Exercises PDEs II: Set 5 (April 10, 2026)

Nonlinear elliptic equations

1. Based on the results from Section 7.1 of the Lecture Notes show that the following problems have unique weak solution:
   a) We look for a weak solution to the problem

$$\begin{aligned} -\Delta u + (\operatorname{arctg} u + \pi)u &= f \qquad &&\text{in } \Omega \\ u &= 0 \qquad &&\text{on } \partial\Omega, \end{aligned}$$

   where $f \in W^{-1,2}(\Omega)$.

   b) We look for a weak solution to the problem

$$\begin{aligned} -\operatorname{div}(\mathbb{A}\nabla u) + \Big(\frac{u}{u^2+1} + 3\Big)u &= f \qquad &&\text{in } \Omega \\ u &= 0 \qquad &&\text{on } \partial\Omega, \end{aligned}$$

   where $f \in W^{-1,2}(\Omega)$ and the matrix $\mathbb{A}$ satisfies the standard ellipticity conditions.

2. Based on the results from Section 7.2 of the Lecture Notes show that the following problem has a unique weak solution:
   We look for a weak solution to the problem

$$\begin{aligned} -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\Big(|\nabla u|^{r-2}\frac{\partial u}{\partial x_i}\Big) + |u|^{r-2}u &= f \qquad &&\text{in } \Omega \\ u &= 0 \qquad &&\text{on } \partial\Omega, \end{aligned}$$

   where $f \in W^{-1,r}(\Omega)$ and $1 < r < \infty$.

3. Try to generalize the results from Section 7.3 in the sense that we additionally assume (instead of $(7.33)_2$) the condition

$$\int_\Omega f(v)v\,\mathrm{d}x + \int_{\partial\Omega} g(v)v\,\mathrm{d}S \geq C\big(\|v\|^{p+1}_{L^{p+1}(\Omega)} + \|v\|^{q+1}_{L^{q+1}(\partial\Omega)} - 1\big)$$

   and show that in such a situation, we can consider any $1 < p < \infty$ and $1 < q < \infty$.

4. Proof the following generalization of the Banach fixed point theorem: Let X, Y by Banach spaces, X reflexive and $X \hookrightarrow Y$. Let H by a non-empty, closed, convex and bounded subset of X and let $T\colon H \to H$ be a mapping such that

$$\|T(u) - T(v)\|_Y \leq \alpha \|u - v\|_Y$$

for any u, $v \in H$ and $0 \leq \alpha < 1$. Then T possesses a unique fixed point in H.