

Nonlinear elliptic equations

1. Prove the following generalization of the Banach fixed point theorem:
Let X, Y be Banach spaces, X reflexive and $X \hookrightarrow Y$. Let H be a non-empty, closed, convex and bounded subset of X and let $T: H \rightarrow H$ be a mapping such that

$$\|T(u) - T(v)\|_Y \leq \alpha \|u - v\|_Y$$

for any $u, v \in H$ and $0 \leq \alpha < 1$. Then T possesses a unique fixed point in H .

2. Prove the following variant of the Schaeffer fixed point Theorem; the proof follows the same idea as the proof of the Schaeffer fixed point Theorem.
Let $K \subset X$ be convex, $0 \in K$ and $T: K \rightarrow K$ be continuous, compact such that

$$\{u \in K \mid u = \lambda T(u), 0 \leq \lambda \leq 1\}$$

is bounded. Then T has a fixed point in K .

3. Prove the following more general version of the weak lower semicontinuity result.

Let $\Omega \subset \mathbb{R}^d$ be measurable, $f: \Omega \times (\mathbb{R}^N \times \mathbb{R}^M) \rightarrow \mathbb{R}$ a Carathéodory function. Moreover, assume that:

- (a) f is bounded from below by an integrable minorant, i.e., there exists $g \in L^1(\Omega)$ such that for all $(\vec{u}, \vec{v}) \in \mathbb{R}^N \times \mathbb{R}^M$ and almost every $x \in \Omega$ it holds

$$f(x, \vec{u}, \vec{v}) \geq -g$$

- (b) f is convex in the last variables, i.e., for each $\vec{u} \in \mathbb{R}^N$, each $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^M$, each $\alpha \in [0, 1]$ and almost every $x \in \Omega$ it holds

$$f(x, \vec{u}, \alpha \vec{v}_1 + (1 - \alpha) \vec{v}_2) \leq \alpha f(x, \vec{u}, \vec{v}_1) + (1 - \alpha) f(x, \vec{u}, \vec{v}_2).$$

Then for every sequence $\{\vec{u}^n\}_{n=1}^\infty = \{u_1^n, \dots, u_N^n\}_{n=1}^\infty$ which fulfils for each $i = 1, \dots, N$

$$u_i^n \rightarrow u_i \quad \text{strongly in } L^1(\Omega)$$

and every sequence $\{\vec{v}^n\}_{n=1}^\infty = \{v_1^n, \dots, v_M^n\}_{n=1}^\infty$ which fulfils for each $i = 1, \dots, M$

$$v_i^n \rightharpoonup v_i \quad \text{weakly in } L^1(\Omega)$$

it holds

$$\int_{\Omega} f(x, \vec{u}(x), \vec{v}(x)) \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f(x, \vec{u}^n(x), \vec{v}^n(x)) \, dx.$$