Cvičení 1

$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_N}\right)^T$ f.. skalární funkce

$\vec{g}$.. vektorová funkce $(\nabla \vec{g})_{ij} = \frac{\partial g_j}{\partial x_i}$

$\vec{g}: \mathbb{R}^3 \to \mathbb{R}^3$ $\operatorname{rot} \vec{g} = \left(\frac{\partial g_3}{\partial x_2} - \frac{\partial g_2}{\partial x_3}, \frac{\partial g_1}{\partial x_3} - \frac{\partial g_3}{\partial x_1}, \frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_2}\right)$ $\operatorname{div} \vec{g} = \sum_{i=1}^N \frac{\partial g_i}{\partial x_i} = Tr(\nabla \vec{g})$

Symbolika: ε_{ijk} – Levi-Civitův tenzor

$\varepsilon_{ijk} = 0$ $i=j$ nebo $i=k$ nebo $j=k$ (dva indexy stejné)

$\varepsilon_{ijk} = 1$ třeba 123, 312, 231

$\varepsilon_{ijk} = -1$ 132, 213, 321

Potom

$(\operatorname{rot} \vec{g})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} \frac{\partial}{\partial x_j} g_k$ $\left(\vec{u}, \vec{v} \in \mathbb{R}^3 \quad (\vec{u} \times \vec{v})_i = \sum_{jk} \varepsilon_{ijk} u_j v_k\right)$

$(\vec{\nabla} \times \vec{g})$

Teď potřebujeme

$\operatorname{div}(\nabla f) = \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right) = \sum_{i=1}^N \frac{\partial^2 f}{\partial x_i^2} = \Delta f$

Pro vektorovou fci

$\Delta \vec{g} = (\Delta g_1, \Delta g_2, \dots, \Delta g_N)$

Ukažme $\forall\, u, v$ v $\mathbb{R}^3$

$\operatorname{rot} \operatorname{rot} \vec{g} = \nabla(\operatorname{div} \vec{g}) - \Delta \vec{g}$

K tomu potřebujeme jednu identitu:

$\sum_{i=1}^N \varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ Kroneckerovo delta

Dk: výměna pořadí j a k $(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} = \delta_{kl}\delta_{jm} - \delta_{km}\delta_{jl} = 0)$

Jestli $j \neq k$ a $l \neq m$

např.: $j=2$ $k=3$ $l=1$ i $m=1$ $\varepsilon_{123}\varepsilon_{112} + \varepsilon_{223} \cdot \varepsilon_{123}$ – vše 0 ✓

$l=2$ $m=3$

$\varepsilon_{123}\varepsilon_{123} = 1$ shodné

$\varepsilon_{123}\varepsilon_{132} = -1$ opačné pořadí.

Potom

$(\operatorname{rot} \operatorname{rot} \vec{g})_i = \sum_{\substack{j,k=1 \\ l,m=1}}^{3} \varepsilon_{ijk} \frac{\partial}{\partial x_j}\left(\varepsilon_{klm} \frac{\partial g_m}{\partial x_l}\right) = \sum_{\substack{j=1 \\ k,l,m}}^{3} \varepsilon_{kij}\, \varepsilon_{klm} \frac{\partial}{\partial x_j}\left(\frac{\partial g_m}{\partial x_l}\right)$

$= \sum_{j,l,m} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial}{\partial x_j}\left(\frac{\partial g_m}{\partial x_l}\right)$

$= \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \frac{\partial g_j}{\partial x_i} - \sum_{j=1}^{3} \frac{\partial^2 g_i}{\partial x_j^2} = \frac{\partial}{\partial x_i}(\operatorname{div} \vec{g}) - (\Delta \vec{g})_i$

Ukážeme, že (i) $$\int_\Omega \frac{\partial f}{\partial x_i} g\, dx = \int_{\partial\Omega} f g\, n_i\, dS - \int_\Omega \frac{\partial g}{\partial x_i} f\, dx$$

Zřejmě $$\int_\Omega \frac{\partial}{\partial x_i}(fg)\, dx = \int_{\partial\Omega} fg\, n_i\, dS$$
$$\parallel$$
$$\int_\Omega \frac{\partial f}{\partial x_i} g\, dx + \int_\Omega f \frac{\partial g}{\partial x_i}\, dx = \int_{\partial\Omega} fg\, n_i\, dS$$

(ii) $$\int_\Omega \Delta f\, g\, dx = \int_{\partial\Omega} \frac{\partial f}{\partial n} g\, dS - \int_\Omega \nabla f \cdot \nabla g\, dx$$

$$\int_\Omega \operatorname{div}(\nabla f \cdot g)\, dx = \int_{\partial\Omega} \nabla f \cdot \vec{n}\, g\, dS = \int_{\partial\Omega} \frac{\partial f}{\partial n} g\, dS$$
$$\parallel$$
$$\int_\Omega \Delta f\, g\, dx + \int_\Omega \nabla f \cdot \nabla g\, dx$$
neboli $$\sum_{\alpha=1}^{N} \frac{\partial}{\partial x_\alpha}\left(\frac{\partial f}{\partial x_\alpha} g\right) = \sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_i^2} g + \sum_{\alpha=1}^{N} \frac{\partial f}{\partial x_\alpha}\frac{\partial g}{\partial x_\alpha}$$

(iii) $$\int_\Omega (\Delta f\, g - \Delta g\, f)\, dx = \int_{\partial\Omega} \left(\frac{\partial f}{\partial n} g - \frac{\partial g}{\partial n} f\right) dS$$

Použijeme bod (ii) dvakrát
$$\int_\Omega \Delta f\, g\, dx = \int_{\partial\Omega} \frac{\partial f}{\partial n} g\, dS - \int_\Omega \nabla f \cdot \nabla g\, dx$$
$$\int_\Omega \Delta g\, f\, dx = \int_{\partial\Omega} g\, \frac{\partial g}{\partial n} f\, dS - \int_\Omega \nabla f \cdot \nabla g\, dx$$
a odečtem
$$\int_\Omega (\Delta f\, g\, dx - \Delta g\, f)\, dx = \int_\Omega \left(\frac{\partial f}{\partial n} g - \frac{\partial g}{\partial n} f\right) dS \quad (+0).$$

Příklad: Nalezněte $u: \mathbb{R}^2 \to \mathbb{R}$ řešící $\frac{\partial u}{\partial x} - 6x^2 \frac{\partial u}{\partial y} = 0$ splňující $u(x,0) = u_0(x)$ pro danou funkci $u_0(x)$.

Řešení:

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = \cancel{-6x^2 = -6c_1^2 t^2}\ -6(t+c_1)^2$$
$$x = t + c_1 \qquad \cancel{y = -2c_1^2 t^3 + c_2 \Rightarrow y - c_2 = -2c_1^2 t^3 = -\frac{2}{c_1} x^3}$$
$$y = -2(t+c_1)^3 + c_2$$
$$y = -2x^3 + c_2$$
$$y + 2x^3 = \text{const}$$
$$u(x,y) = U(y + 2x^3)$$
$$u_0(x) = u(x,0) = U(2x^3) \quad \Rightarrow \quad \boxed{u(x,y) = u_0\left(\sqrt[3]{\frac{y+2x^3}{2}}\right)}$$
$$U(z) = u_0\left(\sqrt[3]{\frac{z}{2}}\right)$$
$$\frac{\partial u}{\partial x} = u_0'\left(\sqrt[3]{\frac{y+2x^3}{2}}\right) \cdot \frac{1}{3}\left(\frac{y+2x^3}{2}\right)^{-\frac{2}{3}} \cdot 3x^2 \ \checkmark$$
$$\frac{\partial u}{\partial y} = u_0'\left(\sqrt[3]{\frac{y+2x^3}{2}}\right) \cdot \frac{1}{3}\frac{1}{2}\left(\frac{y+2x^3}{2}\right)^{-\frac{2}{3}} \ \checkmark$$

Příklad Najděte $u: \mathbb{R}^2 \to \mathbb{R}$ řešení $\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ $\quad u(0,y) = \frac{1}{y}$.

Řešení:

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = y$$

$$x = t + C_1 \qquad y = C_2 e^t$$

α) $y>0$ $C_2>0$ $\Rightarrow \lg y = C_2 + t$

$$x - \lg y = \text{const}$$

$$u(x,y) = U(x - \lg y)$$

$$\frac{1}{y} = u(0,y) = U(-\lg y) \Rightarrow U(z) = e^z$$

$u(x,y) = \frac{1}{y}\cdot e^x$ no stejně i pro $y<0$

tj. $u(x,y) = \frac{1}{y}e^x$ $\quad x\in\mathbb{R},\ y>0$
nebo $x\in\mathbb{R}$ $y<0$

Příklad Nalezněte $u: \mathbb{R}^3 \to \mathbb{R}$ řešení **Cvičení 2**

$$(z+y-x)\frac{\partial u}{\partial x} + (z+x-y)\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Obvyklé $u(x,y,1) = \sin(x+y)$

Řešení:

$$\left.\begin{aligned}\frac{dx}{dt} &= z+y-x\\ \frac{dy}{dt} &= z+x-y\\ \frac{dz}{dt} &= z\end{aligned}\right\} \qquad \begin{aligned}&\frac{dx}{dt} + \frac{dy}{dt} - 2\frac{dz}{dt} = 0\\ &\frac{d}{dt}(x+y-2z) = 0\end{aligned}$$

$$u(x,y,z) = U(x+y-2z)$$

$$\sin(x+y) = u_0(x,y) = U(x+y-2)$$

$$\Rightarrow \boxed{u(x,y,z) = \sin(x+y-2z+2)}$$

Cvičení 2

Najděte $u: \mathbb{R}^2 \to \mathbb{R}$ řešící

$$x\frac{\partial u}{\partial x} + (x+y)\frac{\partial u}{\partial y} = 0$$

s počáteční podmínkou $u(x,y) = u_0(x)$.

a) na okolí bodu $(1,0)$ $\qquad$ b) na okolí bodu $(0,0)$.

Jsou nějaké nejasné momenty?