Cvičenje č. 5

(1) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad x \in (0, \tfrac{1}{2})$

$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t, \tfrac{1}{2}) = 0 \qquad u(0,x) = \delta_a \qquad a \in (0, \tfrac{1}{2})$

Vzemimo $T = T_{u_0^s} \qquad u_0^s = \delta_a + \delta_{-a}$

$T_t = T_{U(t,\cdot)}$

Rešitev je $T_{U(t,\cdot)} * (T_{u_0^s} * \delta_\Sigma)$

$\Rightarrow u(t,x) = \sum_{n=-\infty}^{\infty} \mathcal{F}(u_0^s)(n) \underbrace{\mathcal{F}(U)(t,n)}_{e^{-4\pi^2 n^2 t}} e^{i 2\pi n x}$

$\langle \mathcal{F}(\delta_a), T\varphi \rangle = \langle \delta_a, \mathcal{F}(\varphi) \rangle = \langle \delta_a, \int_R e^{-2\pi i \xi x} \varphi(x)\,dx \rangle =$

$= \int_R e^{-2\pi i a x} \varphi(x)\,dx = \langle T_{e^{-2\pi i a x}}, \varphi \rangle$

Tedy

$$F(u_0^s) = \mathcal{F}(\delta_a + \delta_{-a}) = e^{-2\pi i a x} + e^{2\pi i a x} = 2\cos(2\pi a x)$$

$$u(t,x) = 2\sum_{m=-\infty}^{\infty} \underbrace{\cos(2\pi a m)}_{\text{sudé}}\, \underbrace{e^{-4\pi^2 m^2 t}}_{\text{sudé}}\, \underbrace{e^{2\pi i m x}}_{\text{liché mezi liché}}$$

$$= \boxed{4\sum_{m=1}^{\infty} \cos(2\pi a m)\, e^{-4\pi^2 m^2 t} \cos(2\pi m x) + \underbrace{2}_{m=0}} \qquad x \in (0, \tfrac{1}{2}),\ t > 0$$

(2) $\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0 \qquad x \in (0, \tfrac{1}{4}) \quad t > 0$

$u(0,x) = \delta_a \qquad a \in (0, \tfrac{1}{4})$

$u(t,0) = \dfrac{\partial u}{\partial x}(t, \tfrac{1}{4}) = 0$

Najdi $u(t, \tfrac{1}{4})$ a $\dfrac{\partial u}{\partial x}(t,0)$.

Provedeme prodloužení liché kolem $x=0$ a sudé kolem $x = \tfrac{1}{4}$

$$\tilde{u}_0^{sl} = \delta_a - \delta_{-a} + \delta_{\frac{1}{2}-a} - \delta_{-\frac{1}{2}+a}$$

Tedy $F(\tilde{u}_0^{sl}) = \mathcal{F}(\delta_a - \delta_{-a} + \delta_{\frac{1}{2}-a} - \delta_{-\frac{1}{2}+a}) = e^{-2\pi i a x} - e^{2\pi i a x} + e^{-2\pi i(\frac{1}{2}-a)x} - e^{-2\pi i(a-\frac{1}{2})x}$

$$= -2i \sin(2\pi a x) - 2i \sin(2\pi(\tfrac{1}{2}-a)x)$$

$$u(t,x) = -2i \sum_{n=-\infty}^{\infty} \left(\sin(2\pi a n) + \sin(\pi(1-2a)n)\right) e^{-4\pi^2 n^2 t} e^{2\pi i n x}$$

$$= \boxed{4 \sum_{n=1}^{\infty} \left(\sin 2\pi a n + \sin(\pi(1-2a)n)\right) e^{-4\pi^2 n^2 t} \sin(2\pi n x)}$$

$\sin \pi(1-2a)n = \sin((\pi n) - 2\pi a n) = \sin \pi n \cos 2\pi a n - \cos \pi n \sin 2\pi a n = -(-1)^n \sin 2\pi a n$

liché n $\qquad \sin \dfrac{2\pi(2k-1)}{4} \qquad (-1)^k$

$$u(t,x) = 8 \sum_{k=1}^{\infty} \sin(2\pi a(2k-1))\, e^{-4\pi^2(2k-1)^2 t} \sin(2\pi(2k-1)x)$$

$$u(t,\tfrac{1}{4}) = 8 \sum_{k=1}^{\infty} (-1)^k \sin(2\pi a(2k-1))\, e^{-4\pi^2(2k-1)^2 t}$$

$$\frac{\partial u}{\partial x}(t,0) = 16\pi \sum_{k=1}^{\infty} (2k-1) \sin(2\pi a(2k-1))\, e^{-4\pi^2(2k-1)^2 t} \cdot 1$$

(3) Radialne symetryczne równ.

$$\frac{\partial u}{\partial t} - \Delta u = 0 \qquad x \in B_{\frac{1}{2}}(0) \subset \mathbb{R}^3$$

$$u(0,x) = \chi_{[0,a]}(r) \qquad r = |x| \in [0, \tfrac{1}{2}) \qquad a \in (0, \tfrac{1}{2})$$

$$u(t,x) = 0 \qquad |x| = \tfrac{1}{2}$$

Szukamy radialnie symetrycznego równ.

$$u(t,x) = v(t, \overset{r}{|x|})$$

Pokaż $\Delta u = \frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r}$, tzn.

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial r^2} - \frac{1}{r}\frac{\partial v}{\partial r} = 0$$

$$w := r v$$

$$\frac{\partial w}{\partial r} = v + r\frac{\partial v}{\partial r}$$

$$\frac{\partial^2 w}{\partial r^2} = 2\frac{\partial v}{\partial r} + r\frac{\partial^2 v}{\partial r^2}$$

Tzn $\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial r^2} = 0$

$$w(0,x) = r\chi_{[0,a]} \qquad 0 < a < \tfrac{1}{2}$$

$$w(t,\tfrac{1}{2}) = 0 \quad \text{a dodatkowo} \quad w(t,0) = 0 \qquad \left(\text{bo } w(t,0) = r\tilde{v}(t,r)\big|_{r=0} = 0\right)$$

Tzn liczę podobnie

$$\tilde{w}_0^e = r\chi_{[-a,a]} \quad \text{na} \quad [-\tfrac{1}{2}, \tfrac{1}{2}]$$

$$F(\tilde{w}_0^e) = \int_{-a}^{a} e^{-2\pi i x \xi}\, \underset{\text{liczę}}{x}\, dx = \int_{-a}^{a} -i \sin(2\pi x\xi)\, x\, dx = +i\left[\frac{x\cos(2\pi x\xi)}{2\pi\xi}\right]_{-a}^{a} - i\int_{-a}^{a}\frac{\cos(2\pi x\xi)}{2\pi\xi}$$

$$-\frac{2i\, a\cos(2\pi a\xi)}{2\pi\xi}$$

$$= \frac{-i}{2\pi\xi}\left[+\frac{\sin 2\pi x\xi}{2\pi\xi}\right]_{-a}^{a} + \dots = \frac{i}{2\pi^2\xi^2}\sin(2\pi a\xi) + \frac{i a}{\pi\xi}\cos(2\pi a\xi)$$

$$w(t,r) = \frac{-i}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{n^2}\cancel{\sin(2\pi a n)}\, e^{-4\pi^2 n^2 t}\, e^{2\pi i n r}\; \frac{\sin(2\pi a n) - 2\pi a n\cos(2\pi a n)}{2}$$

$$= \frac{-i\cdot 2i}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cancel{\sin(2\pi a n)}\, e^{-4\pi^2 n^2 t}\sin(2\pi n r)\left(\frac{\sin(2\pi a n) - 2\pi a n\cos(2\pi a n)}{2}\right)$$

$$= \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cancel{\sin(2\pi a n)}\, e^{-4\pi^2 n^2 t}\sin(2\pi n r)\left(\sin(2\pi a n) - (2\pi a n)\cos(2\pi a n)\right)$$

Tzn $$\boxed{u(t,x) = \frac{1}{\pi^2|x|}\sum_{n=1}^{\infty}\frac{1}{n^2}\cancel{\sin(2\pi a n)}\, e^{-4\pi^2 n^2 t}\sin(2\pi n|x|)\left(\sin 2\pi a n - (2\pi a n)(\cos 2\pi a n)\right)}$$

4) Vlnová rovnica

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad x\in\mathbb{R}^+,\ t\in\mathbb{R}^+$$

$u(0,x)=0$

$\frac{\partial u}{\partial t}(0,x) = g_{per}(x)$

a) g_{per} 1-per. rozdelenie $g(x)=\sin^3(2\pi x)$

$\sin 3z = \sin z\cos 2z + \cos z \sin z = \sin z(\cos^2 z - \sin^2 z) + 2\sin z\cos^2 z$

$= 3\sin z(1-\sin^2 z) - \sin^3 z = 3\sin z - 4\sin^3 z$

$\Rightarrow \underline{\sin^3 z = \frac{1}{4}(3\sin z - \sin 3z)}$

$$u(t,x) = \frac{1}{2a}\int_{x-at}^{x+at} \sin^3_{per}(2\pi x)\,dx =$$

$$= \frac{1}{2a}\cdot\frac{1}{4}\int_{x-at}^{x+at}\left(3\sin(2\pi x) - \sin(6\pi x)\right)dx = \frac{1}{8a\cdot 2\pi}\left(\left[\frac{\cos 6\pi x}{3}\right]_{x-at}^{x+at} + \left[3\cos 2\pi x\right]_{x+at}^{x-at}\right)$$

$$= \frac{1}{16\pi a}\left[\frac{1}{3}\left(\cos 6\pi(x+at) - \cos 6\pi(x-at)\right) + 3\left(\cos 2\pi(x+at) - \cos 2\pi(x-at)\right)\right]$$

$$\underline{\frac{1}{16\pi a}\left[\frac{2}{3}\sin(6\pi x)\sin(6\pi a t) - 3\cdot 2\sin(2\pi x)\sin(2\pi a t)\right]}$$

Odpovedá riešeniu pomocou F.Rad

b) g_{per}: 1-per. rozdelenie $g(x) = \cos^4(\pi x)$

$\cos^4(\pi x) = \left(\frac{1+\cos(2\pi x)}{2}\right)^2 = \frac{1}{4}\left(1 + 2\cos(2\pi x) + \frac{1+\cos 4\pi x}{2}\right)$

$= \frac{1}{8}\left(3 + 4\cos(2\pi x) + \cos(4\pi x)\right)$

$$u(t,x) = \frac{1}{2a}\int_{x-at}^{x+at}\cos^4(\pi x)\,dx = \frac{1}{8\cdot 2a}\int_{x-at}^{x+at}\left(3 + 4\cos(2\pi x) + \cos(4\pi x)\right)dx$$

$$= \frac{1}{16a}\left(6at + \frac{2}{\pi}\left[+\sin 2\pi x\right]_{x-at}^{x+at} + \frac{1}{4\pi}\left[+\sin(4\pi x)\right]_{x-at}^{x+at}\right)$$

$$= \frac{1}{16a}\left(6at + \frac{2}{\pi}\left(-\sin 2\pi(x-at) + \sin 2\pi(x+at)\right) + \frac{1}{4\pi}\left(-\sin(4\pi(x-at)) + \sin(4\pi(x+at))\right)\right)$$

$$\underline{= \frac{1}{16a}\left(6at + \frac{4}{\pi}\sin 2\pi a t\cos 2\pi x + \frac{1}{2\pi}\left(\sin(4\pi a t)\cos(4\pi x)\right)\right)}$$

Odpovedá F.R.

5) Vlnová rovnice

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad x \in \mathbb{R}^+ \quad t \in \mathbb{R}^+$$

$$u(0,x) = 0$$

$$\frac{\partial u}{\partial t}(0,x) = \delta_b \qquad b > 0$$

$$u(t,0) = 0$$

Problém na $\mathbb{R}$: $\tilde{g}_1^{\ell}(x) = \delta_b - \delta_{-b}$ ... liché rozšíření

Řešení $u(t,x) = T_{\frac{1}{2a}\chi[-at,at]} * (\delta_b - \delta_{-b})$

$$\langle T_{\frac{1}{2a}\chi[-at,at]} * (\delta_b-\delta_{-b}), \varphi \rangle = \langle (\delta_b - \delta_{-b})(x), \langle T_{\frac{1}{2a}\chi[-at,at]}(y), \varphi(x+y) \rangle\rangle$$

$$= \langle (\delta_b - \delta_{-b})(x), \int_{\mathbb{R}} \frac{1}{2a}\chi_{[-at,at]}(y)\,\varphi(x+y) \rangle = \frac{1}{2a}\int_{-at}^{at} (\varphi(b+y) - \varphi(y-b))\,dy$$

$$= \frac{1}{2a}\int_{-at+b}^{at+b} \varphi(y)\,dy - \frac{1}{2a}\int_{-at-b}^{at-b} \varphi(y)\,dy \Rightarrow$$

$$\boxed{u(t,x) = \frac{1}{2a}\left(\chi_{[-at+b,at+b]}(x) - \chi_{[-at-b,at-b]}(x)\right)}$$