$$= -\int_0^\infty \int_{\mathbb{R}} \frac{\partial \varphi}{\partial t} u \, dx dt - \int_{\mathbb{R}} u(0,x)\varphi(0,x)\,dx - \frac{1}{2}\int_0^\infty\int_{\mathbb{R}} u^2 \frac{\partial \varphi}{\partial x}\,dx dt$$

Definice

Mějme $u \in L^2_{loc}([0,\infty)\times\mathbb{R})$ je taková, že zobrazení $t \mapsto \int_{\mathbb{R}} u(t,\cdot)\varphi(t,\cdot)\,dx$ je spojitá funkce v čase $t=0$ $\forall \varphi \in C^1_0([0,\infty)\times\mathbb{R})$. Pak je u slabé řešení Burgersovy rovnice, jestliže $\forall \varphi \in C^\infty_0([0,\infty)\times\mathbb{R})$

$$\int_0^\infty \int_{\mathbb{R}} u \frac{\partial \varphi}{\partial t}\,dx dt + \frac{1}{2}\int_0^\infty\int_{\mathbb{R}} u^2 \frac{\partial \varphi}{\partial x}\,dx dt = -\int_{\mathbb{R}} u_0(x)\varphi(0,x)\,dx.$$

Důsledky plynou: (i) Je-li u klasické řešení Burgersovy rovnice, pak je i slabým (s danou poč. podmínkou).
(ii) Je-li u slabé řešení dle definice výše a $u \in C^1([0,\infty)\times\mathbb{R})$, pak je též klasickým řešením téže úlohy (s poč. podmínkou).

$\boxed{DÚ}$ – dokažte

Co potřebujeme: $\iint f\varphi\,dx = 0$ $\forall \varphi \in C^\infty_0$ a $f \in L^1_{loc} \Rightarrow f = 0$ s.v.
f spoj. $\Rightarrow f \equiv 0$.

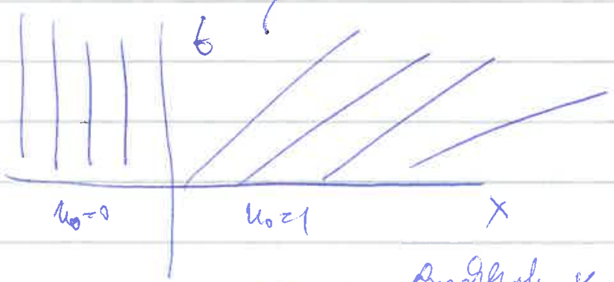
Příklad: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ $u \in (0,\infty)\times\mathbb{R}$

$u(0,x) = u_0(x) = \begin{cases} 0 & x<0 \\ 1 & x\geq 1 \end{cases}$ nebo $\begin{cases} 1 & x<0 \\ 0 & x\geq 0 \end{cases}$

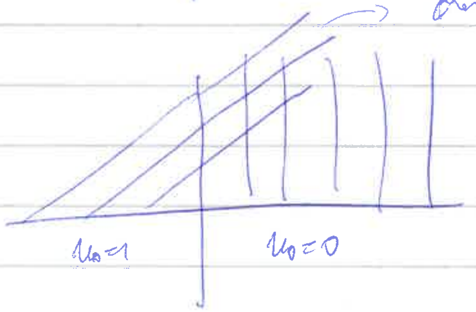
(tzv. Riemannův problém)

Problém je v pol. funkce 0: charakteristiky $x = x_0$, $t = s$; $x = s + x_0$, $t = s$

nedef. 1



Charakteristiky se protínají!!



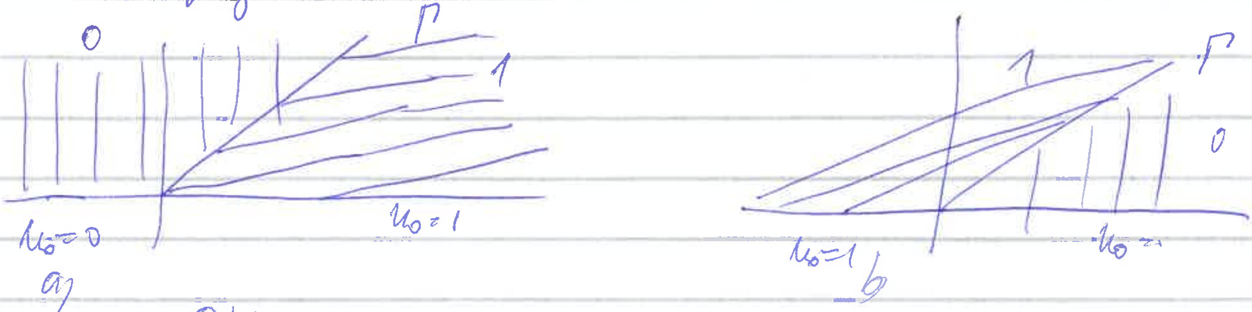
Vzniká tzv. rázová vlna!

PR

[illegible]

Rešenje: [illegible] $\Gamma = \{(t,x) \in [0,\infty) \times [0,\infty),\ t = \alpha x\}$

a definicija rešenja



Odakle je jedina vrednost α, da bi ovo rešenje bilo slabo rešenje

[illegible] a)

$$0 = -\int_0^\infty \int_{\mathbb{R}} \left(u\frac{\partial\varphi}{\partial t} + \frac{u^2}{2}\frac{\partial\varphi}{\partial x} \right) dx\,dt + \int_{\mathbb{R}} u_0\,\varphi(0,\cdot)\,dx$$

$$= \iint_{\substack{\{t<\alpha x\} \\ t>0}} \left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}\frac{\partial\varphi}{\partial x} \right) dx\,dt - \int_0^\infty \varphi(0,x)\,dx$$

$$I_1 = \iint_{\substack{\{t<\alpha x\} \\ t>0}} \frac{\partial\varphi}{\partial t}\,dx\,dt = -\int_0^\infty \int_0^{\alpha x} \frac{\partial\varphi}{\partial t}\,dt\,dx = -\int_0^\infty (\varphi(\alpha x, x) - \varphi(0,x))\,dx$$

$$I_2 = \iint_{\substack{\{t<\alpha x\} \\ t>0}} \frac{1}{2}\frac{\partial\varphi}{\partial x}\,dx\,dt = -\int_0^\infty \int_{\frac{t}{\alpha}}^\infty \frac{1}{2}\frac{\partial u}{\partial x}\,dx\,dt = \frac{1}{2}\int_0^\infty u\left(t, \frac{t}{\alpha}\right) dt \qquad \frac{t}{\alpha} = x$$

$$= \frac{\alpha}{2}\int_0^\infty u(\alpha x, x)\,dx.$$

Ako bi ovo rešenje bilo slabo, mora biti $\left(\frac{\alpha}{2} - 1\right) \int_0^\infty \varphi(\alpha x, x)\,dx = 0 \quad \forall \varphi \in C_0^1([0,\infty)\times\mathbb{R})$

$\Rightarrow \boxed{\alpha = 2}$

Analogno za b) — slabo [illegible]

$$0 = -\iint_{\{\alpha x < t,\ t>0\}} \left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}\frac{\partial\varphi}{\partial x} \right) dx\,dt - \int_{-\infty}^0 \varphi(0,\cdot)\,dx$$

$$I_1 = -\iint_{\{\alpha x < t,\ t>0\}} \frac{\partial\varphi}{\partial t}\,dx\,dt = -\int_{-\infty}^0 \int_0 \frac{\partial\varphi}{\partial t}\,dt\,dx - \int_0^\infty \int_{\alpha x}^\infty \frac{\partial\varphi}{\partial t}\,dt\,dx = \int_{-\infty}^0 \varphi(0,x)\,dx + \int_0^\infty \varphi(\alpha x, x)\,dx$$

$$I_2 = -\iint_{\{\alpha x<t,\ t>0\}} \frac{1}{2}\frac{\partial\varphi}{\partial x}\,dx\,dt = -\int_0^\infty \int_{-\infty}^{\frac{t}{\alpha}} \frac{1}{2}\frac{\partial\varphi}{\partial x}\,dx\,dt = -\frac{1}{2}\int_0^\infty \varphi\left(t, \frac{t}{\alpha}\right) dt = -\frac{\alpha}{2}\int_0^\infty \varphi(\alpha x, x)\,dx$$

$\Rightarrow$ Opet $\left(1 - \frac{\alpha}{2}\right) \int_0^\infty \varphi(\alpha x, x)\,dx = 0 \Rightarrow \boxed{\alpha = 2}$

Cvičení 5

Převod na kanonický tvar

Předmluva: připomenutí,

~~$y = P^T y$~~ $\quad y = P^T x \iff y_k = \sum_{i=1}^{N} p_{ik} x_i$

Potom

$$\sum_{k,l=1}^{N} (P^T A P)_{kl} \frac{\partial^2 v}{\partial y_k \partial y_l} \quad \ldots \qquad A\text{ - symetrická}$$

D_{ij} . diagonální matice

1, Nalézt vlastní čísla a vlastní vektory

2, Využít spektrální rozkladu A - symetrická matice ?

Nalézt transformaci $\quad y_i = \sum b_{ij} x_j$

převádí kvadratickou formu

$$\sum_{i=1}^{N} a_{ij} x_i x_j$$

na kanonický tvar

$$\sum_{k,l} b_{ij} x_j \, b_{kl} x_l \, \underset{\text{diagonál matice}}{d_{ik}} = \underset{\pm 1}{\sum a_{ij} x_i x_j}$$

$$(B^T D B) = A$$

$$D = (B^T A B^{-1})$$

⇓ Nalézt matici B není těžké, pak rovněž inverzi transponované

Příklad:

$u_{xx} + 4u_{xy} + 8u_{yy} \cdot ! \quad u_y + u_y = 0$

a) Matice $\begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}$

Hledáme vlastní čísla $\quad \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 8-\lambda \end{pmatrix} = (1-\lambda)(8-\lambda) - 4 = 8 - 9\lambda + \lambda^2 - 4 = 0$

$$\lambda^2 - 9\lambda + 4 = 0$$

$$\lambda_{1,2} = \frac{9 \pm \sqrt{81-16}}{2} = \frac{9 \pm \sqrt{65}}{2} \quad \ldots \text{ příliš komplikované}$$

Jinou metodou

$$x^2 + 4xy + 8y^2 = (x+2y)^2 + (2y)^2 \Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \text{ a hledáme její inverzi}$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right)$$

$$B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix} \Rightarrow B^{-T} = \begin{pmatrix} 1 & 0 \\ -1 & \frac{1}{2} \end{pmatrix}$$

$$\boxed{\begin{aligned} \eta &= x \\ \mu &= -x + \tfrac{1}{2}y \end{aligned}}$$

Teď $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \mu}$

$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\partial u}{\partial \mu}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial^2 u}{\partial \eta \partial \mu} + \frac{\partial^2 u}{\partial \mu^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}\frac{\partial^2 u}{\partial \eta \partial \mu} - \frac{1}{2}\frac{\partial^2 u}{\partial \mu^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{4}\frac{\partial^2 u}{\partial \mu^2}$$

$$\Rightarrow u_{xx} + 4u_{xy} + 8u_{yy} + u_x + u_y = \frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial^2 u}{\partial \eta \partial \mu} + \frac{\partial^2 u}{\partial \mu^2} + 2\frac{\partial^2 u}{\partial \eta \partial \mu} - 2\frac{\partial^2 u}{\partial \mu^2} + 2\frac{\partial^2 u}{\partial \mu^2}$$

$$+ \frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \mu} + \frac{1}{2}\frac{\partial u}{\partial \mu} = \boxed{\underbrace{\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \mu^2}}_{\Delta_{\eta\mu}} + \frac{\partial u}{\partial \eta} - \frac{1}{2}\frac{\partial u}{\partial \mu}}$$

Zespodu dolů zjednodušit teď nemůže, to už dělat nebudeme.

Příklad:

$$u_{xx} + 2u_{xy} + 2u_{yy} + 4u_{yz} + 5u_{zz} + u_x + u_y = 0$$

$$x^2 + 2xy + 2y^2 + 4yz + 5z^2 = (x+y)^2 + (y+2z)^2 + z^2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Tj. hledaná matice je $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

$$\begin{aligned} \eta &= x \\ \xi &= -y + x \\ \mu &= -2y + z \end{aligned}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \xi}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial \xi} - 2\frac{\partial u}{\partial \mu}$$

$\Rightarrow$ Původní rovnice se transformuje na

$$\Delta_{\eta\xi\mu} u + \frac{\partial u}{\partial \eta} - 2\frac{\partial u}{\partial \mu} = 0$$

Příklad

$$u_{xx} - 2u_{xy} - 3u_{yy} + u_y = 0$$

$$x^2 - 2xy - 3y^2 = (x-y)^2 - (2y)^2$$

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & +2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & +\frac{1}{2} \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & +\frac{1}{2} \\ 0 & 1 & 0 & +\frac{1}{2} \end{array}\right)$$

Hledaná matice $\begin{pmatrix} 1 & 0 \\ +\frac{1}{2} & +\frac{1}{2} \end{pmatrix}$

$\eta = x$

$\mu = \frac{1}{2}x + \frac{1}{2}y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta} + \frac{1}{2}\frac{\partial u}{\partial \mu}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\partial u}{\partial \mu}$$

$$\boxed{\frac{\partial^2 u}{\partial \eta^2} - \frac{\partial^2 u}{\partial \mu^2} + \frac{1}{2}\frac{\partial u}{\partial \mu} = 0}$$

Příklad

$$u_{xx} + 4u_{xy} + 2u_{xz} + 2u_{xw} + 3u_{yy} + 6u_{yz} - 2u_{yw} + 5u_{zz} + 2u_{zw} + 4u_{yw}$$

$$u_x + u_y + u_w = 0$$

Kvadratická forma

$$x^2 + 4xy + 2xz + 2xw + 3y^2 + 6yz - 2yw + \cancel{5}3z^2 + 2zw + 4yw$$

$$= (x + 2y + z + w)^2 - 4yz - 4yw - 2zw - 4y^2 - z^2 - w^2 + 3y^2 + 6yz - 2yw + 3z^2 + 2zw + 4yw$$

$$= (x+2y+z+w)^2 - y^2 + 2yz + 3z^2 - w^2$$

$$= (x+2y+z+w)^2 - (y-z)^2 + z^2 + 3z^2 - w^2$$

$$= (x+2y+z+w)^2 - (y-z)^2 + (2z)^2 - w^2$$

$$u_{\eta\eta} - u_{\mu\mu} + u_{\xi\xi} - u_{\lambda\lambda} \ ?$$

$\eta = x + 2y + z + w$

$\mu = y - z$

$\xi = 2z$

$\lambda = w$

19

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & -\frac{1}{2} & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Hledaná matice J

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

- $\alpha = x$
  $\beta = -2x + y$
  $\gamma = -\frac{3}{2}x + \frac{1}{2}y + \frac{1}{2}z$
  $\delta = w$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha} - 2\frac{\partial u}{\partial \beta} - \frac{3}{2}\frac{\partial u}{\partial \gamma}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \beta} + \frac{1}{2}\frac{\partial u}{\partial \gamma}$$
$$\frac{\partial u}{\partial z} = \frac{1}{2}\frac{\partial u}{\partial \gamma}$$
$$\frac{\partial u}{\partial w} = \frac{\partial u}{\partial \delta}$$

$$\boxed{u_{\alpha\alpha} - u_{\beta\beta} + u_{\gamma\gamma} - u_{\delta\delta} + \frac{3}{2}\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta} - \frac{\partial u}{\partial \gamma} + \frac{\partial u}{\partial \delta} = 0}$$