Homework PDEs II: Set 2 (Deadline May 15, 2026, 23 59)
Return either personally at the exercises or
send a reasonably readable form to `pokorny@karlin.mff.cuni.cz`

Nonlinear elliptic equations

1. (10 pts) Consider the problem

$$\begin{aligned} -\Delta u + b(\nabla u) &= f \qquad && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

   where $\Omega \in C^{1,1}$ and $f \in L^2(\Omega)$. Show existence and uniqueness of a strong solution to this problem provided b is a Lipschitz continuous function with a sufficiently small Lipschitz constant.

2. (20 pts) Consider the problem

$$\begin{aligned} -\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) + g(u) &= f \qquad && \text{in } \Omega \\ u &= u_0 && \text{on } \partial\Omega. \end{aligned}$$

   Formulate the problem weakly. Show existence of a weak solution to this problem under the assumption that $p > 1$, $f \in (W^{1,p}(\Omega))^*$, $u_0 \in W^{1-\frac{1}{p},p}(\partial\Omega)$ and $|g(u)| \le C(1+|u|^q)$, $q < p-1$, g a continuous function. It is enough to justify the corresponding different steps in the proof of the corresponding theorem, no details for standard steps are needed.

3. (20 pts) Consider the problem

$$I[u] = \min_{w \in W_0^{1,2q}(\Omega)} \int_\Omega \Big[\big(1 + |\nabla w|^2\big)^q - w f(x) \Big] \, \mathrm{d}x.$$

   Show existence and uniqueness of such u under the assumption that $q > 1$ and f is a given function, integrable with a suitable power. (It will be necessary to revisit the proof of the corresponding result concerning the uniqueness.) Show that the corresponding minimizer satisfies the weak formulation of the corresponding Euler–Lagrange equations and formulate for each problem the minimal assumption on the integrability of f.