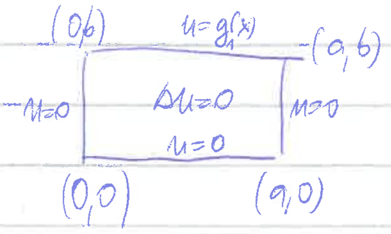
Άσκηση 12



$\Delta u = 0$ στο $(0,a)\times(0,b)$

$u=0$ στο $\{0\}\times(0,b) \cup \{a\}\times(0,b) \cup (0,a)\times\{0\}$

$u = g_1(x)$ στο $(0,a)\times\{b\}$ $\qquad g_1(0)=g_1(b)=0$

$u(x,y) = X(x)\,Y(y)$

$X''(x)\,Y(y) + Y''(y)\,X(x) = 0$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

$X''(x) + \lambda X(x) = 0 \qquad X(0)=X(a)=0$

$\underline{X_k(x) = \sin\frac{k\pi x}{a}} \qquad k\in\mathbb{N}$

$Y_k''(y) - \left(\frac{k\pi}{a}\right)^2 Y_k(y) = 0$

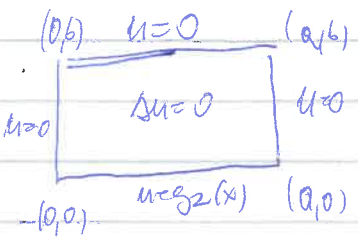
$Y_k(0)=0 \qquad Y_k(b) = A_k$

$Y_k(y) = C_1 e^{\frac{k\pi}{a}y} + C_2 e^{-\frac{k\pi}{a}y} \qquad Y_k(0)=0 \Rightarrow C_1+C_2=0$

$Y_k(y) = C_k \sinh\left(\frac{k\pi y}{a}\right)$

$\underline{Y_k(y) = \frac{A_k}{\sinh\left(\frac{k\pi b}{a}\right)} \sinh\left(\frac{k\pi y}{a}\right)}$

$$\underline{u(x,y) = \sum_{k=1}^{\infty} \frac{A_k}{\sinh\left(k\pi \frac{b}{a}\right)} \sinh\left(\frac{k\pi y}{a}\right) \sin\left(\frac{k\pi}{a}x\right)} \qquad g_1(x) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi}{a}x\right)$$



$g_2(0) = g_2(a) = 0$

$u(x,y) = X(x)\,Y(y)$

Ομοί. $X_k(x) = \sin\frac{k\pi x}{a}$

$$T_k''(y) - \left(\frac{k\pi}{a}\right)^2 T_k(y) = 0$$
$$T_k(0) = A_k$$
$$T_k(b) = 0$$

$$T_k(y) = C_1 e^{\frac{k\pi}{a}y} + C_2 e^{-\frac{k\pi}{a}y}$$

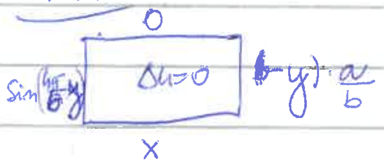
$$T_k(b) = 0 \Rightarrow \quad C_1 e^{\frac{k\pi}{a}b} + C_2 e^{-\frac{k\pi}{a}b} = 0$$
$$C_1 = -C_2 e^{-\frac{2k\pi}{a}b}$$

$$T_k(y) = C_2 \left(e^{\frac{k\pi}{a}y - \frac{2k\pi}{a}b} - e^{-\frac{k\pi}{a}y} \right)$$
$$= 2C_2 e^{-\frac{k\pi}{a}b} \sinh \frac{k\pi}{a}(y-b)$$

$$T_k(y) = -\frac{A_k}{\sinh \frac{k\pi}{a}b} \sinh \frac{k\pi}{a}(y-b)$$

$$u(x,y) = -\sum_{k=1}^{\infty} \frac{A_k}{\sinh(\frac{k\pi}{a}b)} \sinh \frac{k\pi}{a}(y-b) \sin\left(\frac{k\pi}{a}x\right) \qquad g_2(x) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi}{a}x\right)$$

Příklad



Nejprve hledejme tzv. harmonickou funkci, aby $\quad V(0,0) = V(0,b) = V(a,b) = 0$
$$V(a,0) = a$$

$V(x,y) = A + Bx + Cy + Dxy$
$$A = 0$$
$$C = 0$$
$$Ba + Dab = 0 \qquad D = -\frac{1}{b}$$
$$Ba = a \Rightarrow \qquad \underline{B = 1}$$

$$V(x,y) = x - \frac{xy}{b}$$

$u(x,y) = v(x,y) + V(x,y) \Rightarrow$
$$\Delta v = 0$$
$$v(x,0) = x - x = \cancel{x(1-\tfrac{a}{a})}\ 0$$
$$v(a,y) = \cancel{a(1-y) - 1 + \tfrac{y}{b} = y(\tfrac{1}{b} - 1)\ a-1} \qquad a - \frac{a}{b}y - (b-y)\frac{a}{b} = 0$$
$$v(x,b) = 0$$
$$v(0,y) = \sin\left(\frac{k\pi}{a}y\right)$$

Nyní použijeme podmínku z nenulové části

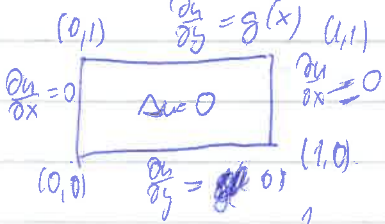
$$u(x,y) = -\sum_{k=1}^{\infty} \frac{A_k}{\sinh\left(\frac{k\pi a}{b}\right)} \sinh\frac{k\pi}{b}(a-x)\, \sin\left(\frac{k\pi}{b} y\right)$$

Odtud vidíme $k=4$ $A_4 = 1$

$$\boxed{u(x,y) = -\frac{\sinh\frac{4\pi}{b}(a-x)}{\sinh\frac{4\pi a}{b}} \sin\frac{4\pi y}{b} + x\left(1-\frac{y}{b}\right)}$$

Obecně se rozpadne úloha, každou řešíme zvlášť dle postupu výše a výsledek je součet 4 úloh a jediné pomocné úlohy. polynom

17. Podmínky pro tuto homogenní, pak nalezněme pomocí klasického řešení



Podmínka je $\underline{\int_0^1 g(x)\,dx = 0}$

$u(x,y) = X(x)\,Y(x)$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = 0$$

$X'(0) = X'(1) = 0$

$X_k(x) = \cos(k\pi x)$ $k = 0,1,2,\dots$

$Y_k''(y) - (k\pi)^2 Y_k(y) = 0$

$Y_k'(0) = 0$ $Y_k'(1) = B_k$

$Y_k(y) = C_1 e^{k\pi y} + C_2 e^{-k\pi y}$

$Y_k(y) = \frac{B_k}{k\pi} \frac{\cosh(k\pi y)}{\sinh(k\pi)}$

$$\underline{u(x,y) = \sum_{k=1}^{\infty} \frac{B_k}{k\pi} \frac{\cosh(k\pi y)}{\sinh(k\pi)} \cos(k\pi x) + C}$$

$k=0$, $Y_0''(y) = 0$, $Y_0'(0) = 0$, $Y_0'(1) = B_0$ — nenulové ale musí podmínka řešení je 0-4 tak není!

kdy $\underline{Y_0 = C}$

$$\underline{g(x) = \sum_{k=1}^{\infty} B_k \cos(k\pi x)}$$

Podmínky ke 2 kombinované

$(1,0)$ na $\frac{\partial u}{\partial y} = g(x)$ $(1,1)$

$u=0$ $\Delta u = 0$ $u=0$

$(0,0)$ $u=0$ $(1,0)$

$$g(x) = \sum_{k=1}^{\infty} A_k \sin(k\pi x)$$

$$u(x,y) = \sum_{k=1}^{\infty} X_k(x)\, Y_k(y)$$

$$\frac{X_k''}{X_k} + \frac{Y_k''}{Y_k} = 0 \qquad \frac{X_k''}{X_k} = -\lambda_k$$

$$X_k = \sin(k\pi x) \qquad k \in \mathbb{N}$$

$$Y_k'' - (k\pi)^2 Y_k = 0$$
$$Y_k(0) = 0$$
$$Y_k'(1) = A_k$$

$$C_1 e^{k\pi y} + C_2 e^{-k\pi y}$$

$$Y_k(y) = \cancel{C_1 \cos(k\pi y) + C_2 \sin(k\pi y)} \qquad \cancel{C_1 = 0} \qquad Y_k(0) = 0$$

$$Y_k(y) = C_k \sinh(k\pi y)$$

$$A_k = (k\pi)\, C_k \cosh(k\pi)$$
$$C_k = \frac{A_k}{\cosh(k\pi)} \frac{1}{k\pi}$$

$$\boxed{u(x,y) = \sum_{k=1}^{\infty} \frac{1}{(k\pi)} \frac{A_k}{\cosh(k\pi)} \sinh(k\pi y) \sin(k\pi x)}$$

(Napojení ve výsledku apod.)