Homework PDEs II: Set 1 (Deadline April 7, 2026, 23 59)
Return either personally at the exercises or
send a reasonably readable form to `pokorny@karlin.mff.cuni.cz`

Sobolev spaces

1. (10 pts) Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of functions bounded in $W^{1,1}(\Omega) \cap L^p(\Omega)$, $\Omega \subset \mathbb{R}^d$, $d \geq 2$, $\Omega \in C^0$, $p > 1$. Show that there exists a subsequence strongly convergent in $L^r(\Omega)$, $1 \leq r < p$.

2. (10 pts) Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of functions bounded in $W^{1,p}(\Omega) \cap L^q(\partial\Omega)$, $1 < p < d$, $\Omega \subset \mathbb{R}^d$, $d \geq 2$, $q > 1$, $\Omega \in C^{0,1}$. Show that there exists a subsequence strongly convergent in $L^r(\partial\Omega)$, $1 \leq r < \max\{\frac{dp-p}{d-p}, q\}$.

3. (15 pts) Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, $\Omega \in C^{2,1}$. Show the existence of the extension operator from $W^{3,p}(\Omega)$ to $W^{3,p}(\mathbb{R}^d)$, $1 \leq p \leq \infty$.

4. (15 pts) Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of positive functions such that

$$\|\ln u_n\|_{W^{1,2}(\Omega)} \leq C < \infty$$
$$\|\nabla u_n^{\frac{m}{2}}\|_{L^2(\Omega)} \leq C < \infty$$
$$\int_{\partial\Omega} u_n \, \mathrm{d}S \leq C < \infty.$$

Let $m > \frac{1}{3}$, $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain.
a) Show that the sequence $u_n^{\frac{m}{2}}$ is bounded in $W^{1,2}(\Omega)$.
b) Show that the sequence u_n is bounded in some $W^{1,r}(\Omega)$, $1 < r < \infty$.
c) Show that there exists a subsequence u_{n_k} which converges a.e. in Ω and in $L^p(\Omega)$ for any $1 \leq p < 3m$.