax) + \left(C_2 - \tfrac{1}{a}\right) \sin ax \end{aligned}} \qquad a > 0$$

2) $a = 0$

$y^+(x) = C_1^+ x + C_2^+$ v $\mathcal{S}'(\mathbb{R})$ a $\mathcal{D}'(\mathbb{R})$ taktéž

$y^-(x) = C_1^- x + C_2^-$

$C_2^+ = C_2^-$

$C_1^+ - C_1^- = 1$

$$\Rightarrow \boxed{\begin{aligned} y^+(x) &= C_1 x + C_2 \qquad x \geq 0 \\ y^-(x) &= (C_1 - 1)x + C_2 \qquad x < 0 \end{aligned}}$$

1) $a > 0.$

$$\lambda^4 + a^4 = 0$$

$$\underline{\lambda = a e^{i\frac{\pi}{4} + k\frac{\pi}{2}}} \qquad k = 0,1,2,3$$

$$\lambda_1 = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)a$$

$$\lambda_2 = \left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)a$$

$$\lambda_3 = \left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)a$$

$$\lambda_4 = \left(+\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)a$$

$$y^+(x) = C_1^+ e^{\frac{\sqrt{2}}{2}ax} \cos\left(\frac{\sqrt{2}}{2}ax\right) + C_2^+ e^{\frac{\sqrt{2}}{2}ax} \sin\left(\frac{\sqrt{2}}{2}ax\right) + C_3^+ e^{-\frac{\sqrt{2}}{2}ax} \cos\left(\frac{\sqrt{2}}{2}ax\right) + C_4^+ e^{-\frac{\sqrt{2}}{2}ax} \sin\left(\frac{\sqrt{2}}{2}ax\right)$$

$$y^-(x) = C_1^- e^{\frac{\sqrt{2}}{2}ax} \cos\left(\frac{\sqrt{2}}{2}ax\right) + C_2^- e^{\frac{\sqrt{2}}{2}ax} \sin\left(\frac{\sqrt{2}}{2}ax\right) + C_3^- e^{-\frac{\sqrt{2}}{2}ax} \cos\left(\frac{\sqrt{2}}{2}ax\right) + C_4^- e^{-\frac{\sqrt{2}}{2}ax} \sin\left(\frac{\sqrt{2}}{2}ax\right)$$

<u>Podmínky:</u>

$$C_1^+ + C_3^+ = C_1^- + C_3^-$$

$$C_1^+\left(\cancel{\frac{\sqrt{2}}{2}a}\right) + C_2^+\left(\cancel{\frac{\sqrt{2}}{2}a}\right) - C_3^+\left(\cancel{\frac{\sqrt{2}}{2}a}\right) + C_4^+\left(\cancel{\frac{\sqrt{2}}{2}a}\right) = C_1^-\left(\cancel{\frac{\sqrt{2}}{2}a}\right) + C_2^-\left(\cancel{\frac{\sqrt{2}}{2}a}\right) - C_3^+\left(\cancel{\frac{\sqrt{2}}{2}a}\right) + C_4^-\left(\cancel{\frac{\sqrt{2}}{2}a}\right)$$

$$\cancel{C_1^+}\left(\cancel{\frac{1}{2}a^2 - \frac{1}{2}a^2}\right) + 2\cdot C_2^+\left(\frac{1}{2}a^2\right) + \cancel{C_3^+}\left(\cancel{\frac{1}{2}a^2 - \frac{1}{2}a^2}\right) - 2C_4^+\left(\cancel{\frac{1}{2}a^2}\right) =$$

$$= \cancel{C_1^-}\left(\cancel{\frac{1}{2}a^2 - \frac{1}{2}a^2}\right) + 2C_2^-\left(\cancel{\frac{1}{2}a^2}\right) + \cancel{C_3^-}\left(\cancel{\frac{1}{2}a^2 - \frac{1}{2}a^2}\right) - 2C_4^-\left(\cancel{\frac{1}{2}a^2}\right)$$

$$C_1^+\left(\tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a - 3\cdot\tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a\right) + C_2^+\left(3\tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a - \tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a\right) + C_3^+\left(-\tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a + 3\tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a\right)$$

$$+ C_4^+\left(3\tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a - \tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a\right)$$

$$= C_1^-\left(\tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a - 3\tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a\right) + C_2^-\left(3\tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a - \tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a\right) + C_3^-\left(-\tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a + 3\tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a\right)$$

$$+ C_4^-\left(3\tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a - \tfrac{1}{2}a^2\tfrac{\sqrt{2}}{2}a\right) + 1$$

Celkem tedy

$$C_1^+ + C_3^+ = C_1^- + C_3^-$$

$$C_1^+ + C_2^+ - C_3^+ + C_4^+ = C_1^- + C_2^- - C_3^- + C_4^-$$

$$C_2^+ - C_4^+ = C_2^- - C_4^-$$

$$\underbrace{-C_1^+ + C_2^+ + C_3^+ + C_4^+ = -C_1^- + C_2^- + C_3^- + C_4^- + \frac{2}{\sqrt{2}a^3}}$$

v $S'(\mathbb{R})$: musí $C_1^+ = C_2^+ = C_3^- = C_4^- = 0$

v $D'(\mathbb{R})$: $\cancel{2}(C_2^+ + C_4^+) = \cancel{2}(C_2^- + C_4^-) + \frac{\cancel{2}\,1}{\sqrt{2}a^3}$

$$C_2^+ - C_4^+ = C_2^- - C_4^-$$

$$\Rightarrow \quad \underline{C_2^+ = C_2^- + \frac{1}{\sqrt{2}a^3}} \qquad \underline{C_4^+ = C_4^- + \frac{1}{\sqrt{2}a^3}}$$

$$\underline{C_1^+ = C_1^- - \frac{2}{\sqrt{2}a^3}} \qquad \underline{C_3^+ = C_3^- + \frac{2}{\sqrt{2}a^3}}$$

$$\boxed{\text{v } S'(\mathbb{R}):\quad \begin{array}{llll} C_1^+ = 0 & C_2^+ = 0 & C_3^+ = \frac{2}{\sqrt{2}a^3} & C_4^+ = \frac{1}{\sqrt{2}a^3} \\ C_1^- = \frac{2}{\sqrt{2}a^3} & C_2^- = -\frac{1}{\sqrt{2}a^3} & C_3^- = 0 & C_4^- = 0 \end{array}}$$

Je-li $a=0$

$y^{(4)} = \delta$

$\Rightarrow \; y^+ = C_1^+ + C_2^+ x + C_3^+ x^2 + C_4^+ x^3$ v $\mathcal{D}'$, $\mathcal{D}'$ kolem 0

$y^- = C_1^- + C_2^- x + C_3^- x^2 + C_4^- x^3$

$$\boxed{\begin{aligned} C_1^+ &= C_1^- \\ C_2^+ &= C_2^- \\ C_3^+ &= C_3^- \\ 6C_4^+ &= 6C_4^- + 1 \end{aligned}}$$

5 $\; -y^{(4)} + a^2 y'' = \delta$

$-\lambda^4 + a^2\lambda^2 = 0$

$\lambda^2 (a^2 - \lambda^2) = 0$

$\lambda_1 = \lambda_2 = 0 \quad \lambda_3 = a \quad \lambda_4 = -a$

$y^+(x) = C_1^+ + C_2^+ x + C_3^+ e^{ax} + C_4^+ e^{-ax}$

$y^-(x) = C_1^- + C_2^- x + C_3^- e^{ax} + C_4^- e^{-ax}$

$C_1^+ + C_3^+ + C_4^+ = C_1^- + C_3^- + C_4^-$

$C_2^+ + a C_3^+ - a C_4^+ = C_2^- + a C_3^- - a C_4^-$

$a^2 C_3^+ + a^2 C_4^+ = a^2 C_3^- + a^2 C_4^-$

$a^3 C_3^+ - a^3 C_4^+ = a^3 C_3^- - a^3 C_4^- + 1$

$\Downarrow$

$C_3^+ = C_3^- + \frac{2}{a^3} \qquad C_1^+ = C_1^-$

$C_4^+ = C_4^- - \frac{2}{a^3} \qquad C_2^+ = C_2^- - \frac{4}{a^2}$

v $\mathcal{D}'(\mathbb{R})$

v $y'(\mathbb{R})$ je nutné

$C_3^+ = C_4^- = 0$

(6) $y^{(4)} + y'' + y = \delta$ (22)

$$\lambda^4 + \lambda^2 + 1 = 0$$

$$\mu^2 + \mu + 1 = 0$$

$$\mu_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\mu_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\frac{2}{3}\pi} \qquad \mu_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{-i\frac{2}{3}\pi} = e^{i\frac{4}{3}\pi}$$

$$\lambda_1 = e^{i\frac{\pi}{3}} \qquad \lambda_2 = e^{i\frac{4}{3}\pi} \qquad \lambda_3 = e^{i\frac{2}{3}\pi} \qquad \lambda_4 = e^{i\frac{5}{3}\pi}$$

$$y^+ = C_1^+ e^{\frac{1}{2}x}\left(\cos\frac{\sqrt{3}}{2}x\right) + C_2^+ e^{\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x + C_3^+ e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x + C_4^+ e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x$$

$$y^- = C_1^- e^{\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x + C_2^- e^{\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x + C_3^- e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x + C_4^- e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x$$

v $y'(x)$ je nač $\boxed{C_1^+ = C_2^+ = C_3^- = C_4^- = 0}$

Rovnice vedení tepla

(1) a) Řešte

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad \text{na } (0,\infty)\times\mathbb{R}^N$$

$$u(0,x) = |x|^2\cos(\beta,x)$$

Uvažujme $u_0(x) = \operatorname{Re}|x|^2 e^{i(\beta,x)} \Rightarrow u(t,x) = \operatorname{Re}(U * u_0)$.

Tedy pro $\tilde{g}(x) = |x|^2 e^{i(\beta,x)}$ je

$$\tilde{u}(t,x) = U * \tilde{g}. \qquad \underset{\hat{U}}{e^{-4\pi^2|\xi|^2 t}}$$

Tedy $\mathcal{F}(\tilde{u})(t,\xi) = \mathcal{F}(U)\cdot\mathcal{F}(\tilde{g})$

$$\mathcal{F}(\tilde{g}) = \mathcal{F}(|x|^2) * \mathcal{F}(e^{i(\beta,x)}) = -\frac{\Delta\delta}{4\pi^2} * \delta_{\frac{\beta}{2\pi}}$$

$$\mathcal{F}(\tilde{u})(t,\xi) = \left(-\frac{1}{4\pi^2}\Delta\delta * \delta_{\frac{\beta}{2\pi}}\right)e^{-4\pi^2|\xi|^2 t}$$

[illegible]

$$\left\langle \mathcal{F}\left(\frac{1}{4\pi^2}\Delta\delta * \delta_{\frac{\beta}{2\pi}}\right)e^{-4\pi^2|\xi|^2 t}, \varphi\right\rangle = -\frac{1}{4\pi^2}\left\langle \Delta\delta * \delta_{\frac{\beta}{2\pi}}, e^{-4\pi^2|\xi|^2 t}\varphi(\xi)\right\rangle$$

$$= -\frac{1}{4\pi^2}\underbrace{\left\langle \delta_{\frac{\beta}{2\pi}}(\xi), \left\langle \Delta\delta(\eta), e^{-4\pi^2|\xi+\eta|^2 t}\varphi(\xi+\eta)\right\rangle\right\rangle}_{J}$$

$$J = \left\langle \delta_{\frac{\beta}{2\pi}}(\xi), \left\langle \delta(\eta)\, \Delta_\eta\left(\varphi(\xi+\eta)\, e^{-4\pi^2|\xi+\eta|^2 t}\right)\right\rangle\right\rangle$$

$$= \left\langle \delta_{\frac{\beta}{2\pi}}(\xi), e^{-4\pi^2|\xi|^2 t}\left(\Delta\varphi(\xi) - 16\pi^2 t\, \xi\cdot\nabla\varphi(\xi) + 64\pi^4 t^2|\xi|^2\varphi(\xi) - 8\pi^2 N t\, \varphi(\xi)\right)\right\rangle$$

$$= e^{-|\beta|^2 t}\left[\Delta\varphi\left(\frac{\beta}{2\pi}\right) + 8\pi t\sum_{k=1}^{N} -\beta_k\frac{\partial\varphi}{\partial\xi_k}\left(\frac{\beta}{2\pi}\right) + \varphi\left(\frac{\beta}{2\pi}\right)\left(|\beta|^2\cdot 16\pi^2 t^2 - 8\pi^2 N t\right)\right]$$

$$= e^{-|\beta|^2 t}\left\langle\left[\Delta\delta + 8\pi t\sum_{k=1}^{N}\beta_k\frac{\partial}{\partial\xi_k}\delta + \left(|\beta|^2 16\pi^2 t^2 - 8\pi^2 N t\right)\delta\right] * \delta_{\frac{\beta}{2\pi}}, \varphi\right\rangle$$

Proto $\mathcal{F}(\tilde{u}) = -\frac{1}{4\pi^2}e^{-|\beta|^2 t}\left(\Delta\delta + 8\pi t\sum_{k=1}^{N}\beta_k\frac{\partial}{\partial\xi_k}\delta + \left(|\beta|^2 16\pi^2 t^2 - 8\pi^2 N t\right)\delta\right) * \delta_{\frac{\beta}{2\pi}}$

Tedy $\tilde{u} = \mathcal{F}_{-1}\left(\mathcal{F}\tilde{u}\right) = -\frac{1}{4\pi^2}e^{-|\beta|^2 t}\,\mathcal{F}_{-1}(\ )\,\mathcal{F}_{-1}\left(\delta_{\frac{\beta}{2\pi}}\right)$

Wiemy $\mathcal{F}\left(e^{i(2\pi(b_1x_1+\dots+b_Nx_N))}\right) = \mathcal{F}\left(e^{i2\pi b_1x_1}\right)\mathcal{F}\left(e^{i2\pi b_2x_2}\right)\dots\mathcal{F}\left(e^{i2\pi b_Nx_N}\right)$

$= \delta_{b_1}\circ\delta_{b_2}\circ\dots\circ\delta_{b_N}$ [illegible] !

Proto $\mathcal{F}_{-1}\left(\delta_{\frac{\beta}{2\pi}}\right) = e^{i(\beta,x)}$

$$\Rightarrow \tilde{u}(t,x) = -\frac{1}{4\pi^2}e^{-|\beta|^2 t}e^{i(\beta,x)}\left(-4\pi^2|x|^2 + 8\pi t\sum_{k=1}^{N}(-2\pi i x_k)\beta_k + 16\pi^2 t^2|\beta|^2 - 8\pi^2 N t\right)$$

Tedy $\boxed{u(t,x) = \operatorname{Re}\tilde{u}(t,x) = e^{-|\beta|^2 t}\left[\left(|x|^2 + 2Nt - 4t^2|\beta|^2\right)\cos(\beta,x) - 4t(\beta,x)\sin(\beta,x)\right]}$

1 b) $u(0,x) = e^{-\alpha|x|^2} \cos(\beta, x)$

Analogicky jako výše

$$\mathcal{F}(\tilde{u}) = \mathcal{F}(U) \times \mathcal{F}(\tilde{g})$$

kde $\mathcal{F}(U) = e^{-4\pi^2|\xi|^2 t}$

$$\mathcal{F}(\tilde{g}) = \mathcal{F}(e^{-\alpha|x|^2}) * \mathcal{F}(e^{i(\beta,x)})$$

$$= \left(\left(\frac{\pi}{\alpha}\right)^{\frac{N}{2}} e^{-\frac{\pi^2|\xi|^2}{\alpha}}\right) * \delta_{\frac{\beta}{2\pi}}$$

Proto

$$\mathcal{F}(\tilde{u}) = \left(\frac{\pi}{\alpha}\right)^{N/2} e^{-\pi^2\frac{|\xi-\frac{\beta}{2\pi}|^2}{\alpha}} e^{-4\pi^2|\xi|^2 t}$$

$$= \left(\frac{\pi}{\alpha}\right)^{N/2} e^{\left(-\frac{\pi^2|\xi|^2}{\alpha} + \frac{\pi}{\alpha}(\xi,\beta) - \frac{|\beta|^2}{4\alpha} - 4\pi^2|\xi|^2 t\right)}$$

$$= \left(\frac{\pi}{\alpha}\right)^{N/2} e^{-\left(\frac{\pi^2}{\alpha}+4\pi^2 t\right)\left(\xi - \frac{\beta}{2\pi(1+4\alpha t)}\right)^2} e^{\left(\frac{\pi^2}{\alpha}\frac{1+4\alpha t}{4\pi^2(1+4\alpha t)^2}|\beta|^2 - \frac{|\beta|^2}{4\alpha}\right)}$$

$$= \left(\frac{\pi}{\alpha}\right)^{N/2} e^{-\left(\frac{\pi^2}{\alpha}+4\pi^2 t\right)\left|\xi - \frac{\beta}{2\pi(1+4\alpha t)}\right|^2} \underbrace{e^{-\frac{|\beta|^2}{4\alpha}\left(1-\frac{1}{4\alpha t+1}\right)}}_{e^{-\frac{|\beta|^2 t}{(1+4\alpha t)}}}$$

$$= \left(\frac{\pi}{\alpha}\right)^{N/2} e^{-\left(\frac{\pi^2}{\alpha}+4\pi^2 t\right)|\xi|^2} e^{-\frac{|\beta|^2 t}{(1+4\alpha t)}} * \delta_{\frac{\beta}{2\pi(1+4\alpha t)}}.$$

Proto $\tilde{u}(t,x) = \left(\frac{\pi}{\alpha}\right)^{N/2} e^{-\frac{|\beta|^2 t}{1+4\alpha t}} \left(\frac{\pi}{\frac{\pi^2}{\alpha}+4\pi^2 t}\right)^{N/2} \cdot e^{-\frac{\pi^2|x|^2}{\frac{\pi^2}{\alpha}+4\pi^2 t}} e^{\frac{i(\beta,x)}{1+4\alpha t}}$

Tedy $u = \mathrm{Re}\,\tilde{u} = \dfrac{1}{(1+4\alpha t)^{N/2}} e^{-\frac{|\beta|^2 t}{1+4\alpha t}} e^{-\frac{\alpha|x|^2}{1+4\alpha t}} \cos\dfrac{(\beta,x)}{1+4\alpha t}$

(2) $\frac{\partial u}{\partial t} - \Delta u = \delta(x) \otimes T_{e^{pt}}$ $\quad p>0 \quad x \in \mathbb{R}^3, t \in \mathbb{R}$

$v := \mathcal{F}(u)$

$$\frac{\partial v}{\partial t} + (4\pi^2|s|^2) v = 1 \otimes T_{e^{pt}}$$

$\Downarrow$

$$\frac{\partial v}{\partial t} + (4\pi^2|s|^2) v = e^{pt}$$

$$\frac{d}{dt}\left(v_1 e^{4\pi^2|s|^2 t}\right) = 0$$

$$v_1 = C e^{-4\pi^2|s|^2 t}$$

$$v_2 = Ae^{pt} \Rightarrow \quad A\left(p + 4\pi^2|s|^2\right) = 1$$

$$A = \frac{1}{p+4\pi^2|s|^2}$$

$$v = Ce^{-4\pi^2|s|^2 t} + \frac{e^{pt}}{p+4\pi^2|s|^2}$$

Teď $u(t,x) = \mathcal{F}^{-1}\left(Ce^{-4\pi^2|s|^2 t} + \frac{e^{pt}}{p+4\pi^2|s|^2}\right) = C\frac{1}{(t)^{n/2}} e^{-\frac{|x|^2}{4t}} + e^{pt}\mathcal{F}^{-1}\left(\frac{1}{p+4\pi^2|s|^2}\right)$

$$\mathcal{F}^{-1}\left(\frac{1}{p+4\pi^2|s|^2}\right) = \frac{2}{r}\int_0^\infty \frac{\rho \sin(2\pi\rho r)}{p+4\pi^2\rho^2}\, d\rho = \frac{1}{r}\,\mathrm{Im}\int_{-\infty}^{\infty}\frac{\rho e^{i2\pi\rho r}}{p+4\pi^2\rho^2}\,d\rho$$

$$= \frac{1}{r}\,\mathrm{Im}\left(2\pi i \operatorname{Res}_{i\frac{\sqrt{p}}{2\pi}} \frac{z e^{i2\pi r z}}{p+4\pi^2 z^2}\right) = \frac{1}{r}\,\mathrm{Im}\left(2\pi i \frac{e^{i2\pi r i\frac{\sqrt{p}}{2\pi}}}{8\pi^2}\right)$$

$$= \frac{e^{-r\sqrt{p}}}{4\pi r}$$

Celk. $\boxed{u(t,x) = \frac{C}{t^{n/2}} e^{-\frac{|x|^2}{4t}} + \frac{e^{pt-\sqrt{p}|x|}}{4\pi|x|}}$

(3) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad$ na $(0,\infty)\times(0,\infty)$

$u(t,0) = 0 \quad t>0 \qquad u(0,x) = U_0 = \text{konst} \qquad x \in (0,\infty)$

Prodloužíme liše poč. podmínku na $\mathbb{R}$ a máme lich

$$u(t,x) = \frac{-U_0}{2\sqrt{\pi}\sqrt{t}} \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t}}\,dy + \frac{U_0}{2\sqrt{\pi}\sqrt{t}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4t}}\,dy$$

$$\frac{x-y}{2\sqrt{t}} = z$$

$$= \frac{-U_0}{\sqrt{\pi}} \int_{+\infty}^{\frac{x}{2\sqrt{t}}} e^{-z^2}\,dz + \frac{U_0}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{t}}} e^{-z^2}\,dz$$

$$= \frac{U_0}{\sqrt{\pi}} \left(\int_{-\infty}^{0} e^{-z^2}\,dz + \cancel{\int_{0}^{\frac{x}{2\sqrt{t}}} e^{-z^2}\,dz \Big) + \frac{U_0}{\sqrt{\pi}} \Big(\int_{0}^{\frac{x}{2\sqrt{t}}} e^{-z^2}\,dz} - \cancel{\int_{0}^{\infty} e^{-z^2}\,dz} + \int_{0}^{\frac{x}{2\sqrt{t}}} e^{-z^2}\,dz \right)$$

$$= \frac{2U_0}{\sqrt{\pi}} \int_{0}^{\frac{x}{2\sqrt{t}}} e^{-z^2}\,dz$$