Exercises PDEs II: Set 4 (March 27, 2026)

Sobolev spaces

1. Based on Theorem 6.9.2 in the Lecture Notes and on the embedding $W^{1,2}((a,b)) \hookrightarrow C([a,b])$ construct a function $V \in W_0^{1,2}((-1,1))$ such that
$$\varphi(0) = \langle \delta, \varphi \rangle_{W_0^{1,2}((-1,1))} = \int_{-1}^{1} V\varphi \,\mathrm{d}x + \int_{-1}^{1} V'\varphi' \,\mathrm{d}x$$
for all $\varphi \in W_0^{1,2}((-1,1))$.

2. Using the identification of the Beppo Levi spaces with the Sobolev ones give a direct proof of the following claims:
a) In one space dimension any Sobolev function is absolutely continuous.
b) If $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is such that the weak derivative is zero almost everywhere, then the function is constant in Ω.
c) If $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, then also u^+, u^- and $|u|$ belong to $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$.

3. Based on the Fredholm alternative for elliptic operators show that the problem to find $u \in W_0^{1,2}(\Omega)$ such that
$$\int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + bu\varphi + \sum_{i=1}^{d} c_i \frac{\partial u}{\partial x_i} \varphi \Big) \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x$$
for all $\varphi \in W_0^{1,2}(\Omega)$ has a unique solution, provided the matrix $\mathbb{A}$ is bounded and strictly positive definite (ellipticity), $b \in L^\infty(\Omega)$, $b(x) \geq b_0 > 0$ a.e. in Ω and $\mathbf{c}$ is bounded. In particular, no assumption on the divergence of the vector $\mathbf{c}$ is needed!

4. Based on the results from Section 7.1 of the Lecture Notes show that the following problems have unique weak solution:
a) We look for a weak solution to the problem
$$\begin{aligned} -\Delta u + (\operatorname{arctg} u + \pi)u &= f \qquad && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$
where $f \in W^{-1,2}(\Omega)$.

a) We look for a weak solution to the problem

$$
\begin{aligned}
-\mathrm{div}\,(\mathbb{A}\nabla u) + \Big(\frac{u}{u^2+1} + 3\Big)u &= f \qquad \text{in } \Omega \\
u &= 0 \qquad \text{on } \partial\Omega,
\end{aligned}
$$

where $f \in W^{-1,2}(\Omega)$ and the matrix $\mathbb{A}$ satisfies the standard ellipticity conditions.

5. Based on the results from Section 7.2 of the Lecture Notes show that the following problem has a unique weak solution:
We look for a weak solution to the problem

$$
\begin{aligned}
-\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\Big(|\nabla u|^{r-2}\frac{\partial u}{\partial x_i}\Big) + |u|^{r-2}u &= f \qquad \text{in } \Omega \\
u &= 0 \qquad \text{on } \partial\Omega,
\end{aligned}
$$

where $f \in W^{-1,r}(\Omega)$ and $1 < r < \infty$.