Příklad Najděte $u: \mathbb{R}^2 \to \mathbb{R}$ řešení $\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ $\quad u(0,y) = \frac{1}{y}$.

Řešení:

$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = y$

$x = t + C_1 \qquad y = C_2 e^t$

α) $y>0 \quad C_2 > 0 \Rightarrow \lg y = C_2 + t$

$x - \lg y = \text{const}$

$u(x,y) = U(x - \lg y)$

$\frac{1}{y} = u(0,y) = U(-\lg y) \Rightarrow U(z) = e^z$

$u(x,y) = \frac{1}{y} \cdot e^x$ má smysl i pro $y<0$

$$\boxed{\text{Tj} \quad u(x,y) = \frac{1}{y} e^x \quad x \in \mathbb{R},\ y>0 \quad \text{nebo} \quad x\in\mathbb{R}\ y<0}$$

Příklad Nalezněte $u: \mathbb{R}^3 \to \mathbb{R}$ řešení $\quad$ $\boxed{\text{Cvičení 2}}$

$$(z+y-x)\frac{\partial u}{\partial x} + (z+x-y)\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Okrajová $u(x,y,1) = \sin(x+y)$

Řešení:

$\frac{dx}{dt} = z+y-x$

$\frac{dy}{dt} = z+x-y$

$\frac{dz}{dt} = z$

$\frac{dx}{dt} + \frac{dy}{dt} - 2\frac{dz}{dt} = 0$

$\frac{d}{dt}(x+y-2z) = 0$

$u(x,y,z) = U(x+y-2z)$

$\sin(x+y) = u_0(x,y) = U(x+y-2)$

$$\Rightarrow \boxed{u(x,y,z) = \sin(x+y-2z+2)}$$

Cvičení 2

Najděte $u: \mathbb{R}^2 \to \mathbb{R}$ řešící

$$x\frac{\partial u}{\partial x} + (x+y)\frac{\partial u}{\partial y} = 0$$

s počáteční podmínkou $u(x,y) = u_0(x)$.

a) na okolí bodu $(1,0)$ $\qquad$ b) na okolí bodu $(0,0)$.

Jsou někde nějaká omezení?

Řešme:

$\frac{dx}{dt} = x \Rightarrow x = C_1 e^t$

$\frac{dy}{dt} = y + x$

tedy $y = C_2 e^t + C_1 t e^t$ (řešíme lineární ODR)

Všimněme si, že $\frac{y}{x} = \frac{C_2 + C_1 t}{C_1} \Rightarrow x e^{-\frac{y}{x}} = C_1 e^t \cdot e^{-\frac{C_2 + C_1 t}{C_1}} = \text{const}$

Proto řešení hledáme jako
$u(x,y) = U\left(x e^{-\frac{y}{x}}\right)$
má být $u(x,0) = u_0(x) \Rightarrow u(x,0) = u_0(x) = U(x)$

$\Rightarrow \boxed{u(x,y) = u_0\left(x e^{-\frac{y}{x}}\right)}$ a má smysl na okolí $(x,0)$, $x \neq 0$

na okolí $(0,0)$ má smysl jen pokud $u_0 = \text{const}$

$U = \text{const}$ vůbec řeší naši rovnici!

Ukažme si jiný postup.
Vyjádřeme si ze vzorců
$x(t) = C_1 e^t$
$y(t) = C_2 e^t + C_1 t e^t$
a ~~[illegible]~~ fixujme $x_0 y_0$ na okolí $(1,0)$. Pro jaké hodnoty C_1, C_2 projde bodem (x,y) v $t = 0$?

$\Rightarrow \cancel{C_1 = x}$ $x_0 = C_1$
$y_0 = C_2 + C_1 0 = C_2$

Tedy charakteristika má tvar
$x(t) = x_0 \cdot e^t$
$y(t) = (y_0) e^t + x_0 t e^t$
Kdy tato charakteristika projde $y = 0$?
$0 = e^t (y_0 + x_0 t)$
tedy $t = -\frac{y_0}{x_0}$ $\quad x(t) = x_0 e^{-\frac{y_0}{x_0}}$ (jednoznačně)

Gričar PDE

Tedaj iskana rešitev je
$u(x,y) = u_0(x_0 e^{-\frac{y_0}{x_0}})$ ali po prejšnje

$$\boxed{u(x,y) = u_0(x e^{-\frac{y}{x}})}$$

Pr. $x^2 \frac{\partial u}{\partial x} + xy \frac{\partial u}{\partial y} = 0$ $u: \mathbb{R}^2 \to \mathbb{R}$

a) $u(x,1) = u_0(x)$ na delu $(1,1)$

b) $u(x,0)$ na delu $(0,0)$

$$\left.\begin{aligned} \frac{dx}{dt} &= x^2 \\ \frac{dy}{dt} &= xy \end{aligned}\right\} \quad \begin{aligned} \frac{d}{dt}(\ln x) &= x \\ \frac{d}{dt}\ln y &= x \end{aligned}$$

$$\Downarrow$$

$$\frac{d}{dt}(\ln x - \ln y) = 0$$

$$\underline{\frac{d}{dt} \ln\left(\frac{x}{y}\right) = 0}$$

Rešitev je tedaj $u(x,y) = \tilde{\tilde{U}}(\ln(\frac{x}{y})) = \tilde{U}\left(\frac{x}{y}\right) = \underline{U\left(\frac{y}{x}\right)}$ (velja v okolici (1,0))

$U\left(\frac{1}{x}\right) = u(x,1) = u_0(x) \Rightarrow U(z) = u_0\left(\frac{1}{z}\right)$

tedaj $\underline{u(x,y) = u_0\left(\frac{x}{y}\right)}$ na delu $(1,1)$

na delu $(0,0)$ — ostal je pokoj $\underline{u_0 = \text{const.}}$

Pr. $(3x+4y)\frac{\partial u}{\partial x} + (4x+3y)\frac{\partial u}{\partial y} = 0$ $u(x,0) = u_0(x)$ na delu $(1,0)$

$$\frac{dx}{dt} = 3x + 4y$$
$$\frac{dy}{dt} = 4x + 3y$$

$$A = \begin{pmatrix} 3-\lambda & 4 \\ 4 & +3-\lambda \end{pmatrix}$$

$$(3-\lambda)^2 - 16 = 0$$
$$9 - 6\lambda + \lambda^2 - 16 = 0$$
$$\lambda^2 - 6\lambda - 7 = 0$$
$$(\lambda-7)(\lambda+1) = 0 \qquad \underline{\lambda_1 = 7}, \ \underline{\lambda_2 = -1}$$

$$\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\vec{a}_1 = (1,1)$

$$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\vec{a}_2 = (+1,-1)$

Tedy: $\begin{pmatrix} e^{-7t} C_1 + e^{t} C_2 \\ e^{-7t} C_1 - e^{t} C_2 \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

je obecné řešení naší soustavy

nebo $\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 7(x+y)$ $\qquad \frac{dx}{dt} - \frac{dy}{dt} = -1(x-y)$

$$\frac{\frac{d}{dt}(x+y)}{x+y} = 7 \qquad \frac{\frac{d}{dt}(x-y)}{x-y} = -1$$

$$\frac{d}{dt}\ln(x+y) = 7 \qquad \frac{d}{dt}\ln|x-y| = -1$$

$$\frac{d}{dt}\left(\ln(x+y) + 7\ln(x-y)\right) = 0$$

$$\frac{d}{dt}\ln\left[(x+y)\cdot(x-y)^7\right] = 0$$

$$(x+y)(x-y)^7 = \text{const}$$

$$u(x,y) = U\left((x+y)(x-y)^7\right)$$

$$u_0(x) = u(x,0) = U(x^8) \qquad x>0$$

$$U(z) = u_0(z^{\frac{1}{8}})$$

$$\boxed{u(x,y) = u_0\left((x+y)^{\frac{1}{8}}(x-y)^{\frac{7}{8}}\right)}$$

Nalezněme prvotní integrály soustavy rovnic

$$\frac{\partial u}{\partial x} + xz\frac{\partial u}{\partial y} - xy\frac{\partial u}{\partial z} = 0$$

$$\left.\begin{aligned} \frac{dx}{dt} &= 1 \\ \frac{dy}{dt} &= xz \\ \frac{dz}{dt} &= -xy \end{aligned}\right\} \Rightarrow \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = 2xyz - 2xyz = 0$$

$$\boxed{\Psi_1(x,y,z) = y^2 + z^2}$$

řešení dané rovnice

Hledejme druhé nezávislé řešení. Pomocí věty o vedení

$$\tilde{x} = x \qquad \tilde{y} = y^2 + z^2 \qquad \tilde{z} = z$$

$\left(\tilde{x}_1 = \tilde{y},\ \tilde{x}_2 = \tilde{x},\ \tilde{x}_3 = \tilde{z} \right.$ z Věty o vedení $\left.\right)$

Hledejme teď řešení rovnice (pro $y > 0$)

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} - \tilde{x}\sqrt{\tilde{y} - \tilde{z}^2}\,\frac{\partial \tilde{u}}{\partial \tilde{z}} = 0$$

($\tilde{y}$ jakoby parametr)

$$\frac{d\tilde{x}}{dt} = 1 \qquad \frac{d\tilde{z}}{dt} = -\tilde{x}\sqrt{\tilde{y} - \tilde{z}^2}$$

$$\frac{1}{\sqrt{\tilde{y}}}\,\frac{d\tilde{z}}{\sqrt{1 - \frac{\tilde{z}^2}{\tilde{y}}}} = -\tilde{x}\,d\tilde{x}$$

$$\arcsin\left(\frac{\tilde{z}}{\sqrt{\tilde{y}}}\right) + \frac{1}{2}\tilde{x}^2 = C \qquad / \sin, \quad \tilde{z} > 0)$$

$$\frac{\tilde{z}}{\sqrt{\tilde{y}}}\cos\left(\frac{\tilde{x}^2}{2}\right) + \sqrt{1 - \frac{\tilde{z}^2}{\tilde{y}}}\,\sin\left(\frac{1}{2}\tilde{x}^2\right) = C$$

Tedy v původních proměnných

$$\frac{z}{\sqrt{y^2+z^2}}\cos\left(\frac{x^2}{2}\right) + \frac{y}{\sqrt{y^2+z^2}}\sin\frac{x^2}{2} = C$$

$$u(x,y,z) = \frac{z}{\sqrt{y^2+z^2}}\cos\frac{x^2}{2} + \frac{y}{\sqrt{y^2+z^2}}\sin\frac{x^2}{2}$$

ale též

$$\boxed{\psi_2 = z\cos\frac{x^2}{2} + y\sin\frac{x^2}{2}} \qquad \text{nad celým } \mathbb{R}^3$$

Na okolí $[1,1,0]$ řešení splňující $u(x,y,0) = x \cdot y$

$$\psi_1(x,y,0) = y^2 \Rightarrow y = \sqrt{\psi_1}$$

$$\psi_2(x,y,0) = y\sin\frac{x^2}{2} \Rightarrow \sin\frac{x^2}{2} = \frac{\psi_2}{\sqrt{\psi_1}}$$

$$\frac{x^2}{2} = \arcsin\frac{\psi_2}{\sqrt{\psi_1}}$$

$$x = \sqrt{2\arcsin\frac{\psi_2}{\sqrt{\psi_1}}}$$

$$x \cdot y = \sqrt{\psi_1} \cdot \sqrt{2\arcsin\frac{\psi_2}{\sqrt{\psi_1}}}$$

$$\boxed{u(x,y,z) = \sqrt{y^2+z^2} \cdot \sqrt{2\arcsin\frac{\left(z\cos\frac{x^2}{2} + y\sin\frac{x^2}{2}\right)}{\sqrt{\psi_1}}}}$$

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$\boxed{Pr.}$ $x\frac{\partial u}{\partial x} + (\ln x)\frac{\partial u}{\partial y} = 0$ $\quad u: \mathbb{R}^2 \to \mathbb{R}$

$u(x,0) = u_0(x)$ pro x ne oblast $(2,0)$

$$\frac{dx}{dt} = x \qquad \frac{dy}{dt} = \ln x \implies x = C_1 e^t, \quad \frac{dy}{dt} = \ln C_1 + t$$

$$\underline{y(t) = \ln C_1 t + \tfrac{1}{2}t^2 + C_2}$$

~~$\frac{dy}{dt}$~~

Fixujme x_0, y_0 ne oblast $(2,0)$ — po platí C_1, C_2 když $t=0$?

$$x_0 = C_1 \quad (>0)$$
$$y_0 = C_2$$
$$x(t) = x_0 e^t$$
$$y(t) = \ln x_0\, t + \tfrac{1}{2}t^2 + y_0$$

Kdy nastane $y=0$?

$$0 = \ln x_0 \cdot t + \tfrac{1}{2}t^2 + y_0$$

$$t^2 + 2\ln x_0 \cdot t + 2y_0 = 0$$

$$t_{1,2} = \frac{-2\ln(x_0) \pm \sqrt{4\ln^2 x_0 - 8y_0}}{2} = -\ln x_0 \pm \sqrt{\ln^2 x_0 - 2y_0}$$

$$x(t) = x_0 e^{-\ln x_0 + \sqrt{\ln^2 x_0 - 2y_0}}$$

$$\underline{u(x,y) = u_0\left(x e^{-\ln x + \sqrt{\ln^2 x - 2y}}\right)}$$

Příklad: $\sqrt{x}\frac{\partial f}{\partial x} + \sqrt{y}\frac{\partial f}{\partial y} + \sqrt{z}\frac{\partial f}{\partial z} = 0$

$$f(1,y,z) = y - z$$

$$\frac{dx}{dt} = \sqrt{x} \qquad \frac{dy}{dt} = \sqrt{y} \qquad \frac{dz}{dt} = \sqrt{z}$$

$$\frac{x^{\frac{1}{2}}}{\frac{1}{2}} = t + C_1 \qquad \frac{y^{\frac{1}{2}}}{\frac{1}{2}} = t + C_2 \qquad \frac{z^{\frac{1}{2}}}{\frac{1}{2}} = t + C_3$$

Tedy
$$x^{\frac{1}{2}} - y^{\frac{1}{2}} = \text{const}$$
$$x^{\frac{1}{2}} - z^{\frac{1}{2}} = \text{const}$$

$$u(x,y,z) = U\left(\sqrt{x} - \sqrt{y},\ \sqrt{x} - \sqrt{z}\right)$$

$$u(1,y,z) = y - z \implies U(\varphi_1, \varphi_2) = (1-\varphi_1)^2 - (1-\varphi_2)^2$$

$$\underline{u(x,y,z) = (1 - \sqrt{x} - \sqrt{y})^2 - (1 - \sqrt{x} - \sqrt{z})^2} \qquad x,y,z > 0.$$