

(4)

10.10.18

Hledíme rovnici

$$\Delta u = 0 \text{ na } B_1(0) \cap \mathbb{R}^2 \cong \mathbb{R}^2$$

$$u = g \text{ na } \partial \mathbb{R}^2$$

Podle úlohy je podmínka souměrnosti

$$\partial_{rr} u + \frac{1}{r} \partial_r u + \frac{1}{r^2} \partial_{\varphi\varphi}^2 u = 0$$

$$v(r, \varphi) = u(x, y)$$

$$v(b, \varphi) = g_1(\varphi)$$

$$v(0, \varphi) = g_2(\varphi)$$

Hledíme rovnici na dvou separačních proměnných

$$v(r, \varphi) = R(r) \Phi(\varphi)$$

Podle

$$(R''(r) + \frac{1}{r} R'(r)) \Phi(\varphi) + \frac{1}{r^2} \Phi''(\varphi) R(r) = 0$$

$$\text{tedy } \frac{r^2 R''(r) + r R'(r)}{R(r)} + \frac{\Phi''(\varphi)}{\Phi(\varphi)} = 0$$

$$\text{Některé tedy } \frac{\Phi''(\varphi)}{\Phi(\varphi)} = -\lambda$$

$$\text{a z úplné rovnice } \Phi(0) = \Phi(2\pi) \\ \Phi'(0) = \Phi'(2\pi)$$

$$\text{tedy } \Phi''(\varphi) + \lambda \Phi(\varphi) = 0 \text{ + podmínka dyfuzní podmínky}$$

a) $\lambda < 0$ exponenciální rovnice

b) $\lambda = 0$ $\Phi''(\varphi) = 0 \Rightarrow \Phi(\varphi) = C_0$ (konstanta pro φ , Φ'' konstanta rovná se 0)

c) $\lambda > 0$

$$\Phi(\varphi) = A \sin(\sqrt{\lambda} \varphi) + B \cos(\sqrt{\lambda} \varphi)$$

2-Periodičnost rovnice na proměnné φ , $\sqrt{\lambda} = k \in \mathbb{N}$ (znamená a vždy do kladných)

tedy

$$\lambda_k = k^2$$

$$k = 0, 1, 2, \dots$$

$$\text{mohou být } \Phi_k(\varphi) = \begin{cases} \cos k\varphi \\ \sin k\varphi \end{cases}$$

Podobně se můžeme dostat rovnici

$$r^2 R''(r) + r R'(r) - k^2 R(r) = 0$$

Rovnici můžeme hledat ve tvaru

$$R_\alpha(r) = r^\alpha$$

$$k(\alpha-1) + \alpha - k^2 = 0$$

Prado

$$x^2 - k^2 = 0$$

Jika $k > 0 \Rightarrow$ $x_1 = k$ $x_2 = -k$
 $k=0 \Rightarrow x_1=0$ dan $x_2=0$ dan x_1 dan x_2 dan x_3 dan x_4 dan x_5 dan x_6 dan x_7 dan x_8 dan x_9 dan x_{10} dan x_{11} dan x_{12} dan x_{13} dan x_{14} dan x_{15} dan x_{16} dan x_{17} dan x_{18} dan x_{19} dan x_{20} dan x_{21} dan x_{22} dan x_{23} dan x_{24} dan x_{25} dan x_{26} dan x_{27} dan x_{28} dan x_{29} dan x_{30} dan x_{31} dan x_{32} dan x_{33} dan x_{34} dan x_{35} dan x_{36} dan x_{37} dan x_{38} dan x_{39} dan x_{40} dan x_{41} dan x_{42} dan x_{43} dan x_{44} dan x_{45} dan x_{46} dan x_{47} dan x_{48} dan x_{49} dan x_{50} dan x_{51} dan x_{52} dan x_{53} dan x_{54} dan x_{55} dan x_{56} dan x_{57} dan x_{58} dan x_{59} dan x_{60} dan x_{61} dan x_{62} dan x_{63} dan x_{64} dan x_{65} dan x_{66} dan x_{67} dan x_{68} dan x_{69} dan x_{70} dan x_{71} dan x_{72} dan x_{73} dan x_{74} dan x_{75} dan x_{76} dan x_{77} dan x_{78} dan x_{79} dan x_{80} dan x_{81} dan x_{82} dan x_{83} dan x_{84} dan x_{85} dan x_{86} dan x_{87} dan x_{88} dan x_{89} dan x_{90} dan x_{91} dan x_{92} dan x_{93} dan x_{94} dan x_{95} dan x_{96} dan x_{97} dan x_{98} dan x_{99} dan x_{100}

Nilai riil no dan $k \in \mathbb{N}_0$

$$P_0 \Phi_0 = B_0^1 + B_0^2 \text{ lnr}$$

$$R_k \Phi_k = (A_k^1 \sin(ky) + B_k^1 \cos(ky)) r^k + (A_k^2 \sin(ky) + B_k^2 \cos(ky)) r^{-k} \text{ kEM}$$

Zatun jsmu ukap/bts diagonal podung. jk li'

$$g_1 = a_k^1 \sin(ky) + d_k^1 \cos(ky) \text{ kEM}$$

$$g_2 = a_k^2 \sin(ky) + d_k^2 \cos(ky)$$

dododum smdan vonsu

$$b^k (A_k^1 \sin(ky) + B_k^1 \cos(ky)) + b^{-k} (A_k^2 \sin(ky) + B_k^2 \cos(ky)) =$$

$$= a_k^1 \sin(ky) + d_k^1 \cos(ky)$$

$$a^k (A_k^1 \sin(ky) + B_k^1 \cos(ky)) + a^{-k} (A_k^2 \sin(ky) + B_k^2 \cos(ky)) =$$

$$= c_a^2 \sin(ky) + d_a^2 \cos(ky)$$

dododum smdan vonsu

$$b^k (A_k^1 + B_k^1) \quad b^k A_k^1 + b^{-k} A_k^2 = C_k^1 \quad | b^k$$

$$a^k A_k^1 + a^{-k} A_k^2 = C_k^2 \quad | a^{-k}$$

(Sudu ji' replant) . Ndm

$$A_k^2 (b^{-2k} - a^{-2k}) = b^{-k} C_k^1 - a^{-k} C_k^2$$

$$A_k^2 = \frac{b^k C_k^1 - a^k C_k^2}{b^{-2k} - a^{-2k}} = \frac{a^{2k} b^k C_k^1 - a^k b^{2k} C_k^2}{a^{2k} - b^{2k}}$$

$$= \frac{a^k b^k}{b^{2k} - a^{2k}} \cdot (b^k C_k^2 - a^k C_k^1)$$

Analogi $A_k^1 = C_k^1 b^{-k} - b^{-k} A_k^2 = C_k^1 b^{-k} \left[1 + \frac{a^k b^k}{b^{2k} - a^{2k}} \cdot \frac{a^k}{b^k} \right]$

$$= \frac{b^k}{b^{2k} - a^{2k}} C_k^1 - \frac{a^k C_k^2}{b^{2k} - a^{2k}}$$

Analogi

$$B_k^1 = \frac{b^k}{b^{2k} - a^{2k}} d_k^1 - \frac{a^k d_k^2}{b^{2k} - a^{2k}}$$

$$B_k^2 = \frac{a^k b^k}{b^{2k} - a^{2k}} (b^k d_k^2 - a^k d_k^1)$$

Gitu

Prinsip
 $k=0$

$$B_0^1 + B_0^2 \ln b = d_0^1$$

$$B_0^1 + B_0^2 \ln a = d_0^2$$

$$B_0^2 \left(\ln \frac{b}{a} \right) = d_0^1 - d_0^2$$

$$B_0^2 = \frac{d_0^1 - d_0^2}{\ln \left(\frac{b}{a} \right)}$$

$$B_0^1 = d_0^1 - \frac{d_0^1 - d_0^2}{\ln \left(\frac{b}{a} \right)} \ln b = d_0^1 - \frac{d_0^1 - d_0^2}{\ln b - \ln a} \ln b$$

$$= \frac{1}{\ln b - \ln a} (d_0^1 (\ln b - \ln a - \ln b) + d_0^2 \ln b) = \frac{d_0^2 \ln b - d_0^1 \ln a}{\ln \frac{b}{a}}$$

allu led

$$u(r, \varphi) = \frac{d_0^2 \ln b - d_0^1 \ln a}{\ln \frac{b}{a}} + \sum_{k=1}^{\infty} r^k \left[\frac{b^k c_k^1 - a^k c_k^2}{b^k - a^k} \sin(k\varphi) + \frac{b^k d_k^1 - a^k d_k^2}{b^k - a^k} \cos(k\varphi) \right]$$

$$+ \frac{d_0^1 - d_0^2}{\ln \frac{b}{a}} \ln r + \sum_{k=1}^{\infty} r^{-k} \left[\frac{a^k b^k}{b^k - a^k} (b^k c_k^2 - a^k c_k^1) \sin(k\varphi) + \frac{a^k b^k}{b^k - a^k} (b^k d_k^2 - a^k d_k^1) \cos(k\varphi) \right]$$

Pokračujeme seřídit konvergenční podmínky pro g_1, g_2 pro $a < r < b$ řada konverguje. Voleme symetrickou

(nepř. pro řadu lze odhadnout $C \left(\frac{r}{b}\right)^k (C_1 H_k^2) \dots$, analogicky
 druhá $C \left(\frac{a}{r}\right)^k (C_1 + C_2)$)

Tedy součet dostáváme na $u \in C^2(\Delta_b \setminus \{0\} \cup \overline{B_a \setminus \{0\}})$

Pokud ale chceme splnit nějaké podmínky, pak potřebujeme, aby $|a| \neq |b|$ - jinak
 součet konverguje ke stejnému, pokud $g_1 \in ACC(0, \infty) \sim g_1' \in C^1(0, \infty)$.

Co když $b \rightarrow \infty$ nebo $a \rightarrow 0^+$? Pak jedinou podmínkou odpovídající
 oběm stranám pro $b \rightarrow \infty$ nebo $a \rightarrow 0^+$ je $A_k^1 = B_k^1 = 0$
 $A_k^2, B_k^2 = 0$!

Podíváme se například na $a = b = \infty$. Některé ledy

$$u(r, \varphi) = B_0^1 + \sum_{k=1}^{\infty} (A_k^2 \sin(k\varphi) + B_k^2) \cos(k\varphi) r^{-k}$$

Tato rade se da uvidet, ca urceni detel (vyjde vedel na nupstl kug)

Jen ai spaticme:

$$u(\varphi) = g(\varphi) = b_0 + \sum_{k=1}^{\infty} (c_k \sin(k\varphi) + d_k \cos(k\varphi))$$

Tedy $B_0^1 = b_0$
 $A_n^2 \cdot a^{-k} = c_k$
 $B_n^2 \cdot a^{-k} = d_k$

a pole

$$u(r, \varphi) = b_0 + \sum_{k=1}^{\infty} (A_k^2 \sin(k\varphi) + B_k^2 \cos(k\varphi)) \cdot \left(\frac{r}{a}\right)^k$$

Analogy ke podkladu i pro uky me urceni (vrajstl yma, a me mulyba?)

Prklad:

$$\Delta u = 0 \quad \text{na} \quad 0 < r < a \quad - \quad 0 < \varphi < \alpha < 2\pi$$

$$u(a, \varphi) = g(\varphi) \quad \varphi(0) = g(\alpha)$$

$$u(r, 0) = u(r, \alpha) = 0 \quad 0 \leq r \leq a$$

Rovna uky ma $g(\varphi) = \varphi \sin\left(\frac{\pi}{\alpha}\varphi\right)$

Podopy dymu p r = 0 a r = a na polle

$$u(r, \varphi) = R(r) \Phi(\varphi)$$

$$r^2 \frac{R''(r)}{R(r)} + \frac{\Phi''(\varphi)}{\Phi(\varphi)} = 0$$

$$\text{ale lez}^c \quad \Phi'' + \lambda \Phi = 0$$

$$\Phi(0) = \Phi(\alpha) = 0$$

jit nima, co kolle vide na vstov (~~co je~~ $\lambda = \left(\frac{k\pi}{\alpha}\right)^2$ k e n)

$$\Phi_k(\varphi) = \sin\left(\frac{k\pi}{\alpha}\varphi\right)$$

$$r^2 R''(r) + r R'(r) - \left(\frac{k\pi}{\alpha}\right)^2 R(r) = 0$$

$R(r) = r^{\pm \frac{k\pi}{\alpha}}$

ale $-\frac{k\pi}{\alpha}$ nma amud² ke mly \rightarrow NR

Radon kof

$$u(r, \varphi) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\varphi}{\alpha}\right) r^{\frac{k\alpha}{\alpha}}$$

$$u(r, \varphi) = \sum_{k=1}^{\infty} C_k \sin\left(\frac{k\varphi}{\alpha}\right)$$

$$A_k = C_k \cdot r^{-\frac{k\alpha}{\alpha}}$$

$$C_k = \frac{2}{\alpha} \int_0^{\alpha} \varphi \sin\left(\frac{k\varphi}{\alpha}\right) \sin\left(\frac{k\varphi}{\alpha}\right) d\varphi$$

1) $k=1$

$$C_1 = \frac{2}{\alpha} \int_0^{\alpha} \varphi \frac{1 - \cos\left(\frac{2\varphi}{\alpha}\right)}{2} d\varphi = \frac{1}{\alpha} \int_0^{\alpha} \left(\varphi - \varphi \cos\left(\frac{2\varphi}{\alpha}\right)\right) d\varphi$$

$$= \frac{1}{2} \cdot \alpha - \frac{1}{\alpha} \left[\varphi \cdot \frac{\sin\left(\frac{2\varphi}{\alpha}\right)}{\frac{2}{\alpha}} \right]_0^{\alpha} + \frac{1}{2\pi} \int_0^{\alpha} \sin\left(\frac{2\varphi}{\alpha}\right) d\varphi$$

$$= \frac{1}{2} \alpha + \frac{1}{2\pi} \left[-\cos\left(\frac{2\varphi}{\alpha}\right) \right]_0^{\alpha} = \frac{1}{2} \alpha + \frac{1}{\pi}$$

$$C_k = \frac{2}{\alpha} \int_0^{\alpha} \varphi \sin\left(\frac{k\varphi}{\alpha}\right) \sin\left(\frac{k\varphi}{\alpha}\right) d\varphi = \frac{2}{\alpha} \int_0^{\alpha} \varphi \left[\cos\left(\frac{(k-1)\varphi}{\alpha}\right) - \cos\left(\frac{(k+1)\varphi}{\alpha}\right) \right] d\varphi$$

$$= \frac{2}{\alpha} \left(\left[\varphi \cdot \frac{\sin\left(\frac{(k-1)\varphi}{\alpha}\right)}{\frac{k-1}{\alpha}} \right]_0^{\alpha} - \left[\varphi \cdot \frac{\sin\left(\frac{(k+1)\varphi}{\alpha}\right)}{\frac{k+1}{\alpha}} \right]_0^{\alpha} \right)$$

$$+ \frac{2}{\alpha} \int_0^{\alpha} \left(\frac{\cos\left(\frac{(k-1)\varphi}{\alpha}\right)}{\frac{k-1}{\alpha}} - \frac{\cos\left(\frac{(k+1)\varphi}{\alpha}\right)}{\frac{k+1}{\alpha}} \right) d\varphi$$

$$= \frac{2}{\alpha} \cdot \left[\frac{\alpha^2}{(k-1)^2} \left(\left[-\cos\left(\frac{(k-1)\varphi}{\alpha}\right) \right]_0^{\alpha} \right) - \frac{\alpha^2}{(k+1)^2} \left(\left[-\cos\left(\frac{(k+1)\varphi}{\alpha}\right) \right]_0^{\alpha} \right) \right]$$

$$= \frac{2\alpha}{\pi^2} (1 - (-1)^{k+1}) \left(\frac{1}{(k-1)^2} - \frac{1}{(k+1)^2} \right)$$

$$= \frac{2\alpha (1 - (-1)^{k+1})}{\pi^2} \frac{k}{(k^2 - 1)^2}$$

$$u(r, \varphi) = \frac{1}{2} \alpha - \frac{2}{\pi^2} \sum_{k=2}^{\infty} \frac{k (1 - (-1)^{k+1})}{(k^2 - 1)^2} \sin\left(\frac{k\varphi}{\alpha}\right) \left(\frac{r}{\alpha}\right)^{\frac{k\alpha}{\alpha}}$$

Zitat zu Herleitung dieses Resultats!

Delso munda jaan

a) mijaat njae



- mijaat njae, ja mijaat djae laloda mijaat n
a mijaat njae (oia djae ja njae)

b) mijaat njae



- mijaat njae djae mijaat njae
me mijaat njae
(oia djae ja njae)