(23)

$$= \frac{1}{2} \frac{1}{(2n+7)\frac{\pi}{2}} + \frac{1}{2} \frac{1}{(2n-9)\frac{\pi}{2}}$$

Celkem $A_n = \frac{2}{\pi(2n-1)} - \frac{1}{\pi(2n+7)} - \frac{1}{\pi(2n-9)}$ $\left(\sim \frac{1}{n^2}\right)$

$$u(t,x) = 1 + \frac{1}{\pi}\sum_{n=1}^{\infty} \left(\frac{2}{2n-1} - \frac{1}{2n+7} - \frac{1}{2n-9}\right) e^{-\left(\frac{\pi}{2}(2n-1)\right)^2 t} \sin\left(\frac{\pi}{2}(2n-1)x\right)$$

Cvičení č. 7

(1) Hledejme řešení

$$\frac{\partial u}{\partial t} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \quad \text{na } (0,\infty) \times (-a,a)$$

$$u(0,x) = x^2(x-a) \quad t=0 \text{ na } (-a,a)$$

u je $2a$-periodická

Oddělujme proměn. $X'' + \lambda X = 0$

X je $2a$-periodické $\Rightarrow \lambda \leq 0$ ne

$\Rightarrow X = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$ $\quad \sqrt{\lambda} = \frac{\pi n x}{a}$

$T' + c^2\lambda^2 T = 0$ $\quad \lambda = \frac{n\pi x}{a}$

$T(0) = A_n$ $\quad n \in \mathbb{N}_0$

$T_0 = A_0$

$T_n = A_n e^{-c^2\left(\frac{n\pi}{a}\right)^2 t}$

$$u(t,x) = \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi}{a}x\right) + B_n\sin\frac{n\pi}{a}x\right) e^{-c^2\left(\frac{n\pi}{a}\right)^2 t} + A_0$$

a dosadíme do počáteční podm.

$$x^2 = \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi}{a}x\right) + B_n\sin\left(\frac{n\pi}{a}x\right)\right) + A_0$$

$$A_0 = 2\cdot\frac{1}{a}\int_{-a}^{a} x^2(x-a)\,dx = \frac{2}{a}\cdot\left[\ldots - \frac{2}{3}a^4\right] = -\frac{4}{3}a^3$$

dále $A_n = \frac{1}{a}\int_{-a}^{a} \underbrace{x^2(x-a)}_{v}\underbrace{\cos\left(\frac{n\pi}{a}x\right)}_{u'}dx = \frac{1}{a}\left[x^2(x-a)\underbrace{\frac{\sin\frac{n\pi}{a}x}{\frac{n\pi}{a}}}_{0}\right]_{-a}^{a} - \frac{1}{n\pi}\int_{-a}^{a}\underbrace{(3x^2-2xa)}_{v}\underbrace{\sin\left(\frac{n\pi}{a}x\right)}_{u'}dx$

$$= \left[(3x^2-2xa)\frac{-\cos\left(\frac{n\pi}{a}x\right)}{\frac{n\pi}{a}}\right]_{-a}^{a} + \frac{a}{n^2\pi^2}\int(6x-2a)\cos\left(\frac{n\pi}{a}x\right)dx = \frac{a}{n^2\pi^2}\underbrace{\left[(6x-2a)\sin\left(\frac{n\pi}{a}x\right)\right]}_{=0}$$

$$-\frac{a^2}{n^2\pi^2}\int_{-a}^{a} 6\sin\frac{n\pi}{a}x\,dx = \qquad a\cdot 2a\frac{(-1)^{n+1}}{(n\pi)^2}$$

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Analogicky

$$B_n = \frac{2}{2a}\int_{-a}^{a} x^2(x-a)\sin\left(\frac{n\pi}{a}x\right)dx = \frac{1}{n\pi}\left[-x^2(x-a)\cos\left(\frac{n\pi x}{a}\right)\right]_{-a}^{a} + \dots$$

$$= \frac{1}{n\pi}(+2a^3)(-1)^n \dots$$

(2) Hledejme řešení

$$\partial_t u - \partial_{xx} u = 0 \qquad 0<x<1 \quad t\in(0,\infty)$$

$$u(0,x) = g(x) \quad t>0 \qquad x\in(0,1)$$

$$u(t,0) = 0$$

$$\frac{\partial u}{\partial x}(t,1) + h\,u(t,1) = 0 \qquad h>0$$

Řešení:

Separované podmínky $\quad X'' + \lambda X = 0$

$$X(0)=0 \qquad X'(1) + hX(1) = 0$$

a) $\lambda<0$

$$X = C_1 e^{-\sqrt{|\lambda|}x} + C_2 e^{\sqrt{|\lambda|}x} \qquad x>0$$

$X(0)=0 \Rightarrow \quad C_1+C_2=0$

$X'(1)+hX(1)=0 \Rightarrow \quad C_1\,(-\sqrt{|\lambda|})\,e^{-\sqrt{|\lambda|}} + C_2\sqrt{|\lambda|}\,e^{\sqrt{|\lambda|}} + C_1 h\,e^{-\sqrt{|\lambda|}} + C_2 h\,e^{\sqrt{|\lambda|}} = 0$

$$0 = C_1\left[-\sqrt{|\lambda|}e^{-\sqrt{|\lambda|}} - \sqrt{|\lambda|}e^{\sqrt{|\lambda|}} + h e^{-\sqrt{|\lambda|}} - h e^{+\sqrt{|\lambda|}}\right] = 0$$

$C_1\neq 0 \Rightarrow \quad e^{2\sqrt{|\lambda|}} = \dfrac{h-\sqrt{|\lambda|}}{h+\sqrt{|\lambda|}}$ nemá, $h>0$,

LS: roste nemá řešení $\Rightarrow$ kandidáti $\lambda<0$ nejsou

b) $\lambda=0$ $\quad X''=0 \qquad X(0)=0$

$X = Ax+B \qquad \Rightarrow B=0$

$X'(1) + hX(1) = \quad A+hA = 0 \quad$ ne

c) $\lambda>0$

$$X(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$$

$C_1=0 \quad \wedge \quad C_2\left(\sqrt{\lambda}\cos(\sqrt{\lambda}) + h\sin\sqrt{\lambda}\right) = 0$

$$\operatorname{tg}(\sqrt{\lambda}) = -\frac{\sqrt{\lambda}}{h}$$

potom $\exists$ nekonečně mnoho $\lambda_k \qquad 0<\lambda_1<\lambda_2<\dots \qquad \lambda_k\to\infty \quad k\to\infty$

$$X_k = A_k\sin(\sqrt{\lambda_k}\,x)$$

a tedy $\underbrace{u(t,x) = B_k\sin(\sqrt{\lambda_k}x)\,e^{-\lambda_k t}}$ $\qquad u(t,x) = \sum_{k=1}^{\infty} B_k\sin(\mu_k x)\,e^{-\mu_k^2 t}$

Nejde o Fourierovu řadu, ale jde o úplný $\sin(\sqrt{\lambda_n}\,x)$ úplný na $L^2(-1,1)$,

proto rozložme pod. podm. hod. v 0

$$g(x) = \sum_{k=1}^{\infty} B_k\sin(\mu_k x) \qquad \mu_k = \sqrt{\lambda_k}$$

(3)

Počítejme $\int_0^1 \sin(\mu_k x)\sin(\mu_m x)\,dx = \frac{1}{2}\int_0^1 [\cos((\mu_k-\mu_m)x) - \cos((\mu_k+\mu_m)x)]\,dx$

$$= \frac{1}{2}\,\frac{(\mu_k+\mu_m)\sin(\mu_k-\mu_m) - (\mu_k-\mu_m)\sin(\mu_k+\mu_m)}{\mu_k^2-\mu_m^2} = \frac{\mu_m\sin\mu_k\cos\mu_m - \mu_k\sin\mu_m\cos\mu_k}{\mu_k^2-\mu_m^2}$$

Protože $\operatorname{tg}\mu_k = \frac{\sin\mu_k}{\cos\mu_k} = -\frac{\mu_k}{h}$

$\sin\mu_k = -\frac{\mu_k}{h}\cos\mu_k$

$$\Rightarrow \frac{\mu_m\left(-\frac{\mu_k}{h}\cos\mu_k\cos\mu_m\right) + \mu_k\left(-\frac{\mu_m}{h}\right)\cos\mu_k\cos\mu_m}{\mu_k^2-\mu_m^2} = 0$$

$$\int_0^1 \sin^2(\mu_k x)\,dx = \frac{1}{2}\int_0^1 \frac{1-\cos(2\mu_k x)}{2} = \frac{1}{2} - \frac{1}{4\mu_k}\sin(2\mu_k) = \frac{\mu_k - \sin\mu_k\cos\mu_k}{2\mu_k}$$

Proto $B_k = \frac{2\mu_k}{\mu_k - \sin\mu_k\cos\mu_k}\int_0^1 g(x)\sin(\mu_k x)\,dx$

Úplně stejně pak lze řešit danou diferenciální rovnici s danými okrajovými podmínkami.

Příklad

Uvažujme úlohu

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sin(k\pi x)\sin\alpha t \qquad k\in\mathbb{N}$$

$$u(0,x) = u_0(x)$$

$$u(t,0) = u(t,1) = 0$$

Užitím metody řešení hledáme ve tvaru

$$u(t,x) = \sum_{k=1}^{\infty} T_k(t)\,X_k(x) \qquad X_k(x) = \sin(k\pi x)$$

Tedy nová úloha je

$$T_k'(t) + T_k(t)\cdot(k\pi)^2 = \sin(\alpha t)$$

a máme $u(t,x) = u_1(t,x) + T_k(t)\sin(k\pi x)$

$$\sum_{k=1}^{\infty} B_k\sin(k\pi x) \qquad B_k = 2\int_0^1 \sin(k\pi x)\,u_0(x)\,dx$$

Hledejme T_k tak, aby $T_k(0) = 0$

Tedy

$$\left(T_k(t)\cdot e^{(k\pi)^2 t}\right)' = \sin(\alpha t)\,e^{(k\pi)^2 t}$$

$$T_k(t) = e^{-(k\pi)^2 t}\int_0^t \sin(\alpha\tau)\,e^{(k\pi)^2\tau}\,d\tau \qquad \text{(splňuje nulovou podmínku)}$$

(20)

Potřebujeme

$$\int_0^t \sin(\alpha\tau)\, e^{(k\pi)^2\tau}\, d\tau = \operatorname{Im}\left[\int_0^t e^{(i\alpha+(k\pi)^2)\tau}\, d\tau\right] =$$

$$= \operatorname{Im}\left[\frac{e^{(i\alpha+(k\pi)^2)\tau}}{i\alpha+(k\pi)^2}\right]_0^t = \operatorname{Im}\left(\frac{e^{i\alpha t+(k\pi)^2 t}-1}{i\alpha+(k\pi)^2}\right)$$

$$= \operatorname{Im}\left(\frac{\left[(\cos(\alpha t)+i\sin(\alpha t))e^{(k\pi)^2 t}-1\right]\left(-i\alpha+(k\pi)^2\right)}{(k\pi)^2+\alpha^2}\right)$$

$$= \frac{(k\pi)^2\cos(\alpha t)\, e^{(k\pi)^2 t}+\alpha\sin(\alpha t)\, e^{(k\pi)^2 t}+\alpha}{(k\pi)^2+\alpha^2}$$

$$\boxed{T_k(t)\sin(k\pi x) = \frac{(k\pi)^2\cos(\alpha t)+\alpha\sin(\alpha t)+\alpha e^{-(k\pi)^2 t}}{(k\pi)^2+\alpha^2}\sin(k\pi x)}$$

Pozn. řešení je omezené na libovolné volbě $\alpha\in\mathbb{R}$!

Příklad

Řešme úlohu:

$\partial_t u - \Delta u = 0$ na $(0,\infty)\times B_1(0)$ a $B_1(0)\subset\mathbb{R}^3$!

$u(0,x) = v(r)$ $\left(= r\sin(\pi r)\right)$

$u(t,|x|=1) = 0$

Řešení

Hledejme řešení ve tvaru $u(t,x) = v(t,r)$

$$\partial_t v - \left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)v = 0$$

Zavedeme funkci $w(t,x) = rv(t,r)$

$$\partial_t w - \frac{\partial^2}{\partial r^2}(rv) = \partial_t w - \frac{\partial^2}{\partial r^2}(rv(t,r)) = r\frac{\partial v}{\partial t} - r\left(\frac{\partial^2 v}{\partial r^2}+2\frac{\partial v}{\partial r}\right) = 0$$

$w(t,1) = 0$ ale $w(t,0)$?

Chceme u omezené kolem počátku – tedy i v omezené kolem $r=0$ $\Rightarrow$ $rv \to 0$!
$\quad\quad r\to 0+$

$w(t,0) = 0$ – umíme řešit

$w(0,r) = r^2\sin(\pi r)$

$$w(t,r) = \sum_{k=1}^{\infty} B_k \sin(k\pi r)\, e^{-(k\pi)^2 t}$$

$$B_k = 2\int_0^1 r^2\sin(\pi r)\sin(k\pi r)\, dr =$$

a) $k=1$

$$B_1 = 2\int_0^1 r^2\sin^2(\pi r)\, dr = 2\int_0^1 r^2\,\frac{1-\cos 2\pi r}{2}\, dr = \frac{1}{3} - \int_0^1 \underbrace{r^2}_{v}\underbrace{\cos(2\pi r)}_{u'}\, dr$$

(11)

$$= \frac{1}{3} - \Big[\underbrace{r^2 \frac{\sin 2\pi r}{2\pi}}_{=0} \Big]_0^1 + \frac{2}{2\pi} \int_0^1 r \sin(2\pi r)\, dr = \frac{1}{3} + \frac{1}{\pi}\Big[-r \frac{\cos 2\pi r}{2\pi}\Big]_0^1 + \frac{1}{2\pi^2} \underbrace{\int_0^1 \cos(2\pi r)\, dr}_{=0}$$

$$= \boxed{\frac{1}{3} - \frac{1}{2\pi^2}}$$

$k\geq 1$ $$2 \int_0^1 r^2 \underbrace{\sin(\pi r) \sin(k\pi r)}_{-\frac{1}{2}(\cos((k+1)\pi r) - \cos((k-1)\pi r))} dr = \int_0^1 r^2 \cos((k-1)\pi r)\, dr - \int_0^1 r^2 \cos((k+1)\pi r)\, dr$$

$$= \underbrace{\Big[r^2 \frac{\sin((k-1)\pi r)}{(k-1)\pi} \Big]_0^1}_{0} - \underbrace{\Big[r^2 \frac{\sin((k+1)\pi r)}{(k+1)\pi} \Big]_0^1}_{0} - \frac{2}{(k-1)\pi} \int_0^1 r \sin((k-1)\pi r)\, dr + \frac{2}{(k+1)\pi} \int_0^1 r \sin((k+1)\pi r)\, dr$$

$$= -\frac{2}{(k-1)\pi} \Big[r \,\frac{-\cos((k-1)\pi r)}{(k-1)\pi} \Big]_0^1 + \frac{2}{(k+1)\pi} \Big[r \frac{\cos((k+1)\pi r)}{(k+1)\pi} \Big]_0^1 + 0$$

$$= \frac{2}{(k-1)^2\pi^2} (-1)^{k-1} - \frac{2}{(k+1)^2\pi^2} (-1)^{k-1} = \frac{k^2+2k+1 - k^2+2k-1}{(k^2-1)^2 \pi^2} \cdot 2(-1)^{k-1}$$

$$= \frac{8k(-1)^{k-1}}{(k^2-1)^2\pi^2}$$

Celk. $$\underbrace{u(t,x) = \sum_{k=2}^{\infty} \frac{8k(-1)^{k-1}}{(k^2-1)^2\pi} \cdot \frac{\sin(k\pi|x|)}{|x|} e^{-(k\pi)^2 t} + \Big(\frac{1}{3} - \frac{1}{2\pi^2}\Big) \frac{\sin(\pi|x|)}{|x|} e^{-\pi^2 t}}$$

(v 0 .. O.K. – dodefinovat)

Cvičení 0.8

(1) Řešte rovnici

$\partial_{tt} u - \Delta u = 0$ na $(0,\infty)\times\mathbb{R}^3$

$u(0,x) = e^{-a|x|^2}$

$\partial_t u(0,x) = 0$

Určete $u(t,x)$ a $\lim_{t} u(t,0)$

Řešení

Buď $u(t,x) = v(t,r)$ $r = |x|$. Pak

$\partial_{tt} v - (\partial_{rr} v + \frac{2}{r}\partial_r v) = 0$ $/ r$

$\partial_{tt}(rv) - \partial_{rr}(rv) = 0$ $w := rv$

$\partial_{tt} w - \partial_{rr} w = 0$

$w(0,r) = re^{-a|r|^2}$ $w(0,0)$ na $(0,\infty)\times(0,\infty)$

$\partial_t w(0,r) = 0$ na $(0,\infty)$