On the steady compressible Navier–Stokes–Fourier system with temperature dependent viscosities

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results obtained in collaboration with M. Bulíček (Praha), V. Giovangigli (Paris), D. Jesslé (Toulon), A. Jüngel (Wien), O. Kreml (Praha), P.B. Mucha (Warszawa), A. Novotný (Toulon), T. Piasecki (Warszawa), N. Zamponi (Praha), E. Zatorska (London)

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Steady compressible Navier–Stokes–Fourier system I

$\Omega \subset \mathbb{R}^3$, bounded, smooth ($C^2$)

- Balance of mass

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \qquad (1)$$

ϱ: $\Omega \mapsto \mathbb{R}$ . . . density of the fluid
$\mathbf{u}$: $\Omega \mapsto \mathbb{R}^3$ . . . velocity field

- Balance of momentum

$$\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbb{S} + \nabla p = \varrho \mathbf{f} \qquad (2)$$

$\mathbb{S}$ . . . viscous part of the stress tensor (symmetric tensor)
$\mathbf{f}$: $\Omega \mapsto \mathbb{R}^3$ . . . specific volume force (given)
p. . . pressure (scalar quantity)

- Balance of total energy

$$\operatorname{div}\left(\varrho E \mathbf{u}\right) + \operatorname{div}(\mathbf{q} + p\mathbf{u}) = \varrho \mathbf{f} \cdot \mathbf{u} + \operatorname{div}\left(\mathbb{S}\mathbf{u}\right) \qquad (3)$$

$E = \frac{1}{2}|\mathbf{u}|^2 + e$. . . specific total energy
e . . . specific internal energy (scalar quantity)
$\mathbf{q}$ . . . heat flux (vector field)
(no energy sources assumed)

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Steady compressible Navier–Stokes–Fourier system II

- Boundary conditions at $\partial\Omega$: velocity

$$\begin{gathered} \mathbf{u} \cdot \mathbf{n} = 0 \\ (\mathbb{I} - \mathbf{n} \otimes \mathbf{n})(\mathbb{S}\mathbf{n} + \lambda \mathbf{u}) = \mathbf{0}, \end{gathered} \tag{4}$$

$\lambda \geq 0$

or

$$\mathbf{u} = \mathbf{0} \tag{5}$$

- Boundary conditions at $\partial\Omega$: temperature

$$\mathbf{q} + L(\vartheta - \Theta_0) = 0, \tag{6}$$

$L > 0$, $\Theta_0 > 0$

or

$$\vartheta = \Theta_1 \tag{7}$$

- Total mass

$$\int_\Omega \varrho \, \mathrm{d}x = M > 0 \tag{8}$$

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Thermodynamics I

We will work with basic quantities: density ϱ and temperature ϑ

We assume: $e = e(\varrho, \vartheta)$, $p = p(\varrho, \vartheta)$

Gibbs' relation

$$\frac{1}{\vartheta}\left(De(\varrho,\vartheta) + p(\varrho,\vartheta) D\left(\frac{1}{\varrho}\right)\right) = Ds(\varrho,\vartheta) \tag{9}$$

with $s(\varrho, \vartheta)$ the specific entropy.

The specific entropy fulfills formally the entropy balance

$$\operatorname{div}(\varrho s \mathbf{u}) + \operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma = \frac{\mathbb{S} : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2} \tag{10}$$

Second law of thermodynamics

$$\sigma = \frac{\mathbb{S} : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2} \geq 0 \tag{11}$$

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Constitutive relations I

- Newtonian fluid

$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla \mathbf{u}) = \mu(\vartheta)\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \tfrac{2}{3}\,\mathrm{div}\,\mathbf{u}\mathbb{I}\right] + \xi(\vartheta)\,\mathrm{div}\,\mathbf{u}\mathbb{I} \qquad (12)$$

$\mu(\cdot)\colon \mathbb{R}^+ \to \mathbb{R}^+$,
$\xi(\cdot)\colon \mathbb{R}^+ \to \mathbb{R}^+_0$: viscosity coefficients

- Fourier's law

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla\vartheta) = -\kappa(\vartheta)\nabla\vartheta \qquad (13)$$

$\kappa(\cdot)\colon \mathbb{R}^+ \mapsto \mathbb{R}^+ \dots$ heat conductivity

- Pressure law

$$\begin{aligned} p = p(\varrho, \vartheta) &= \varrho^\gamma + \varrho\vartheta \\ \text{or} \quad &= (\gamma - 1)\varrho e(\varrho, \vartheta) \end{aligned} \qquad (14)$$

(we will not consider the latter, due to additional technicalities)

- Internal energy

$$e(\varrho, \vartheta) = c_v\vartheta + \frac{\varrho^{\gamma-1}}{\gamma - 1} \qquad (15)$$

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- Heat conductivity

$$\kappa(\vartheta) \sim (1+\vartheta)^m \tag{16}$$

$m \in \mathbb{R}^+$

- Viscosity coefficients

$$\begin{gathered} C_1(1+\vartheta)^\alpha \leq \mu(\vartheta) \leq C_2(1+\vartheta)^\alpha \\ 0 \leq \xi(\vartheta) \leq C_2(1+\vartheta)^\alpha \end{gathered} \tag{17}$$

$\mu(\cdot)$ globally Lipschitz continuous, $\xi(\cdot)$ continuous,
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Weak solution I

We consider Navier boundary conditions for the velocity and the Neumann boundary conditions for the temperature.

- Weak formulation of the continuity equation

$$\int_\Omega \varrho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x = 0 \qquad \forall \psi \in C^1(\overline{\Omega}) \tag{18}$$

- Renormalized continuity equation
  ϱ extended by zero outside Ω, $\mathbf{u}$ extended outside Ω so that it remains in the $W^{1,p}$ space

$$\int_\Omega b(\varrho)\mathbf{u} \cdot \nabla \psi \, \mathrm{d}x + \int_\Omega \big(\varrho b'(\varrho) - b(\varrho)\big) \operatorname{div} \mathbf{u} \psi \, \mathrm{d}x = 0 \forall \psi \in C_0^1(\mathbb{R}^3) \tag{19}$$

  for all $b \in C^1([0,\infty))$ with $b'(z) = 0$ for $z \geq K > 0$.

- Weak formulation of the momentum equation

$$\begin{aligned} &\int_\Omega \big(-\varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\varphi} - p(\varrho, \vartheta) \operatorname{div} \boldsymbol{\varphi} + \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \boldsymbol{\varphi} \big) \, \mathrm{d}x \\ &+ \lambda \int_{\partial\Omega} \mathbf{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}\sigma = \int_\Omega \varrho \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \quad \forall \boldsymbol{\varphi} \in C_\mathbf{n}^1(\overline{\Omega}; \mathbb{R}^3) \end{aligned} \tag{20}$$

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Weak solution II

Weak formulation of the total energy balance

$$\int_\Omega -\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\Big)\mathbf{u}\cdot\nabla\psi\,\mathrm{d}x$$
$$= \int_\Omega \left(\varrho\mathbf{f}\cdot\mathbf{u}\psi + p(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\right)\mathrm{d}x$$
$$-\int_\Omega \left(\big(\mathbb{S}(\vartheta,\nabla\mathbf{u})\mathbf{u}\big)\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta\cdot\nabla\psi\right)\mathrm{d}x \tag{21}$$
$$-\int_{\partial\Omega}(L(\vartheta-\Theta_0)+\lambda|\mathbf{u}|^2)\psi\,\mathrm{d}\sigma$$
$$\forall\psi\in C^1(\overline{\Omega})$$

Definition

The triple $(\varrho,\mathbf{u},\vartheta)$ is called a renormalized weak solution to our system (1)–(8) if $\varrho\geq 0$, $\vartheta>0$, $\mathbf{u}\cdot\mathbf{n}=0$, $\int_\Omega\varrho\,\mathrm{d}x = M$, (18), (19), (20) and (21) hold true.

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&- \int_\Omega \left(\left(\mathbb{S}(\vartheta,\nabla\mathbf{u})\mathbf{u}\right)\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta\cdot\nabla\psi\right)\mathrm{d}x \\
&- \int_{\partial\Omega} (L(\vartheta-\Theta_0) + \lambda|\mathbf{u}|^2)\psi\,\mathrm{d}\sigma \\
&\qquad \forall\psi\in C^1(\overline{\Omega})
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Variational entropy solution I

- Weak formulation of the entropy inequality

$$\int_\Omega \Big(\frac{\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{u}}{\vartheta} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\Big)\psi\,\mathrm{d}x + \int_{\partial\Omega}\frac{L}{\vartheta}\Theta_0\psi\,\mathrm{d}\sigma$$
$$\le \int_{\partial\Omega} L\psi\,\mathrm{d}\sigma + \int_\Omega \Big(\kappa(\vartheta)\frac{\nabla\vartheta\cdot\nabla\psi}{\vartheta} - \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x \tag{22}$$
$$\forall \text{ nonnegative } \psi \in C^1(\overline{\Omega})$$

- Global total energy balance

$$\int_{\partial\Omega}\big(L(\vartheta-\Theta_0)+\lambda|\mathbf{u}|^2\big)\,\mathrm{d}\sigma = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x \tag{23}$$

Definition

The triple $(\varrho,\mathbf{u},\vartheta)$ is called a renormalized variational entropy solution to our system (1)–(8), if $\varrho \ge 0$, $\vartheta > 0$, $\mathbf{u}\cdot\mathbf{n} = 0$, $\int_\Omega \varrho\,\mathrm{d}x = M$ (18), (19) and (20) are satisfied in the same sense as in Definition 1, and we have the entropy inequality (22) together with the global total energy balance (23).

Both definitions are reasonable in the sense that any smooth weak or entropy variational solution is actually a classical solution to (1)–(8) (weak-strong compatibility).

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&\leq \int_{\partial\Omega} L\psi \, \mathrm{d}\sigma + \int_\Omega \left(\kappa(\vartheta) \frac{\nabla\vartheta \cdot \nabla\psi}{\vartheta} - \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla\psi \right) \mathrm{d}x \\
&\qquad \forall \text{ nonnegative } \psi \in C^1(\overline{\Omega})
\end{aligned}
\tag{22}
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\int_{\partial\Omega} (L(\vartheta - \Theta_0) + \lambda |\mathbf{u}|^2) \, \mathrm{d}\sigma = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x \tag{23}
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$$\leq \int_{\partial\Omega} L\psi \, \mathrm{d}\sigma + \int_\Omega \Big(\kappa(\vartheta) \frac{\nabla\vartheta \cdot \nabla\psi}{\vartheta} - \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla\psi \Big) \, \mathrm{d}x \qquad (22)$$

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Mathematical results

Until 2009, in the literature there was no existence results except for small data results or one result by P.L. Lions, where, however, the fixed mass was replaced by the finite L^p norm of the density for p sufficiently large.

- P.B. Mucha, M.P.: *On the steady compressible Navier–Stokes–Fourier system*, Communications in Mathematical Physics **288** (2009), 349–377.
- P.B. Mucha, M.P.: *Weak solutions to equations of steady compressible heat conducting fluids*, Mathematical Models & Methods in Applied Sciences **20** (2010), 785–813.
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Approximate system I

We consider for simplicity Ω not axially symmetric and $\lambda \geq 0$. We have in this case Korn's inequalities of the form

$$\|\mathbf{u}\|_{1,p} \leq C\Big(\int_\Omega \frac{1}{\vartheta}\mathbb{S}(\vartheta, \nabla\mathbf{u}) : \nabla\mathbf{u}\ \mathrm{d}x\Big)^{\frac{1}{2}} \|\vartheta\|_{3m}^{\frac{1-\alpha}{2}},$$

where $p = \frac{6m}{3m+1-\alpha} < 2$ if $0 \leq \alpha < 1$, $p = 2$ if $\alpha = 1$. We first consider the easier case $\alpha = 1$.

We can prove existence of a solution to the following system for arbitrary $\delta > 0$ provided β, $B \gg 1$.
Continuity equation:

$$\int_\Omega \varrho_\delta \mathbf{u}_\delta \cdot \nabla\psi\ \mathrm{d}x = 0 \tag{24}$$

for all $\psi \in W^{1,\frac{30\beta}{25\beta-18}}(\Omega;\mathbb{R})$, as well as in the renormalized sense
Momentum equation:

$$\begin{aligned}&\int_\Omega \Big(-\varrho_\delta(\mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \nabla\boldsymbol{\varphi} + \mathbb{S}(\vartheta_\delta, \nabla\mathbf{u}_\delta) : \nabla\boldsymbol{\varphi}\\ &-\big(p(\varrho_\delta, \vartheta_\delta) + \delta\varrho_\delta^\beta + \delta\varrho_\delta^2\big)\,\mathrm{div}\,\boldsymbol{\varphi}\Big)\ \mathrm{d}x = \int_\Omega \varrho_\delta \mathbf{f} \cdot \boldsymbol{\varphi}\ \mathrm{d}x\end{aligned} \tag{25}$$

for all $\boldsymbol{\varphi} \in W_{\mathbf{n}}^{1,\frac{5}{2}}(\Omega;\mathbb{R}^3)$

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Approximate system II

Total energy balance:

$$\int_\Omega \Big(\big(- \frac{1}{2}\varrho_\delta |\mathbf{u}_\delta|^2 - \varrho_\delta e(\varrho_\delta, \vartheta_\delta) \big) \mathbf{u}_\delta \cdot \nabla \psi + \big(\kappa(\vartheta_\delta) + \delta \vartheta_\delta^B + \delta \vartheta_\delta^{-1} \big) \nabla \vartheta_\delta \cdot \nabla \psi \Big) \, \mathrm{d}x$$
$$+ \int_{\partial\Omega} (L + \delta \vartheta_\delta^{B-1})(\vartheta_\delta - \Theta_0) \psi \, \mathrm{d}\sigma = \int_\Omega \varrho_\delta \mathbf{f} \cdot \mathbf{u}_\delta \psi \, \mathrm{d}x + \int_\Omega \Big(\big(- \mathbb{S}(\vartheta_\delta, \nabla \mathbf{u}_\delta) \mathbf{u}_\delta$$
$$+ \big(p(\varrho_\delta, \vartheta_\delta) + \delta \varrho_\delta^\beta + \delta \varrho_\delta^2 \big) \mathbf{u}_\delta \big) \cdot \nabla \psi + \delta \vartheta_\delta^{-1} \psi \Big) \, \mathrm{d}x + \delta \int_\Omega \Big(\frac{1}{\beta - 1} \varrho_\delta^\beta + \varrho_\delta^2 \Big) \mathbf{u}_\delta \cdot \nabla \psi \, \mathrm{d}x \tag{26}$$

for all $\psi \in C^1(\overline{\Omega}; \mathbb{R})$

Entropy inequality:

$$\int_\Omega \Big(\vartheta_\delta^{-1} \mathbb{S}(\vartheta_\delta, \mathbf{u}) : \nabla \mathbf{u}_\delta + \delta \vartheta_\delta^{-2} + \big(\kappa(\vartheta_\delta) + \delta \vartheta_\delta^B + \delta \vartheta_\delta^{-1} \big) \frac{|\nabla \vartheta_\delta|^2}{\vartheta_\delta^2} \Big) \psi \, \mathrm{d}x$$
$$\leq \int_\Omega \Big(\big(\kappa(\vartheta_\delta) + \delta \vartheta_\delta^B + \delta \vartheta_\delta^{-1} \big) \frac{\nabla \vartheta_\delta : \nabla \psi}{\vartheta_\delta} - \varrho s(\varrho_\delta, \vartheta_\delta) \mathbf{u}_\delta \cdot \nabla \psi \Big) \, \mathrm{d}x \tag{27}$$
$$+ \int_{\partial\Omega} \frac{L + \delta \vartheta_\delta^{B-1}}{\vartheta_\delta} (\vartheta_\delta - \Theta_0) \psi \, \mathrm{d}\sigma,$$

for all $\psi \in C^1(\overline{\Omega}; \mathbb{R})$ nonnegative

Estimates independent of δ I

Use in the entropy inequality and in the total energy balance test functions $\psi \equiv 1$:

$$\int_\Omega \left(\kappa(\vartheta_\delta) + \delta\vartheta_\delta^B + \delta\vartheta_\delta^{-1}\right)\frac{|\nabla\vartheta_\delta|^2}{\vartheta_\delta^2}\,\mathrm{d}x + \int_\Omega \left(\frac{1}{\vartheta_\delta}\mathbb{S}(\vartheta_\delta,\mathbf{u}_\delta) : \nabla\mathbf{u}_\delta + \delta\vartheta_\delta^{-2}\right)\mathrm{d}x$$

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Using suitable estimates of the Bogovskii-type we can get rid of the δ-dependent terms and we conclude:

$$\begin{aligned}&\|\mathbf{u}_\delta\|_{1,2} + \|\nabla\vartheta_\delta^{\frac{m}{2}}\|_2 + \|\nabla\ln\vartheta_\delta\|_2 + \|\vartheta_\delta^{-1}\|_{1,\partial\Omega}\\ &+\delta\left(\|\nabla\vartheta_\delta^{\frac{B}{2}}\|_2^2 + \|\nabla\vartheta_\delta^{-\frac{1}{2}}\|_2^2 + \|\vartheta_\delta\|_{3B}^B + \|\vartheta_\delta^{-2}\|_1\right) \le C\end{aligned} \tag{30}$$

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To estimate the density, we may use the Bogovskii-type estimates, but this leads to the bound $\gamma > \frac{3}{2}$. Therefore we apply another approach based on "potential"estimates of the pressure.

- Define for $1 \leq a \leq \gamma$, $0 < b < 1$

$$\mathcal{A} = \int_\Omega (\varrho_\delta^a |\mathbf{u}_\delta|^2 + \varrho_\delta^b |\mathbf{u}_\delta|^{2b+2}) \, \mathrm{d}x \tag{32}$$

- Using the previous estimates we get, under some conditions on a and b

$$\begin{gathered} \|\mathbf{u}_\delta\|_{1,2} \leq C \\ \|\vartheta_\delta\|_{3m} \leq C\big(1 + \mathcal{A}^{\frac{a-b}{2(ab+a-2b)}}\big) \\ \int_\Omega \big(\varrho_\delta^{s\gamma} + \varrho_\delta^{(s-1)\gamma} p(\varrho_\delta, \vartheta_\delta) + (\varrho_\delta |\mathbf{u}_\delta|^2)^s + \delta \varrho_\delta^{\beta+(s-1)\gamma}\big) \, \mathrm{d}x \\ \leq C(1 + \mathcal{A}^{\frac{sa-b}{ab+a-b}}), \end{gathered} \tag{33}$$

- We use as test function in the momentum equation

$$\varphi_i(x) \sim \frac{(x-y)_i}{|x-y|^A}.$$

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Estimates independent of δ III

Lemma

Let $y \in \Omega$, $R_0 < \frac{1}{3} dist(y, \partial\Omega)$. *Then*

$$\int_{B_{R_0}(y)} \Big(\frac{p(\varrho_\delta, \vartheta_\delta)}{|x-y|^A} + \frac{\varrho_\delta |\mathbf{u}_\delta|^2}{|x-y|^A} \Big) \mathrm{d}x \leq C\Big(1 + \|p(\varrho_\delta, \vartheta_\delta)\|_1 + \|\mathbf{u}_\delta\|_{1,2}(1 + \|\vartheta_\delta\|_{3m}) + \|\varrho_\delta |\mathbf{u}_\delta|^2\|_1\Big), \tag{34}$$

provided $A < \min\Big\{\frac{3m-2}{2m}, 1\Big\}$.

Similar test functions can be used for y near and at the boundary. We obtain a similar result. More complex for the Dirichlet boundary conditions, leads to more restrictions.

- Let us consider

$$-\Delta h = \varrho_\delta^a + \varrho_\delta^b |\mathbf{u}_\delta|^{2b} - \frac{1}{|\Omega|}\int_\Omega (\varrho_\delta^a + \varrho_\delta^b |\mathbf{u}_\delta|^{2b}) \mathrm{d}x, \quad \frac{\partial h}{\partial \mathbf{n}}|_{\partial\Omega} = 0. \tag{35}$$

The unique strong solution can be written

$$h(y) = \int_\Omega G(x,y)(\varrho_\delta^a + \varrho_\delta^b |\mathbf{u}_\delta|^{2b})\, \mathrm{d}x + l.o.t. \tag{36}$$

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Estimates independent of δ IV

As $G(x, y) \leq C|x-y|^{-1}$, we get

$$\|h\|_\infty \leq C(1 + \mathcal{A}^\eta), \tag{37}$$

where $\eta = \eta(a, b, \gamma, m)$

- Next

$$\mathcal{A} = \int_\Omega -\Delta h \mathbf{u}_\delta^2 \, \mathrm{d}x = \int_\Omega \nabla h \cdot \nabla |\mathbf{u}_\delta|^2 \, \mathrm{d}x \leq 2\|\nabla \mathbf{u}_\delta\|_2 B^{\frac{1}{2}}, \tag{38}$$

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Employing once more integration by parts

$$B = -\int_\Omega h\Delta h |\mathbf{u}_\delta|^2 \, \mathrm{d}x - \int_\Omega h \nabla h \cdot \nabla \mathbf{u}_\delta \cdot \mathbf{u}_\delta \, \mathrm{d}x$$
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- Then,

$$\mathcal{A} \leq C(1 + \mathcal{A}^{\tilde{\eta}})$$

and we require $\tilde{\eta} < 1$. This leads to a set of conditions.

- Analyzing these conditions, we finally have

Lemma

Let $(\varrho_\delta, \mathbf{u}_\delta, \vartheta_\delta)$ solve our approximate problem. Let $\gamma > 1$ and $m > \frac{2}{4\gamma - 3}$. Then there exists $s > 1$ such that

$$\begin{aligned}
\sup_{\delta>0} \|\varrho_\delta\|_{\gamma s} &< +\infty \\
\sup_{\delta>0} \|\varrho_\delta \mathbf{u}_\delta\|_{s} &< +\infty \\
\sup_{\delta>0} \|\varrho_\delta |\mathbf{u}_\delta|^2\|_{s} &< +\infty \\
\sup_{\delta>0} \|\mathbf{u}_\delta\|_{1,2} &< +\infty \\
\sup_{\delta>0} \|\vartheta_\delta\|_{3m} &< +\infty \\
\sup_{\delta>0} \|\vartheta_\delta^{m/2}\|_{1,2} &< +\infty \\
\sup_{\delta>0} \delta \|\varrho_\delta^{\beta+(s-1)\gamma}\|_{1} &< +\infty.
\end{aligned} \tag{42}$$

Moreover, we can take $s > \frac{6}{5}$ provided $\gamma > \frac{5}{4}$, and $m > \max\{1, \frac{2\gamma+10}{17\gamma-15}\}$.

Estimates independent of δ V

- Then,
$$\mathcal{A} \leq C(1 + \mathcal{A}^{\tilde{\eta}})$$
and we require $\tilde{\eta} < 1$. This leads to a set of conditions.
- Analyzing these conditions, we finally have

Lemma

Let $(\varrho_\delta, \mathbf{u}_\delta, \vartheta_\delta)$ solve our approximate problem. Let $\gamma > 1$ and $m > \frac{2}{4\gamma - 3}$. Then there exists $s > 1$ such that

$$\begin{array}{lcl} \sup_{\delta>0} \|\varrho_\delta\|_{\gamma s} & < & +\infty \\ \sup_{\delta>0} \|\varrho_\delta \mathbf{u}_\delta\|_s & < & +\infty \\ \sup_{\delta>0} \|\varrho_\delta |\mathbf{u}_\delta|^2\|_s & < & +\infty \\ \sup_{\delta>0} \|\mathbf{u}_\delta\|_{1,2} & < & +\infty \\ \sup_{\delta>0} \|\vartheta_\delta\|_{3m} & < & +\infty \\ \sup_{\delta>0} \|\vartheta_\delta^{m/2}\|_{1,2} & < & +\infty \\ \sup_{\delta>0} \delta \|\varrho_\delta^{\beta+(s-1)\gamma}\|_1 & < & +\infty. \end{array} \tag{42}$$

Moreover, we can take $s > \frac{6}{5}$ provided $\gamma > \frac{5}{4}$, and $m > \max\{1, \frac{2\gamma+10}{17\gamma-15}\}$.

Limit passage $\delta \to 0^+$ I

Continuity equation

$$\int_\Omega \rho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x = 0 \tag{43}$$

for all $\psi \in C^1(\overline{\Omega}; \mathbb{R})$

Momentum equation

$$\int_\Omega \Big(- \varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla \varphi + \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \varphi - \overline{p(\varrho, \vartheta)} \operatorname{div} \varphi \Big) \, \mathrm{d}x = \int_\Omega \varrho \mathbf{f} \cdot \varphi \, \mathrm{d}x \tag{44}$$

for all $\varphi \in C^1_{\mathbf{n}}(\overline{\Omega}; \mathbb{R}^3)$

Entropy inequality

$$\begin{aligned} &\int_\Omega \Big(\vartheta^{-1} \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \kappa(\vartheta) \frac{|\nabla \vartheta|^2}{\vartheta^2} \Big) \psi \, \mathrm{d}x \\ &\leq \int_\Omega \Big(\kappa(\vartheta) \frac{\nabla \vartheta : \nabla \psi}{\vartheta} - \overline{\varrho s(\varrho, \vartheta)} \mathbf{u} \cdot \nabla \psi \Big) \, \mathrm{d}x + \int_{\partial\Omega} \frac{L}{\vartheta} (\vartheta - \Theta_0) \psi \, \mathrm{d}\sigma, \end{aligned} \tag{45}$$

for all $\psi \in C^1(\overline{\Omega}; \mathbb{R})$, nonnegative

Global total energy balance

$$\int_{\partial\Omega} L(\vartheta - \Theta_0) \, \mathrm{d}\sigma = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x \tag{46}$$

(total energy balance with test function $\psi \equiv 1$)

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Momentum equation

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(total energy balance with test function $\psi \equiv 1$)

Limit passage $\delta \to 0^+$ II

Total energy balance

$$\int_\Omega \Big(\big(-\frac{1}{2}\varrho|\mathbf{u}|^2 - \overline{\varrho e(\varrho,\vartheta)}\big)\mathbf{u}\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta : \nabla\psi \Big)\,\mathrm{d}x$$

$$+ \int_{\partial\Omega} (L(\vartheta - \Theta_0)\psi\,\mathrm{d}\sigma = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\psi\,\mathrm{d}x + \int_\Omega \big(-\mathbb{S}(\vartheta,\nabla\mathbf{u})\mathbf{u} + \overline{p(\varrho,\vartheta)}\mathbf{u}\big)\cdot\nabla\psi\,\mathrm{d}x \tag{47}$$

for all $\psi \in C^1(\overline{\Omega};\mathbb{R})$. We can pass only in certain situations, when we have better a priori estimates! We need $s > \frac{6}{5}$ and $m > 1$.

We need to show the strong convergence of the density!

Main ingredients:

- Effective viscous flux identity
- Oscillation defect measure estimate
- Renormalized continuity equation

Limit passage $\delta \to 0^+$ II

Total energy balance

$$\int_\Omega \Big(\big(- \frac{1}{2}\varrho|\mathbf{u}|^2 - \overline{\varrho e(\varrho,\vartheta)}\big)\mathbf{u}\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta : \nabla\psi \Big)\,\mathrm{d}x$$
$$+ \int_{\partial\Omega} \big(L(\vartheta - \Theta_0)\psi\,\mathrm{d}\sigma = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\psi\,\mathrm{d}x + \int_\Omega \big(-\mathbb{S}(\vartheta,\nabla\mathbf{u})\mathbf{u} + \overline{p(\varrho,\vartheta)}\mathbf{u}\big)\cdot\nabla\psi\,\mathrm{d}x \tag{47}$$

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Limit passage $\delta \to 0^+$ II

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We need to show the strong convergence of the density!

Main ingredients:

- Effective viscous flux identity
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Limit passage $\delta \to 0^+$ III

Item 1: Effective viscous flux

Using as test function $\zeta(x)\nabla\Delta^{-1}(1_\Omega T_k(\varrho_\delta))$ with $T_k(z) = kT(\frac{z}{k})$, $k \in N$ for

$$T(z) = \begin{cases} z \text{ for } 0 \le z \le 1, \\ \text{concave on } (0,\infty), \\ 2 \text{ for } z \ge 3, \end{cases}$$

in the approximative balance of momentum, and $\zeta(x)\nabla\Delta^{-1}(1_\Omega \overline{T_k(\varrho)})$ in its limit version we can deduce

$$\begin{aligned} &\overline{p(\varrho,\vartheta)T_k(\varrho)} - \left(\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)\right)\overline{T_k(\varrho)\operatorname{div}\mathbf{u}} \\ &= \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)} - \left(\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)\right)\overline{T_k(\varrho)}\operatorname{div}\mathbf{u} \end{aligned} \tag{48}$$

a.e. in Ω.

Limit passage $\delta \to 0^+$ IV

Item 2: Oscillation defect measure

We do not have L^2-bound on the density and thus we do not know whether the renormalized continuity equation for the limit holds. To show it, we introduce:

Oscillation defect measure

$$\mathbf{osc_q}[\varrho_\delta \to \varrho](Q) = \sup_{k>1} \left(\limsup_{\delta \to 0^+} \int_Q |T_k(\varrho_\delta) - T_k(\varrho)|^q \, \mathrm{d}x \right) \qquad (49)$$

We have

$$\begin{aligned} \varrho_\delta &\rightharpoonup \varrho && \text{in } L^1(\Omega;\mathbb{R}), \\ \mathbf{u}_\delta &\rightharpoonup \mathbf{u} && \text{in } L^p(\Omega;\mathbb{R}^3), \\ \nabla \mathbf{u}_\delta &\rightharpoonup \nabla \mathbf{u} && \text{in } L^p(\Omega;\mathbb{R}^{3\times 3}) \end{aligned}$$

and

$$\mathbf{osc_q}[\varrho_\delta \to \varrho](\Omega) < \infty \qquad (50)$$

for $q > p'$, then the limit density and velocity satisfy the renormalized continuity equation.

Assuming $m > \max\{\frac{2}{3(\gamma-1)}, \frac{2}{3}\}$, it can be verified that (50) holds true with some $2 < q < \gamma + 1$.

Limit passage $\delta \to 0^+$ IV

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Assuming $m > \max\{\frac{2}{3(\gamma-1)}, \frac{2}{3}\}$, it can be verified that (50) holds true with some $2 < q < \gamma + 1$.

Limit passage $\delta \to 0^+$ V

We also get

$$\limsup_{\delta\to0^+}\int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^{\gamma+1}\,\mathrm{d}x \le C\int_\Omega\left(\overline{p(\varrho,\vartheta)T_k(\varrho)}-\overline{p(\varrho,\vartheta)}\;\overline{T_k(\varrho)}\right)\mathrm{d}x, \tag{51}$$

$$\limsup_{\delta\to0^+}\int_\Omega \frac{1}{1+\vartheta}|T_k(\varrho_\delta)-T_k(\varrho)|^{\gamma+1}\,\mathrm{d}x \le C\int_\Omega\frac{1}{1+\vartheta}\left(\overline{p(\varrho,\vartheta)T_k(\varrho)}-\overline{p(\varrho,\vartheta)}\;\overline{T_k(\varrho)}\right)\mathrm{d}x. \tag{52}$$

Item 3: Application of the renormalization As $(\varrho_\delta,\mathbf{u}_\delta)$ and $(\varrho,\mathbf{u})$ verify the renormalized continuity equation, we have:

$$\int_\Omega T_k(\varrho)\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}x=0$$

and

$$\int_\Omega T_k(\varrho_\delta)\,\mathrm{div}\,\mathbf{u}_\delta\,\mathrm{d}x=0,\qquad \text{i.e.}\int_\Omega\overline{T_k(\varrho)\,\mathrm{div}\,\mathbf{u}}\,\mathrm{d}x=0$$

To this aim, use

$$\mathrm{div}(b(\varrho)\mathbf{u})+(\varrho b'(\varrho)-b(\varrho))\,\mathrm{div}\,\mathbf{u}=0 \text{ in } \mathcal{D}'(\mathbb{R}^3)$$

with

$$b(\varrho)=\varrho\int_1^\varrho\frac{T_k(z)}{z^2}\,\mathrm{d}z.$$

Limit passage $\delta \to 0^+$ V

We also get

$$\limsup_{\delta\to 0^+} \int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1}\,\mathrm{d}x \le C \int_\Omega \left(\overline{p(\varrho,\vartheta)\,T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)} \right) \mathrm{d}x, \tag{51}$$

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$$b(\varrho) = \varrho \int_1^\varrho \frac{T_k(z)}{z^2}\,\mathrm{d}z.$$

Limit passage $\delta \to 0^+$ V

We also get

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$$b(\varrho) = \varrho \int_1^\varrho \frac{T_k(z)}{z^2}\, \mathrm{d}z.$$

Limit passage $\delta \to 0^+$ V

We also get

$$\limsup_{\delta\to 0^+} \int_\Omega |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1} \, \mathrm{d}x \leq C \int_\Omega \left(\overline{p(\varrho,\vartheta) T_k(\varrho)} - \overline{p(\varrho,\vartheta)} \; \overline{T_k(\varrho)} \right) \mathrm{d}x, \tag{51}$$

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To this aim, use

$$\operatorname{div}(b(\varrho)\mathbf{u}) + (\varrho b'(\varrho) - b(\varrho)) \operatorname{div} \mathbf{u} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3)$$

with

$$b(\varrho) = \varrho \int_1^\varrho \frac{T_k(z)}{z^2} \, \mathrm{d}z.$$

Limit passage $\delta \to 0^+$ V

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with

$$b(\varrho) = \varrho \int_1^\varrho \frac{T_k(z)}{z^2} \, \mathrm{d}z.$$

Limit passage $\delta \to 0^+$ VI

Using the effective viscous flux identity we get that

$$\int_\Omega \frac{1}{\frac{4}{3}\mu(\vartheta)+\xi(\vartheta)}\left(\overline{p(\varrho,\vartheta)T_k(\varrho)}-\overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\right)\,\mathrm{d}x = \int_\Omega \left(T_k(\varrho)-\overline{T_k(\varrho)}\right)\operatorname{div}\mathbf{u}\,\mathrm{d}x. \tag{53}$$

As $\lim_{k\to\infty}\|T_k(\varrho)-\varrho\|_1 = \lim_{k\to\infty}\|\overline{T_k(\varrho)}-\varrho\|_1 = 0$, the definition of the oscillation defect measure together with (50)

$$\lim_{k\to\infty}\int_\Omega \frac{1}{\frac{4}{3}\mu(\vartheta)+\xi(\vartheta)}\left(\overline{p(\varrho,\vartheta)T_k(\varrho)}-\overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\right)\,\mathrm{d}x = 0.$$

Hence

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$$\lim_{k\to\infty}\limsup_{\delta\to 0^+}\int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^{q}\,\mathrm{d}x = 0$$

with some $q>2$, the same as for the oscillation defect measure. Now, as

$$\|\varrho_\delta-\varrho\|_1 \le \|\varrho_\delta - T_k(\varrho_\delta)\|_1 + \|T_k(\varrho_\delta)-T_k(\varrho)\|_1 + \|T_k(\varrho)-\varrho\|_1,$$

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$$\varrho_\delta \to \varrho \quad \text{in } L^p(\Omega;\mathbb{R}) \quad \forall 1\le p< s\gamma.$$

Limit passage $\delta \to 0^+$ VI

Using the effective viscous flux identity we get that

$$\int_\Omega \frac{1}{\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)} \left(\overline{p(\varrho,\vartheta) T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\; \overline{T_k(\varrho)} \right) \mathrm{d}x = \int_\Omega \left(T_k(\varrho) - \overline{T_k(\varrho)} \right) \operatorname{div} \mathbf{u} \, \mathrm{d}x. \tag{53}$$

As $\lim_{k\to\infty} \| T_k(\varrho) - \varrho\|_1 = \lim_{k\to\infty} \| \overline{T_k(\varrho)} - \varrho\|_1 = 0$, the definition of the oscillation defect measure together with (50)

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Results I (Navier b.c.)

We proved:

Theorem

Let $\Omega \in C^2$ *be a bounded domain in* $\mathbb{R}^3$, $\mathbf{f} \in L^\infty(\Omega;\mathbb{R}^3)$, $\Theta_0 \geq K_0 > 0$ *a.e. at* $\partial\Omega$, $\Theta_0 \in L^1(\partial\Omega)$. *Let* $\gamma > 1$, $m > \max\left\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\right\}$.
Let Ω *be not axially symmetric. Then there exists a variational entropy solution to our problem. Moreover,* $(\varrho, \mathbf{u})$ *is a renormalized solution to the continuity equation.*
Additionally, if $m > 1$ *and* $\gamma > \frac{5}{4}$, *then the solution is a weak solution, i.e. also the weak formulation of the total energy balance is fulfilled.*

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Results II (Dirichlet b.c.)

Theorem

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Changes for $\alpha < 1$

Recall that

$$\|\mathbf{u}\|_{1,p} \leq C\Big(\int_\Omega \frac{1}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\mathbf{u}\,\mathrm{d}x\Big)^{\frac{1}{2}} \|\vartheta\|_{3m}^{\frac{1-\alpha}{2}},$$

i.e., for $\alpha < 1$ we control only $W^{1,p}$-norm of the velocity, $p < 2$.

Moreover, the integration-by-parts argument does not work! We can replace it with certain properties of Bessel kernels and Bessel potential spaces. The proof itself is similar, but more technical and the results are more messy, we have three parameters: α, γ and m. This is a recent project with O. Kreml.

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Dirichlet b.c. for the temperature I

Let us for simplicity assume the Dirichlet boundary condition for the temperature on the whole $\partial\Omega$. We take Ω_1 smooth bounded such that $\Omega \subset\subset \Omega_1$ and consider in Ω_1

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \tag{54}$$

$$\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbb{S} + \nabla p_\lambda = \varrho \mathbf{f} \tag{55}$$

$$\begin{gathered}\operatorname{div}\left(\varrho E \mathbf{u}\right) + \operatorname{div}(\mathbf{q} + p_\lambda \mathbf{u}) \\ = \varrho \mathbf{f} \cdot \mathbf{u} + \operatorname{div}\left(\mathbb{S}\mathbf{u}\right) + \lambda^{-1} 1_{\Omega_1 \setminus \Omega} \Big(\int_{\partial\Omega} |\vartheta - \Theta_1| \mathrm{d}S + \int_{\partial\Omega} |\mathbf{u}| \mathrm{d}S + \Big| \int_\Omega \varrho \, \mathrm{d}x - M \Big| \Big).\end{gathered} \tag{56}$$

with $\lambda > 0$. The existence of such solutions can be shown by modification of the standard procedure for the compressible N-S-F system. We use e.g. the homogeneous boundary conditions for the velocity and the flux condition for the temperature on $\partial\Omega$.

Dirichlet b.c. for the temperature II

From the entropy inequality we read in addition to the estimates for the N-S-F system

$$\int_{\partial\Omega} |\vartheta - \Theta_1| \mathrm{d}S + \int_{\partial\Omega} |\mathbf{u}| \mathrm{d}S + \Big| \int_{\Omega} \varrho \, \mathrm{d}x - M \Big| \leq C\lambda$$

and we can conclude as in the previous situation.

- We treat the homogeneous Dirichlet boundary conditions for the velocity and the results are the same as for the Navier boundary conditions in the case of the N.S.F. system (the better ones).
- We deal only with weak solutions.
- The test functions in the total energy balance are compactly supported.

This is a recent project with T. Piasecki.

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Chemically reacting mixtures I

$$\begin{gathered} \operatorname{div}(\varrho \mathbf{u}) = 0, \\ \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbb{S} + \nabla \pi = \varrho \mathbf{f}, \\ \operatorname{div}(\varrho E \mathbf{u}) + \operatorname{div}(\pi \mathbf{u}) + \operatorname{div} \mathbb{Q} - \operatorname{div}(\mathbb{S}\mathbf{u}) = \varrho \mathbf{f} \cdot \mathbf{u}, \\ \operatorname{div}(\varrho Y_k \mathbf{u}) + \operatorname{div} \mathbf{F}_k = m_k \omega_k, \quad k \in \{1, \dots, n\} \end{gathered} \tag{57}$$

with the boundary conditions

$$\mathbf{u} = \mathbf{0}, \tag{58}$$

$$\mathbf{F}_k \cdot \mathbf{n} = 0, \tag{59}$$

$$-\mathbf{Q} \cdot \mathbf{n} + L(\vartheta - \Theta_0) = 0, \tag{60}$$

and the given total mass

$$\int_\Omega \varrho \, \mathrm{d}x = M > 0. \tag{61}$$

Indeed, $\sum_{k=1}^n \mathbf{F}_k = \mathbf{0}$, $\sum_{k=1}^n m_k \omega_k = 0$ and we must construction solutions such that $\sum_{k=1}^N Y_k = 1$.

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Based on similar ideas presented for the N-S-F system the existence of weak and variational entropy solutions can be established in the case of the same molar masses (closely connected with information from the entropy inequality which plays a central role here).

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THANK YOU VERY MUCH

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