Sobolev spaces

1. Assume that $\Omega \in C^{0,1}$. Simplify the proof of the compact embedding
$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega).$$

2. Construct a radially symmetric function in the form
$$f(x) = \widetilde{f}(r) = r^\alpha \ln^\beta(r)$$
so that the function belongs to $W^{1,2}(B_{\frac{1}{2}}(0))$, but it does not belong to any $W^{1,p}(B_{\frac{1}{2}}(0))$ for $p > 2$, $B_{\frac{1}{2}}(0) \subset \mathbb{R}^3$.

3. Consider a function $\varphi \in W^{1,2}(B_1(0))$ with compact support in $B_1(0)$ and define
$$u_k(x) = k^{\frac{1}{2}}\varphi(kx), \qquad k \in \mathbb{N}.$$
Show that $\nabla u_k \rightharpoonup 0$ in $L^2(B_1(0))$ but it does not converge strongly. Show also that $u_k \rightharpoonup 0$ in $L^6(B_1(0))$, but it does not converge strongly, however, show that $u_k \to 0$ strongly in $L^p(B_1(0))$ for any $1 \le p < 6$.

4. Show that in order to have well defined traces, it is not possible to weaken the assumption that $\Omega \in C^{0,1}$.