Existence analysis of a stationary compressible fluid model for heat-conducting and chemically reacting mixtures

Milan Pokorný

Charles University in Prague
E-mail: pokorny@karlin.mff.cuni.cz

Joint work with Miroslav Bulíček (Praha), Ansgar Jüngel (Wien), Nicola Zamponi (Mannheim)

8th ECM, Portorož (online), June 21-25, 2021

The model I

We consider a model describing steady flow of compressible, heat conducting mixture of chemically reacting gases. We restrict ourselves to the model with one velocity for the whole mixture (barycentric) and use the Navier–Stokes–Fourier system combined with the Maxwell–Stefan cross diffusion equation in the Fick–Onsager form.

The resulting system of PDEs reads

$$\begin{aligned} \operatorname{div}(\rho_i \mathbf{v} + \mathbf{J}_i) &= r_i, \quad i = 1, \dots, N, \\ \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} - \mathbb{S}) + \nabla p &= \rho \mathbf{b}, \\ \operatorname{div}\left(\rho E \mathbf{v} + \mathbf{Q} - \mathbb{S}\mathbf{v} + p\mathbf{v}\right) &= \rho \mathbf{b} \cdot \mathbf{v}. \end{aligned}$$

Above: $\vec{\rho} = (\rho_1, \dots, \rho_N)$ are the densities of the separate species, $\rho = \sum_{i=1}^N \rho_i$ is the total density of the mixture, $\mathbf{v}$ is the barycentric velocity, $\mathbf{J}_i$, $i = 1, 2, \dots, N$ are the partial fluxes, r_i, $i = 1, 2, \dots, N$ are the source terms due to the chemical reactions, $\mathbb{S}$ is the stress tensor, p is the pressure, $\mathbf{b}$ is the external force, E is the specific total energy and $\mathbf{Q}$ is the heat flux.

The model II

We consider the following boundary conditions:

$$\mathbf{J}_i \cdot \boldsymbol{\nu} = 0, \quad i = 1, \dots, N,$$
$$\mathbf{v} \cdot \boldsymbol{\nu} = 0, \quad (\mathbb{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu})(\mathbb{S}\boldsymbol{\nu} + \alpha_1 \mathbf{v}) = \mathbf{0},$$
$$\mathbf{Q} \cdot \boldsymbol{\nu} + \alpha_2(\theta_0 - \theta) = 0.$$

Here, $\boldsymbol{\nu}$ is the external normal to $\partial\Omega$, $\theta_0 \geq T_0 > 0$ is the external temperature, $\alpha_1 \geq 0$ and $\alpha_2 > 0$.

We also prescribe the total mass of the mixture:

$$\frac{1}{|\Omega|} \int_\Omega \rho \, dx = \overline{\rho}.$$

The total energy balance may be formally replaced by the entropy balance

$$-\operatorname{div} \mathcal{J} + \Xi = 0,$$

where $\mathcal{J}$ is the entropy flux and Ξ is the entropy production rate.

Constitutive relations I

- Entropy flux

$$\mathcal{J} = \rho s \mathbf{v} - \sum_{i=1}^{N} \frac{\mu_i}{\theta} \mathbf{J}_i + \frac{1}{\theta} \boldsymbol{Q},$$

where $\vec{\mu}$ are the chemical potentials.

- Entropy production

$$\Xi = -\sum_{i=1}^{N} \mathbf{J}_i \cdot \nabla \frac{\mu_i}{\theta} + \boldsymbol{Q} \cdot \nabla \frac{1}{\theta} + \frac{\mathbb{S} : \nabla \mathbf{v}}{\theta} - \sum_{i=1}^{N} r_i \frac{\mu_i}{\theta}.$$

Due to the Second law of thermodynamics the entropy production Ξ must be non-negative. We will guarantee that

$$-\sum_{i=1}^{N} \mathbf{J}_i \cdot \nabla \frac{\mu_i}{\theta} + \boldsymbol{Q} \cdot \nabla \frac{1}{\theta} \geq 0, \quad \frac{\mathbb{S} : \nabla \mathbf{v}}{\theta} \geq 0, \quad -\sum_{i=1}^{N} r_i \frac{\mu_i}{\theta} \geq 0.$$

- Viscous stress tensor

$$\mathbb{S} = 2\lambda_1(\theta)\left(\mathbb{D}(\mathbf{v}) - \frac{1}{3}\operatorname{div}(\mathbf{v})\mathbb{I}\right) + \lambda_2(\theta)\operatorname{div}(\mathbf{v})\mathbb{I}, \tag{1}$$

where $\lambda_1(\theta)$ and $\lambda_2(\theta)$ are the temperature-dependent shear and bulk viscosity coefficients, respectively.

Constitutive relations II

- Heat flux

$$\boldsymbol{Q} = -\kappa(\theta)\nabla\theta - \sum_{i=1}^{N} M_i \nabla \frac{\mu_i}{\theta}, \tag{2}$$

where $\kappa(\theta)$ is the thermal conductivity and the coefficients M_i depend on $\vec{\rho}$ and θ.

- Diffusion flux

$$\mathbf{J}_i = -\sum_{j=1}^{N} M_{ij} \nabla \frac{\mu_j}{\theta} - M_i \nabla \frac{1}{\theta}, \quad i = 1, \ldots, N, \tag{3}$$

where $M_{ij} = M_{ij}(\vec{\rho}, \theta)$ are diffusion coefficients, by Onsager's principle symmetric.

Constitutive relations III

Due to the mass conservation we require $\sum_{i=1}^{N} \mathbf{J}_i = \mathbf{0}$, $\sum_{i=1}^{N} r_i = 0$. We assume

$$\sum_{i=1}^{N} M_{ij} = \sum_{i=1}^{N} M_i = 0, \quad j = 1, \dots, N, \tag{4}$$

and

$$\exists C_M > 0 : \quad \sum_{i,j=1}^{N} M_{ij} z_i z_j \geq C_M |\Pi \vec{z}|^2 \quad \text{for all } \vec{z} \in \mathbb{R}^N,$$

where $\Pi = \mathbb{I} - \vec{1} \otimes \vec{1}/N$ is the orthogonal projector on $\text{span}\{\vec{1}\}^{\perp}$.

- Source terms

$$\begin{cases} r_i = r_i(\Pi(\vec{\mu}/\theta), \theta), \quad i = 1, \dots, N, \\ \exists C_r > 0,\ \zeta > 0,\ \beta > 0 : \quad -\sum_{i=1}^{N} r_i(\Pi(\vec{q}), \theta) q_i \geq C_r |\Pi \vec{q}|^2, \\ |r_i(\Pi(\vec{q}), \theta)| \leq C_r(|\Pi \vec{q}|^{5(6-\zeta)/6} + \theta^{5(3\beta-\zeta)/6}) \quad \text{for all } \vec{q} \in \mathbb{R}^N, \theta > 0, \end{cases}$$

for some $\zeta > 0$ possibly very small, $\beta > 0$, and all $i = \{1, 2, \dots, N\}$.

Thermodynamics

We denote the Helmholtz free energy as $\varrho\psi$. It determines other thermodynamic quantities:

$$\mu_i = \frac{\partial(\rho\psi)}{\partial\rho_i}, \quad p = -\rho\psi + \sum_{i=1}^{N} \rho_i\mu_i,$$

$$\rho e = \rho\psi - \theta\frac{\partial(\rho\psi)}{\partial\theta}, \quad \rho s = -\frac{\partial(\rho\psi)}{\partial\theta},$$

with e the specific internal energy and s the specific entropy. We denote

$$h_\theta(\vec{\rho}) := \rho\psi(\vec{\rho}, \theta).$$

Under assumptions stated below we define

$$h^*_\theta(\vec{\mu}) = \sup_{\vec{\rho}\in\mathbb{R}^N_+} \left(\vec{\rho}\cdot\vec{\mu} - h_\theta(\vec{\rho})\right)$$

the Legendre transform of h_θ, which in fact equals the pressure p. Further

$$\vec{\mu} = \nabla h_\theta(\vec{\rho}) \quad \text{if and only if} \quad \vec{\rho} = \nabla h^*_\theta(\vec{\mu}).$$

Hypotheses I

We assume $\gamma > \frac{3}{2}$ and set $\nu := \gamma \min\left\{\frac{2\gamma-3}{\gamma}, \frac{3\beta-2}{3\beta+2}\right\}$.

Main hypotheses

(H1) *Domain:* $\Omega \subset \mathbb{R}^3$ is a bounded domain with a C^2 boundary that is not axially symmetric.

The smoothness can be relaxed to Lipschitz domain, however, it is technical. The assumption on the symmetry is connected with Korn's inequality and (H2).

(H2) *Data:* $\alpha_1 = 0$, $\alpha_2 > 0$, $\theta_0 \in L^1(\partial\Omega)$, $\operatorname{ess\,inf}_{\partial\Omega} \theta_0 > 0$, $\boldsymbol{b} \in L^\infty(\Omega; \mathbb{R}^3)$.

It is possible to take $\alpha_1 > 0$ and relax the assumption on Ω, more technicalities.

(H3) *Viscosity and heat conductivity:* λ_1, λ_2, $\kappa \in C^0(\mathbb{R}_+)$ and there exist constants c_1, c_2, κ_1, κ_2, such that for all $\theta > 0$,

$$
\begin{aligned}
c_1(1+\theta) \le \lambda_1(\theta) &\le c_2(1+\theta), \\
0 \le \lambda_2(\theta) &\le c_2(1+\theta), \\
\kappa_1(1+\theta)^\beta \le \kappa(\theta) &\le \kappa_2(1+\theta)^\beta.
\end{aligned}
$$

The linear growth in the viscosity can be relaxed to sublinear, but more technicalities appear.

Hypotheses II

(H4) *Diffusion coefficients:* There exists $\zeta > 0$ (a fixed small positive number) such that for all $i, j = 1, \ldots, N$, the coefficients M_{ij}, $M_i \in C^0(\mathbb{R}^N_{+,0} \times \mathbb{R}_+)$ satisfy and

$$|M_{ij}(\vec{\rho}, \theta)| + \frac{|M_i(\vec{\rho}, \theta)|}{\theta} \leq \widetilde{C}_M(\rho^{(\gamma+\nu-\zeta)/3} + \theta^{(3\beta-\zeta)/3} + 1)$$

for all $(\vec{\rho}, \theta) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ and some constants $\widetilde{C}_M$.

(H5) *Reaction terms:* Additionally to the above $\sum_{i=1}^N r_i = 0$.

(H6) *Free energy density:* $h_\theta \in C^2(\mathbb{R}^N_+)$ is for fixed $\theta > 0$ strictly convex function with respect to $\vec{\rho}$. Furthermore, it fulfils some growth conditions and some conditions on the behaviour next zero for the functions, its first and second gradient.

The conditions are slightly technical, but they in general ensure some integrability properties of the thermodynamic potentials and properties of the Legendre transform.

Hypotheses III

(H7) We assume that there exists $\omega \in (1, \gamma)$, $\tilde{c}_p > 0$ and $\tilde{C}_p > 0$ such that for all $\theta \in \mathbb{R}_+$, all $\vec{\rho} \in \mathbb{R}^N_+$ and all $\vec{x} \in \mathbb{R}^N$ there holds

$$\tilde{c}_p |\vec{x}|^2 \left(\frac{\theta}{\rho} + \rho^{\gamma-2} \right) \leq \sum_{i,j=1}^{N} \frac{\partial^2 h_\theta(\vec{\rho})}{\partial \rho_i \partial \rho_j} x_i x_j \leq \tilde{C}_p |\vec{x}|^2 \left(\frac{\theta}{\rho} + \rho^{\gamma-2} + \rho^{\omega-2} \right).$$

This assumption in particular ensures that the pressure p satisfies for some $c_p, C_p > 0$

$$c_p(\rho\theta + \rho^\gamma) \leq p(\vec{\rho}, \theta) \leq C_p(1 + \rho\theta + \rho^\gamma)$$

for all $(\vec{\rho}, \theta) \in \mathbb{R}^N_{+,0} \times \mathbb{R}_+$.

All hypotheses and previous assumptions can be fulfilled by the choice

$$\rho\psi = \theta \sum_{i=1}^{N} n_i \log n_i + \overline{n}^\gamma - c_W \rho\theta \log\theta,$$

where $n_i = \rho/m_i$, $\overline{n} = \sum_{i=1}^N n_i$. There also exists suitable choice of terms describing chemical reactions.

Weak and variational entropy solutions I

Weak formulation of the species balance

$$\sum_{i=1}^{N}\int_{\Omega}\Big(-\rho_i\mathbf{v}+\sum_{j=1}^{N}M_{ij}\nabla\frac{\mu_j}{\theta}+M_i\nabla\frac{1}{\theta}\Big)\cdot\nabla\phi_i\,dx=\sum_{i=1}^{N}\int_{\Omega}r_i\phi_i\,dx$$

for all $\phi_1,\dots,\phi_N\in W^{1,\infty}(\Omega)$;

Weak formulation of the momentum balance,

$$\int_{\Omega}(-\rho\mathbf{v}\otimes\mathbf{v}+\mathbb{S}):\nabla\boldsymbol{u}\,dx+\int_{\partial\Omega}\alpha_1\mathbf{v}\cdot\boldsymbol{u}\,ds=\int_{\Omega}(p\,\mathrm{div}\,\boldsymbol{u}+\rho\boldsymbol{b}\cdot\boldsymbol{u})\,dx,$$

for all $\boldsymbol{u}\in W^{1,\infty}_{\boldsymbol{\nu}}(\Omega)$;

Weak formulation of the total energy balance

$$\int_{\Omega}(-\rho E\boldsymbol{v}-\boldsymbol{Q}+\mathbb{S}\boldsymbol{v}-p\boldsymbol{v})\cdot\nabla\varphi\,dx+\int_{\partial\Omega}(\alpha_1|\boldsymbol{v}|^2+\alpha_2(\theta-\theta_0))\varphi\,ds=\int_{\Omega}\rho\boldsymbol{b}\cdot\boldsymbol{v}\varphi\,dx,$$

for all $\varphi\in W^{1,\infty}(\Omega)$;

Weak and variational entropy solutions II

Weak formulation of the entropy inequality

$$\int_\Omega \left(\rho s \mathbf{v} + \sum_{i=1}^N \frac{\mu_i}{\theta}\left(\sum_{j=1}^N M_{ij}\nabla\frac{\mu_j}{\theta} + M_i \nabla\frac{1}{\theta}\right) - \frac{1}{\theta}\left(\kappa(\theta)\nabla\theta + \sum_{i=1}^N M_i\nabla\frac{\mu_i}{\theta}\right)\right)\cdot\nabla\Phi\,dx$$

$$+\int_\Omega \left(\sum_{i,j=1}^N M_{ij}\nabla\frac{\mu_i}{\theta}\cdot\nabla\frac{\mu_j}{\theta} + \kappa(\theta)|\nabla\log\theta|^2 + \frac{\mathbb{S}:\nabla\mathbf{v}}{\theta} - \sum_{i=1}^N r_i\frac{\mu_i}{\theta}\right)\Phi\,dx$$

$$\leq \alpha_2 \int_{\partial\Omega} \frac{\theta-\theta_0}{\theta}\Phi\,ds,$$

for every $\Phi \in W^{1,\infty}(\Omega)$ with $\Phi \geq 0$ a.e. in Ω

Total energy equality

(It is in fact weak formulation of the total energy balance with $\psi \equiv 1$)

$$\int_{\partial\Omega} (\alpha_1|\mathbf{v}|^2 + \alpha_2(\theta-\theta_0))\,ds = \int_\Omega \rho\mathbf{b}\cdot\mathbf{v}\,dx.$$

Summing weak formulations of mass balances with $\phi_i \equiv 1$ we obtain

$$\int_\Omega \rho\mathbf{v}\cdot\nabla\Phi\,dx = 0,$$

for all $\Phi \in W^{1,\infty}(\Omega)$.

Weak and variational entropy solutions III

We need also Renormalized form of this equation: for $\varphi \in W^{1,\infty}(\Omega)$ and $b \in C^1(\mathbb{R})$, $b(0) = 0$ with b' having compact support,

$$\int_\Omega \big(b(\rho)\mathbf{v}\cdot\nabla\varphi - \varphi(\rho b'(\rho) - b(\rho))\,\mathrm{div}\,\mathbf{v}\big)\,dx = 0.$$

Definition (Weak solutions)

We call the functions

$$\rho_1,\dots,\rho_N \in L^\gamma(\Omega), \quad \mathbf{v} \in H^1_\nu(\Omega), \quad \log\theta, \theta^{\beta/2} \in H^1(\Omega)$$

such that

$$\rho|\mathbf{v}|^2\mathbf{v},\ \mathbb{S}(\theta,\nabla\mathbf{v})\mathbf{v},\ p(\vec{\rho},\theta)\mathbf{v} \in L^1(\Omega;\mathbb{R}^3)$$

a *renormalized weak solution* to our problem if there holds the weak formulations of the species equation, momentum equation, total energy balance and the total density $\rho := \sum_{i=1}^N \rho_i$ is a renormalized solution to the continuity equation.

Weak and variational entropy solutions IV

Definition (Variational entropy solutions)

We call the functions

$$\rho_1, \dots, \rho_N \in L^\gamma(\Omega), \quad \mathbf{v} \in H^1_\nu(\Omega), \quad \log\theta, \theta^{\beta/2} \in H^1(\Omega)$$

such that

$$\rho|\mathbf{v}|^2 \in L^1(\Omega)$$

a *renormalized variational entropy solution* to our problem if there holds the weak formulations of the species equation, momentum equation, entropy inequality and total energy equality, and the total density $\rho := \sum_{i=1}^N \rho_i$ is a renormalized solution to the continuity equation.

Theorem (Large-data existence of solutions)

Let Hypotheses (H1)–(H7) hold. Let $\beta > 2/3$ and $\gamma > 3/2$. Then there exists a renormalized variational entropy solution to our problem. Moreover, if $\beta > 1$ and $\gamma > 5/3$, then the solution is also a renormalized weak solution.

Comments I

The paper contains proof of existence of a solution for large data for chemically reacting mixtures without the assumption that the molar masses are the same for each component as it was the case in previous works based on a mixture model developed by V. Giovangigli:

- V. Giovangigli, M.P., E. Zatorska. On the steady flow of reactive gaseous mixture. *Analysis (Berlin)* **35** (2015), 319–341.
- T. Piasecki, M.P. Weak and variational entropy solutions to the system describing steady flow of a compressible reactive mixture. *Nonlin. Anal.* **159** (2017), 365–392.
- T. Piasecki, M.P. On steady solutions to a model of chemically reacting heat conducting compressible mixture with slip boundary conditions. Mathematical analysis in fluid mechanics–selected recent results, 223–242, *Contemp. Math.* **710**, Amer. Math. Soc., Providence, RI, 2018.

We obtained similar results as in our paper, including the situation when the exponent γ is very close to one as well as we captured differences for the Navier and homogeneous Dirichlet boundary conditions which appear in the situation when the exponent γ is close to one.

Comments II

The model used in the presentation is based on the model developed in

W. Dreyer, P.-E. Druet, P. Gajewski, and C. Guhlke. Analysis of improved Nernst–Planck–Poisson models of compressible isothermal electrolytes. *Z. Angew. Math. Phys.* **71** (2020), Paper No. 119, 68 pp.

An important role is played by the convexity of the Helmholtz free energy and it uses tools of convex analysis in order to verify the strong convergence of partial densities. In the paper above more complex (and evolutionary) problem was studied, including chemical reactions on the boundary, but the temperature was assumed to be constant.

Comments III

The main difficulty is connected with strong convergence of the densities. The strategy (taken from the series of papers [DDGG]) is based on the strong convergence of the projection of chemical potentials combined with the strong convergence of the total density.

One of the developments of our paper is a modified proof of the strong convergence of the total density introduced briefly below uses a modification of the Feireisl's idea of oscillation defect measure estimates. It allows to get rid of some restrictions for the parameters γ and β with respect to the results for the single constituent heat-conducting fluids.

The result can be extended for arbitrary $\gamma > 1$ based on similar idea as described in

A. Novotný, M.P. Weak and variational solutions to steady equations for compressible heat conducting fluids. *SIAM J. Math. Anal.* **43** (2011), 1158–1188.

P.B. Mucha, M.P. E. Zatorska. Existence of stationary weak solutions for compressible heat conducting flows. In: *Handbook of mathematical analysis in mechanics of viscous fluids.* 2595–2662, Springer, Cham, 2018.

Comments IV

The main problem are estimates of the density which must be obtained differently then presented below (by another suitable test function in the momentum balance), it is possible only for the steady problem and differs for the homogeneous Dirichlet and Navier boundary conditions. It would technically complicate the proof, thus it is not considered here.

Main ideas of the proof

We present main ideas of the proof of weak compactness of solutions, the construction of approximate solutions will be mentioned briefly at the end.

- A priori estimates
  - Estimates from entropy inequality and total energy equality
  - Additional density estimates
- Limit passage based on direct compactness
- Strong convergence of densities
  - Effective viscous flux identity
  - A variant of control of oscillations
  - Renormalized continuity equation
  - Strong convergence of the total density
  - Strong convergence of partial densities

A priori estimates I

Step 1: A priori estimates coming from entropy inequality and total energy equality:

We use the entropy inequality with the test function identically one and the total energy equality together with our Hypotheses to get ($\vec{q} = \vec{\mu}/\theta$)

$$\|\mathbf{v}\|_{H^1(\Omega)} + \|\Pi(\vec{q})\|_{H^1(\Omega)} \leq C,$$
$$\|\nabla \log \theta\|_{L^2(\Omega)} + \|\nabla \theta^{\beta/2}\|_{L^2(\Omega)} + \|1/\theta\|_{L^1(\partial\Omega)} \leq C,$$
$$\|\theta\|_{L^1(\partial\Omega)} + \|\log \theta\|_{H^1(\Omega)} + \|\theta^{\beta/2}\|^{2/\beta}_{H^1(\Omega)} + \|\theta\|_{L^{3\beta}(\Omega)} \leq C\big(1 + \|\rho\|_{L^{6/5}(\Omega)}\big).$$

Step 2: Additional density estimates:

We apply as test function in the momentum equation

$$\mathcal{B}\Big(\rho^\nu - \frac{1}{|\Omega|}\int_\Omega \rho^\nu \,\mathrm{d}x\Big)$$

with $\nu := \gamma \min\left\{\frac{2\gamma-3}{\gamma}, \frac{3\beta-2}{3\beta+2}\right\}$. We get

$$\|\rho\|_{L^{\gamma+\nu}(\Omega)} \leq C, \qquad \nu > 0 \text{ provided } \gamma > 3/2, \beta > 2/3$$

and thus we control all norms mentioned above uniformly.

Limit passage I

We now assume that we have a sequence of data

$$
\begin{aligned}
\boldsymbol{b}_\delta \to \boldsymbol{b} \quad & \text{strongly in } L^p(\Omega;\mathbb{R}^3) \text{ for all } p<\infty,\\
\boldsymbol{b}_\delta \rightharpoonup^* \boldsymbol{b} \quad & \text{weakly}^* \text{ in } L^\infty(\Omega;\mathbb{R}^3),\\
\overline{\rho}_\delta \to \overline{\rho} > 0 \quad & \text{in } \mathbb{R},\\
(\theta_0)_\delta \to \theta_0 \quad & \text{strongly in } L^1(\partial\Omega).
\end{aligned}
$$

We assume that for each $\delta > 0$ we have a corresponding solution and the sequence fulfils the a priori estimates presented above. We aim at showing that, modulo subsequence, the limit of solutions given by the estimates is again a solution to the original problem.

The estimates immediately yield the strong convergence of the velocity, projection of the chemical potentials and the temperature, however, only weak convergence of the pressure and the total density (as well as density of each species).

Limit passage II

Limit in the species equation

$$\sum_{i=1}^{N}\int_{\Omega}\left(-\rho_i\mathbf{v}+\sum_{j=1}^{N}\overline{M_{ij}(\vec{\rho_\delta},\theta_\delta)}\nabla q_j-\overline{\frac{M_i(\vec{\rho_\delta},\theta_\delta)}{\theta_\delta}}\frac{\nabla\theta}{\theta}\right)\cdot\nabla\phi_i\,\mathrm{d}x=\sum_{i=1}^{n}\int_{\Omega}r_i(\vec{q},\theta)\phi_i\,\mathrm{d}x,$$

where the bar over a quantity denotes the corresponding weak limit.

Limit in the momentum equation

$$\int_{\Omega}(-\rho\mathbf{v}\otimes\mathbf{v}+\mathbb{S}(\theta,\nabla\mathbf{v})):\nabla\boldsymbol{u}\,\mathrm{d}x+\int_{\partial\Omega}\alpha_1\mathbf{v}\cdot\boldsymbol{u}\,\mathrm{d}s=\int_{\Omega}(\overline{p(\vec{\rho_\delta},\theta_\delta)}\,\mathrm{div}\,\boldsymbol{u}+\rho\boldsymbol{b}\cdot\boldsymbol{u})\,\mathrm{d}x.$$

Limit in the total energy balance

Here we need $\beta > 1$ and $\gamma > 5/3$!

$$\int_{\Omega}\Big(-\frac{1}{2}\rho|\mathbf{v}|^2\mathbf{v}-\overline{\rho_\delta e(\vec{\rho_\delta},\theta_\delta)}\mathbf{v}+\kappa(\theta)\nabla\theta+\sum_{i=1}^{N}\overline{\frac{M_i(\vec{\rho_\delta},\theta_\delta)}{\theta_\delta}}\nabla q_i$$
$$+\mathbb{S}\mathbf{v}-\overline{p(\vec{\rho_\delta},\theta_\delta)}\mathbf{v}\Big)\cdot\nabla\varphi\,\mathrm{d}x+\int_{\partial\Omega}(\alpha_1|\mathbf{v}|^2+\alpha_2(\theta-\theta_0))\varphi\,\mathrm{d}s=\int_{\Omega}\rho\boldsymbol{b}\cdot\mathbf{v}\varphi\,\mathrm{d}x.$$

Limit passage III

Limit in the entropy inequality

$$\int_\Omega \Big[\overline{\rho_\delta s(\vec{\rho_\delta}, \theta_\delta)\mathbf{v}} + \sum_{i=1}^N q_i \Big(\sum_{j=1}^N \overline{\frac{M_{ij}(\vec{\rho_\delta}, \theta_\delta)}{\theta_\delta}} \nabla q_j - \overline{\frac{M_i(\vec{\rho_\delta}, \theta_\delta)}{\theta_\delta}} \frac{\nabla \theta}{\theta} \Big)$$

$$- \Big(\frac{\kappa(\theta)}{\theta} \nabla \theta + \sum_{i=1}^N \overline{\frac{M_i(\vec{\rho_\delta}, \theta_\delta)}{\theta_\delta}} \nabla q_i \Big) \Big] \cdot \nabla \Phi \, \mathrm{d}x$$

$$+ \int_\Omega \Big(\sum_{i,j=1}^N \overline{M_{ij}(\vec{\rho_\delta}, \theta_\delta)} \nabla \frac{\mu_i}{\theta} \cdot \nabla \frac{\mu_j}{\theta} + \kappa(\theta) |\nabla \log \theta|^2 + \frac{\mathbb{S} : \nabla \mathbf{v}}{\theta} - \sum_{i=1}^N r_i \frac{\mu_i}{\theta} \Big) \Phi \, \mathrm{d}x$$

$$\le \alpha_2 \int_{\partial \Omega} \frac{\theta - \theta_0}{\theta} \Phi \, \mathrm{d}s.$$

In order to conclude, we need to show that

$$\vec{\rho_\delta} \to \vec{\rho} \qquad \text{a.e. in } \Omega$$

which implies due to estimates strong convergence of the sequence in $L^q(\Omega; \mathbb{R}^N)$ for any $1 \le q < \nu + \gamma$.

Strong convergence of densities I

Step 1: Effective viscous flux identity

Let T_k be the standard sequence of cut-off functions, i.e.

$$T_1(z) := \begin{cases} z & \text{for } 0 \leq z \leq 1, \\ \text{concave, increasing, } C^1\text{-function} & \text{for } 1 < z < 3, \\ 2 & \text{for } z \geq 3. \end{cases}$$

$$T_k(z) := kT_1(z/k).$$

Then we have the *effective viscous flux identity*

$$\overline{p_\delta T_k(\rho_\delta)} - p\overline{T_k(\rho_\delta)} = \left(\lambda_2(\theta) + \frac{4}{3}\lambda_1(\theta)\right)\left(\overline{T_k(\rho_\delta)\operatorname{div}\mathbf{v}_\delta} - \overline{T_k(\rho_\delta)}\operatorname{div}\mathbf{v}\right),$$

where $p_\delta := p(\vec{\rho}_\delta, \theta_\delta)$ and $p := \overline{p(\vec{\rho}_\delta, \theta_\delta)}$. It follows by testing the momentum equation by

$$\nabla\Delta^{-1}T_k(\rho_\delta) \qquad \text{and} \qquad \nabla\Delta^{-1}\overline{T_k(\rho_\delta)}$$

and exploiting special structure of the momentum equation to be able to apply the Div-Curl lemma.

Strong convergence of densities II

Step 2: Control of oscillations
Part 1:

Set

$$W_k := \overline{p_\delta T_k(\rho_\delta)} - p\,\overline{T_k(\rho_\delta)}.$$

Then for all $k \in \mathbb{N}$,

$$\begin{aligned} 0 \le W_k \le W_{k+1} \quad &\text{a.e. in } \Omega, \\ 0 \le \theta\big(\overline{\rho_\delta T_k(\rho_\delta)} - \rho\,\overline{T_k(\rho_\delta)}\big) \le K_2 W_k \quad &\text{a.e. in } \Omega, \end{aligned}$$

where K_2 is a fixed constant given by properties of h_θ.

The proof is based on convexity of $h_\theta(\cdot)$ and our assumptions on the structural properties of this function.

Strong convergence of densities III

Step 2: Control of oscillations
Part 2:

Furthermore

$$\int_\Omega \frac{W_k}{\lambda_1(\theta) + \lambda_2(\theta)} \, \mathrm{d}x \le C \sup_{\delta > 0} \int_\Omega \frac{\lambda_1(\theta_\delta) + \lambda_2(\theta_\delta)}{\theta_\delta} (\operatorname{div} \mathbf{v}_\delta)^2 \, \mathrm{d}x \le C.$$

Thus, due to the monotonicity of (W_k) and monotone convergence, the sequence $(W_k/(\lambda_1(\theta) + \lambda_2(\theta))_{k \in \mathbb{N}}$ is strongly converging in $L^1(\Omega)$ to a non-negative integrable function.

The proof uses the effective viscous flux and the result from Step 1. It yields $\overline{T_k(\rho_\delta) \operatorname{div} \mathbf{v}_\delta} - \overline{T_k(\rho_\delta)} \operatorname{div} \mathbf{v} \ge 0$. Other steps are straightforward.

Strong convergence of densities IV

Step 2: Control of oscillations
Part 3:

Finally, the quantities

$$Q_k := \overline{p_\delta(T_k(\rho_\delta) - \rho_\delta T_k'(\rho_\delta))} - p\overline{(T_k(\rho_\delta) - \rho_\delta T_k'(\rho_\delta))},$$
$$O_k := T_k(\rho) - \overline{T_k(\rho_\delta)}$$

are non-negative and satisfy for all $k \in \mathbb{N}$,

$$\theta O_k^2 \leq C W_k \quad \text{and} \quad \lim_{k\to\infty} \int_\Omega \frac{Q_k + \theta O_k^2}{\lambda_1(\theta) + \lambda_2(\theta)} \,\mathrm{d}x = 0.$$

The proof is similar to the proof of Part 1 and uses structural properties of $h_\theta(\cdot)$ and the concavity and sub-linearity of the cut-off function $T_k(\cdot)$ as well as the equi-integrability of the sequence ρ_δ.

Strong convergence of densities V

Step 3: Renormalized continuity equation

Due to the estimates above, if $(\rho_\delta, \mathbf{v}_\delta)$ are a renormalized solution to the continuity equation, then also the limit $(\rho, \mathbf{v})$ is a renormalized solution to the continuity equation.

The proof is similar to the proof in the case when we have the control of the *oscillation defect measure.* It uses the renormalization of the limit of the renormalized continuity equation with $b(\rho_\delta) := T_k(\rho_\delta)$ and subsequent limit $k \to \infty$ and uses the above proved results in Step 2. The advantage of this approach in comparison to the standard proof based on oscillation defect measure estimate is the fact that it does not introduce any further restrictions on the exponents β and γ.

Strong convergence of densities VI

Step 4: Strong convergence of the total density

As it is now standard in the mathematical theory of compressible fluids, $\rho_\delta \to \rho$ strongly in $L^1(\Omega)$.

The proof is a slight modification of the approach in case of the control of oscillation defect measure. It is based on suitable application of the renormalized continuity equation combined with effective viscous flux identity and results from Step 2.

Strong convergence of densities VII

Step 5: Strong convergence of partial densities and further properties

Consequently, also $\vec{\rho}_\delta \to \vec{\rho}$ strongly in $L^1(\Omega; \mathbb{R}^N)$.

This follows by the strong convergence of $\Pi\big(\frac{\vec{\mu}_\delta}{\theta_\delta}\big)$, strong convergence of ρ_δ and properties of the Legendre transform.

Finally, we also have

$$\Pi(\vec{\mu}_\delta) \to \theta \Pi_{\vec{q}} \qquad \text{a.e. in } \Omega,$$

where $\Pi\big(\frac{\vec{\mu}_\delta}{\theta_\delta}\big) \to \Pi_{\vec{q}}$ a.e. in Ω.

Approximations

The construction of the approximate solutions is quite complex (based on six parameters). We need to regularize the thermodynamic quantities, improve properties of the pressure and heat conductivity. Instead of the total energy balance we work with internal energy balance and apply Galerkin approximation approximation on the velocity. Altogether it requires 6 parameters. The first limit passages are easy to perform, the last one is based on the ideas from the weak compactness part.

More details can be found in

M. Bulíček, A. Jüngel, M. Pokorný, N. Zamponi. Existence analysis of a stationary compressible fluid model for heat-conducting and chemically reacting mixtures. arXiv:2001.06082.

THANK YOU FOR THE ATTENTION!