Exercises PDEs II: Set 7 (May 15, 2026)

Bochner spaces

1. Show that the spaces $L^p(0,T;L^p(\Omega))$, $1 \le p < \infty$ are isomorphically identical with the spaces $L^p((0,T)\times\Omega)$, $\Omega \subset \mathbb{R}^d$, measurable. Show that, however, for $p = \infty$ this is not true.

Semigroups

2. Consider the second order parabolic equation

$$\begin{aligned} \partial_t u + Lu &= 0 && \text{in } (0,T)\times\Omega \\ u &= 0 && \text{on } (0,T)\times\partial\Omega \\ u(0,\cdot) &= g && \text{in } \Omega. \end{aligned}$$

Show that the operator

$$Au := -Lu = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial j}\Big) - \sum_{i=1}^{d} c_i(x)\frac{\partial u}{\partial x_i} - b(x)u$$

is a generator of the c_0-semigroup on $L^2(\Omega)$ which is γ-contractive, provided the coeficients a_{ij} satisfy the standard elipticity condition and $\Omega \in C^{1,1}$, a_{ij}, b and c_i are sufficiently smooth.

3. Consider the second order hyperbolic equation

$$\begin{aligned} \partial_{tt} u + Lu &= 0 && \text{in } (0,T)\times\Omega \\ u &= 0 && \text{on } (0,T)\times\partial\Omega \\ u(0,\cdot) &= g && \text{in } \Omega \\ \partial_t u(0,\cdot) &= h && \text{in } \Omega. \end{aligned}$$

Assume the second order equation in time rewritten as system of two first order equations in time:

$$\begin{aligned} \partial_t u - v &= 0 && \text{in } (0,T)\times\Omega \\ \partial_t v + Lu &= 0 && \text{in } (0,T)\times\Omega \\ u &= 0 && \text{on } (0,T)\times\partial\Omega \\ u(0,\cdot) &= g && \text{in } \Omega \\ v(0,\cdot) &= h && \text{in } \Omega. \end{aligned}$$

Show that the operator $A(u,v) := (v, -Lu)$, where

$$-Lu = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial j}\right) - b(x)u$$

is a generator of the c_0-semigroup on $W_0^{1,2}(\Omega) \times L^2(\Omega)$ which is γ-contractive, provided the coeficients a_{ij} satisfy the standard elipticity condition, $a_{ij} = a_{ji}$, $b \geq 0$ and $\Omega \in C^{1,1}$, a_{ij} and b are sufficiently smooth.