Weak solutions for some compressible multicomponent fluid models

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joint work with A. Novotný (Toulon)

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Bi-fluid system I

We consider a model of two compressible viscous fluids with the same velocity as introduced e.g. in

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for the viscous case.

The model assumes that the velocity is the same for both fluids and we distinguish only the mass fractions. At each space-point both fluids may be present, there is no interface between the fluids. On the other hand, we assume a certain algebraic relation between the models of each fluid: the pressures for both components are the same and the whole model can be viewed as a result of some kind of averaging (homogenization) and can be obtained physically rigorously. Summing up the balances of linear momentum for each fluid we end up with the model in $Q_T = (0, T) \times \Omega \subset \mathbb{R}^4$

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Bi-fluid system II

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\partial_t(\mathfrak{a}\varrho_+) + \operatorname{div}(\mathfrak{a}\varrho_+\mathbf{u}) &= 0,\\
\partial_t((1-\mathfrak{a})\varrho_-) + \operatorname{div}((1-\mathfrak{a})\varrho_-)\mathbf{u}) &= 0,\\
\partial_t\big((\mathfrak{a}\varrho_+ + (1-\mathfrak{a})\varrho_-))\mathbf{u}\big) + \\
\operatorname{div}\Big(\big(\mathfrak{a}\varrho_+ + (1-\mathfrak{a})\varrho_-\big)\mathbf{u}\otimes\mathbf{u}\Big) + \nabla\mathfrak{P}_+(\varrho_+) &= \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\operatorname{div}\mathbf{u},\\
\mathfrak{P}_+(\varrho_+) &= \mathfrak{P}_-(\varrho_-),
\end{aligned}
\tag{1}
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Above, $\mathbf{u}$ is the (common) velocity, while $\mathfrak{P}_\pm$ are given functions characterizing the species in the mixture. Next, $0 \le \mathfrak{a} \le 1$, $\varrho_+ \ge 0$, $\varrho_- \ge 0$ and $\mathbf{u}$ are unknown functions. They have the following meaning: $\mathfrak{a}$, $\mathfrak{a}\varrho_+$, $(1-\mathfrak{a})\varrho_-$ denote the rate of amount of the first species, density of the first and the second species in the mixture, respectively. The constants μ and λ are the shear and bulk (average) viscosities of the mixture. We assume $\mu > 0$ and $2\mu + 3\lambda \ge 0$.

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Bi-fluid system III

We consider the boundary conditions

$$\mathbf{u} = \mathbf{0} \tag{2}$$

on $(0, T) \times \partial\Omega$, and the initial conditions in Ω

$$\begin{aligned} \mathfrak{a}\varrho_+(0,x) &= \mathfrak{a}_0\varrho_{+,0}(x) := \varrho_0(x), \\ (1-\mathfrak{a})\varrho_-(0,x) &= (1-\mathfrak{a}_0)\varrho_{-,0}(x) := Z_0(x), \\ (\mathfrak{a}\varrho_+ + (1-\mathfrak{a})\varrho_-)\mathbf{u}(0,x) &= (\mathfrak{a}_0\varrho_{0,+} + (1-\mathfrak{a}_0)\varrho_{-,0})\mathbf{u}_0(x) := \mathbf{M}_0. \end{aligned} \tag{3}$$

and call the system (1)–(3) the *bi-fluid system with algebraic closure*.

Bi-fluid system IV

Another system was also recently investigated in

A. Vasseur, H. Wen, C. Yu: Global weak solution to the viscous two-fluid model with finite energy, J. de Math. Pures et Appl. **125**, 247–282, 2019.

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\ \partial_t Z + \operatorname{div}(Z\mathbf{u}) &= 0, \\ \partial_t\big((\varrho + Z)\mathbf{u}\big) + \operatorname{div}\big((\varrho+Z)\mathbf{u}\otimes\mathbf{u}\big) + \nabla P(\varrho, Z) &= \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla \operatorname{div}\mathbf{u} \end{aligned} \tag{4}$$

with the boundary condition

$$\mathbf{u} = \mathbf{0} \tag{5}$$

on $(0,T)\times\partial\Omega$ and the initial conditions in Ω

$$\varrho(0,x) = \varrho_0(x),\ Z(0,x) = Z_0(x),\ (\varrho+Z)\mathbf{u}(0,x) = \mathbf{m}_0(x). \tag{6}$$

We will call this problem the *tranformed two-densities system*. In fact, the authors considered a special case $P(\varrho, Z) = \varrho^\gamma + Z^\beta$ for some $\gamma > \frac{9}{5}$ and $\beta \geq 1$. A similar system (however, as an auxiliary system for another problem) was considered for $\gamma > \frac{3}{2}$ and $\varrho + Z$ replaced by ϱ also in

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Aim of the talk: to present as general as possible existence theory for the bi-fluid system with algebraic closure (with special emphasis on the power-like behaviour of the pressure functions $\mathfrak{P}_\pm$ near zero and near infinity.
It will be achieved by careful study of the transformed two-densities system (based on several non trivial tools presented below) and on a transformation of the bi-fluid system with algebraic closure to the two-densities system.

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Weak solution for the two-densities model I

Definition

The triple $(\varrho, Z, \mathbf{u})$ is a bounded energy weak solution to problem (4–6), if $\varrho, Z \geq 0$ a.e. in $I \times \Omega$, $\varrho \in L^\infty(I; L^\gamma(\Omega))$, $Z \in L^\infty(I; L^\gamma(\Omega))$, $\mathbf{u} \in L^2(I; W_0^{1,2}(\Omega; R^3))$, $(\varrho + Z)|\mathbf{u}|^2 \in L^\infty(I; L^1(\Omega))$, $P(\varrho, Z) \in L^1(I \times \Omega)$, and

$$\begin{aligned} \int_0^T \int_\Omega \left(\varrho \partial_t \psi + \varrho \mathbf{u} \cdot \nabla \psi \right) \mathrm{d}x \, \mathrm{d}t + \int_\Omega \varrho_0 \psi(0, \cdot) \, \mathrm{d}x = 0 \\ \int_0^T \int_\Omega \left(Z \partial_t \psi + Z \mathbf{u} \cdot \nabla \psi \right) \mathrm{d}x \, \mathrm{d}t + \int_\Omega Z_0 \psi(0, \cdot) \, \mathrm{d}x = 0 \end{aligned} \tag{7}$$

for any $\psi \in C_c^1([0, T) \times \overline{\Omega})$,

$$\begin{aligned} &\int_0^T \int_\Omega \left((\varrho + Z)\mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + (\varrho + Z)(\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\varphi} + P(\varrho, Z) \operatorname{div} \boldsymbol{\varphi} \right) \mathrm{d}x \, \mathrm{d}t \\ = &\int_0^T \int_\Omega \left(\mu \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} + (\mu + \lambda) \operatorname{div} \mathbf{u} \operatorname{div} \boldsymbol{\varphi} \right) \mathrm{d}x \, \mathrm{d}t - \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}x \end{aligned} \tag{8}$$

for any $\boldsymbol{\varphi} \in C_c^1([0, T) \times \Omega; R^3)$,

Weak solution for the two-densities model II

and the energy inequality holds

$$\int_\Omega \Big(\frac{1}{2}(\varrho + Z)|\mathbf{u}|^2 + H_P(\varrho, Z)\Big)(\tau,\cdot)\,\mathrm{d}x + \int_0^\tau \int_\Omega \big(\mu|\nabla \mathbf{u}|^2 + (\mu+\lambda)(\operatorname{div}\mathbf{u})^2\big)\,\mathrm{d}x\,\mathrm{d}t \le \int_\Omega \Big(\frac{|\mathbf{m}_0|^2}{2(\varrho_0 + Z_0)} + H_P(\varrho_0, Z_0)\Big)\,\mathrm{d}x \tag{9}$$

for a.a. $\tau \in (0, T)$.

We consider the Helmholtz free energy function $H_P(\varrho, Z)$ corresponding to P is a solution of the partial differential equation of the first order in $(0,\infty)^2$

$$P(\varrho, Z) = \varrho\frac{\partial H_P(\varrho, Z)}{\partial \varrho} + Z\frac{\partial H_P(\varrho, Z)}{\partial Z} - H_P(\varrho, Z). \tag{10}$$

It is not uniquely determined. We take

$$H = H_P(\varrho, Z) = \varrho\int_1^\varrho \frac{P(s, s\frac{Z}{\varrho})}{s^2}\,\mathrm{d}s \text{ if } \varrho > 0,\ H_P(0,0) = 0. \tag{11}$$

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Hypotheses I

Hypothesis (H1). We denote

$$\mathcal{O}_{\underline{a}} := \{(\varrho, Z) \in R^2 \mid \varrho \in [0,\infty), \underline{a}\varrho < Z < \overline{a}\varrho\},\ 0 \le \underline{a} < \overline{a} < \infty \tag{12}$$

and assume

$$(\varrho_0, Z_0) \in \overline{\mathcal{O}_{\underline{a}}} = \{(\varrho, Z) \in R^2 \mid \varrho \in [0,\infty), \underline{a}\varrho \le Z \le \overline{a}\varrho\},\ 0 \le \underline{a} < \overline{a} < \infty. \tag{13}$$

In what follows we will always use the following convention for the calculus of fractions $\frac{Z}{\varrho}$ provided $(\varrho, Z) \in \mathcal{O}_{\underline{a}}$, namely

$$s = \frac{Z}{\varrho} := \begin{cases} \frac{Z}{\varrho} \text{ if } \varrho > 0, \\ 0 \text{ if } \varrho = 0. \end{cases} \tag{14}$$

Hypothesis (H2).

$$\varrho_0 \in L^\gamma(\Omega),\ Z_0 \in L^\beta(\Omega) \ \text{ if } \beta > \gamma, \tag{15}$$

$$\mathbf{m}_0 \in L^1(\Omega; R^3),\ (\varrho_0 + Z_0)|\mathbf{u}_0|^2 \in L^1(\Omega).$$

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Hypotheses II

Hypothesis (H3).
We suppose that pressure $P \in C(\overline{\mathcal{O}_{\underline{a}}}) \cap C^1(\mathcal{O}_{\underline{a}})$ is such that

$$\forall \varrho \in (0,1), \sup_{s\in[\underline{a},\overline{a}]} |P(\varrho, \varrho s)| \leq \overline{C}\varrho^{\alpha} \text{ with some } \alpha > 0. \tag{16}$$

and for all $(\varrho, Z) \in \overline{\mathcal{O}_{\underline{a}}}$

$$\underline{C}(\varrho^{\gamma} + Z^{\beta} - 1) \leq P(\varrho, Z) \leq \overline{C}(\varrho^{\gamma} + Z^{\beta} + 1), \tag{17}$$

with $\gamma \geq \frac{9}{5}$ and $\beta > 0$, where $\underline{C}$, $\overline{C}$ are two positive constants.
We moreover assume

$$|\partial_Z P(\varrho, Z)| \leq C(\varrho^{-\underline{\Gamma}} + \varrho^{\overline{\Gamma}-1}) \text{ in } \mathcal{O}_{\underline{a}} \tag{18}$$

with some $0 \leq \underline{\Gamma} < 1$, and with some $0 < \overline{\Gamma} < \gamma + \gamma_{BOG}$ if $\underline{a} = 0$, $0 < \overline{\Gamma} < \max\{\gamma + \gamma_{BOG}, \beta + \beta_{BOG}\}$ if $\underline{a} > 0$, where $\gamma_{BOG} := \min\{\frac{2}{3}\gamma - 1, \frac{\gamma}{2}\}$ is the improvement of the integrability due to the Bogovskii operator estimates. Similarly for β_{BOG}.

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Hypotheses III

Hypothesis (H4).
We assume

$$P(\varrho, \varrho s) = \mathcal{P}(\varrho, s) - \mathcal{R}(\varrho, s), \tag{19}$$

where $[0, \infty) \ni \varrho \mapsto \mathcal{P}(\varrho, s)$ is non decreasing for any $s \in [\underline{a}, \overline{a}]$, and $\varrho \mapsto \mathcal{R}(\varrho, s)$ is for any $s \in [\underline{a}, \overline{a}]$ a non-negative C^2-function in $[0, \infty)$ uniformly bounded with respect to $s \in [\underline{a}, \overline{a}]$ with compact support uniform with respect to $s \in [\underline{a}, \overline{a}]$. Here, $\underline{a}, \overline{a}$ are the constants from relation (12). Moreover, if $\gamma = \frac{9}{5}$, we suppose that

$$\mathcal{P}(\varrho, s) = f(s)\varrho^{\gamma} + \pi(\varrho, s), \tag{20}$$

where $[0, \infty) \ni \varrho \mapsto \pi(\varrho, s)$ is non decreasing for any $s \in [\underline{a}, \overline{a}]$ and $f \in L^{\infty}(\underline{a}, \overline{a})$, $\operatorname{ess\,inf}_{s \in (\underline{a}, \overline{a})} f(s) \geq \underline{f} > 0$. Finally,

$$\forall \varrho \in (0, 1), \sup_{s \in [\underline{a}, \overline{a}]} P(\varrho, \varrho s) \leq c\varrho^{\alpha} \text{ with some } c > 0 \text{ and } \alpha > 0. \tag{21}$$

Hypotheses IV

Hypothesis (H5).
Function $\varrho \mapsto P(\varrho, Z)$ is for all $Z > 0$ locally Lipschitz on $(0, \infty)$ and function $Z \mapsto \partial_Z P(\varrho, Z)$ is for all $\varrho > 0$ locally Lipschitz on $(0, \infty)$ with Lipschitz constant

$$\begin{aligned} &\widetilde{L}_P(\varrho, Z) \leq C(\underline{r})(1 + \varrho^A) \\ &\widetilde{L}_P(\varrho, Z) \leq C(\underline{r})(1 + Z^A) \\ &\text{for all } \underline{r} > 0,\ (\varrho, Z) \in \mathcal{O}_{\underline{a}} \cap (\underline{r}, \infty)^2 \end{aligned} \tag{22}$$

with some non negative number A. Number $C(\underline{r})$ may diverge to $+\infty$ as $\underline{r} \to 0^+$.

Examples I

We may take

$$P(\varrho, Z) = \varrho^{\gamma} + Z^{\beta} + \sum_{i=1}^{M} F_i(\varrho, Z), \tag{23}$$

where $F_i(\varrho, Z) = C_i \varrho^{r_i} Z^{s_i}$, $0 \leq r_i < \gamma$, $0 \leq s_i < \beta$, $r_i + s_i < \max\{\gamma, \beta\}$. It is an easy matter to check that all Hypotheses (H3–H5) are fulfilled.
Another possibility is

$$P(\varrho, Z) = (\varrho + Z)^{\gamma} + \sum_{i=1}^{M} F_i(\varrho, Z), \tag{24}$$

where F_i are as above (for $\beta = \gamma$).
One more nontrivial example will be presented later, when we consider the real bi-fluid system.

Results I

Theorem

Let $\gamma \geq \frac{9}{5}$, $0 < \beta < \infty$. *Then under Hypotheses (H1–H5) problem* (4–6) *admits at least one weak solution in the sense of Definition* 1. *Moreover, for all* $t \in [0,T]$, $(\varrho(t,x), Z(t,x)) \in \mathcal{O}_{\underline{a}}$ *for a.a.* $x \in \Omega$,
$\varrho \in C_{weak}([0,T]; L^{\gamma}(\Omega)) \cap C([0,T]; L^1(\Omega))$,
$Z \in C_{weak}([0,T]; L^{q_{\gamma,\beta}}(\Omega)) \cap C([0,T]; L^1(\Omega))$,
$(\varrho + Z)\mathbf{u} \in C_{weak}([0,T]; L^q(\Omega; R^3))$ *for some* $q > 1$ *and* $P(\varrho, Z) \in L^q(I \times \Omega)$ *for some* $q > 1$. *In the above*

$$q_{\gamma,\beta} = \gamma \text{ if } \beta < \gamma, \; q_{\gamma,\beta} = \beta \text{ if } \beta \geq \gamma.$$

Definition

The quadruple $(\mathfrak{a}, \varrho_-, \varrho_+, \mathbf{u})$ is a bounded energy weak solution to problem (1–3), if $0 \le \mathfrak{a} \le 1$, $\varrho_\pm \ge 0$ a.a. in $I \times \Omega$, $\varrho^\pm \in L^\infty(I; L^1(\Omega))$, $\mathbf{u} \in L^2(I; W_0^{1,2}(\Omega; R^3))$, $(\mathfrak{a}\varrho_+ + (1-\mathfrak{a})\varrho_-)|\mathbf{u}|^2 \in L^\infty(I; L^1(\Omega))$, $\mathfrak{P}_-(\varrho_-) = \mathfrak{P}_+(\varrho_+) \in L^1(I \times \Omega)$, and:

- Continuity equations (7) are satisfied with $\varrho = \mathfrak{a}\varrho_+$ and $Z = (1-\mathfrak{a})\varrho_-$;
- Momentum equation (8) is satisfied with $\varrho = \mathfrak{a}\varrho_+$, $Z = (1-\mathfrak{a})\varrho_-$ and with function $P(\varrho, Z)$ replaced by $\mathfrak{P}_+(\varrho_+)$;
- There is a non negative function $\mathfrak{H} : (0,1) \times (0,\infty)^2$ such that

$$\int_\Omega \Big(\frac{1}{2}(\varrho + Z)|\mathbf{u}|^2 + \mathfrak{H}(\mathfrak{a}, \varrho_-, \varrho_+)\Big)(\tau, \cdot)\,\mathrm{d}x$$
$$+ \int_0^\tau \int_\Omega \big(\mu|\nabla \mathbf{u}|^2 + (\mu+\lambda)(\operatorname{div}\mathbf{u})^2\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \int_\Omega \Big(\frac{|\mathbf{M}_0|^2}{2(\varrho_0 + Z_0)} + \mathfrak{H}(\mathfrak{a}_0, \varrho_{-,0}, \varrho_{+,0})\Big)\,\mathrm{d}x \qquad (25)$$

for a.a. $\tau \in (0, T)$.

Results for the bi-fluid system with algebraic closure I

Theorem

Let $0 \leq \underline{a} < \overline{a} < \infty$. *Let* $G := \gamma^+ + \gamma^+_{BOG}$ *if* $\underline{a} = 0$ *and* $G := \max\{\gamma^+ + \gamma^+_{BOG}, \gamma^- + \gamma^-_{BOG}\}$ *if* $\underline{a} > 0$. *Assume*

$$0 < \gamma^- < \infty,\ \gamma^+ \geq \frac{9}{5},\ \overline{\Gamma} < G, \tag{26}$$

where

$$\overline{\Gamma} = \left\{ \begin{array}{c} \max\{\gamma^+ - \frac{\gamma^+}{\gamma^-} + 1,\ \gamma^- + \frac{\gamma^-}{\gamma^+} - \frac{\gamma^+}{\gamma^-}\} \textit{ if } \underline{a} = 0 \\ \max\{\gamma^+ - \frac{\gamma^+}{\gamma^-} + 1, \gamma^- + \frac{\gamma^-}{\gamma^+} - 1\} \textit{ if } \underline{a} > 0 \end{array} \right\}.$$

Suppose that

$$0 \leq \mathfrak{a}_0 \leq 1,\ \underline{a}\mathfrak{a}_0\varrho_{+,0} \leq (1 - \mathfrak{a}_0)\varrho_{-,0} \leq \overline{a}\mathfrak{a}_0\varrho_{+,0}, \mathfrak{P}_-(\varrho_{-,0}) = \mathfrak{P}_+(\varrho_{+,0}) \tag{27}$$

$$\varrho_{+,0} \in L^{\gamma^+}(\Omega),\ \frac{|\mathbf{M}_0|^2}{\mathfrak{a}_0\varrho_{+,0} + (1 - \mathfrak{a}_0)\varrho_{-,0}} \in L^1(\Omega).$$

Results for the bi-fluid system with algebraic closure II

Assume further that

$$\mathfrak{P}_\pm \in C([0,\infty)) \cap C^2((0,\infty)),\ \mathfrak{P}_\pm(0)=0,\ \mathfrak{P}'_\pm(s)>0, s>0, \tag{28}$$

$$\underline{a}_- s^{\gamma^- -1} - b_- \le \mathfrak{P}'_-(s),\ \mathfrak{P}_-(s) \le \overline{a}_- s^{\gamma^-} + b_-,$$

$$\underline{a}_+ s^{\gamma^+ -1} - b_+ \le \mathfrak{P}'_+(s) \ \le\ \overline{a}_+ s^{\gamma^+ -1} + b_+,$$

$$|\mathfrak{P}''_\pm(s)| \le d_\pm s^{A_\pm} + e_\pm,\ s \ge r > 0,\ \mathfrak{P}''_+(s) \ge 0,\ s \in (0,\infty)$$

with some positive constants $a_\pm$, $b_\pm$, $d_\pm$, $e_\pm$ *and* $A_\pm$. *Suppose further that*

1.

$$\sup_{s\in(0,1)} s^{\underline{\Gamma}} \frac{\mathfrak{P}'_+(s+\mathfrak{q}^{-1}(\overline{a}s))(s+\mathfrak{q}^{-1}(\overline{a}s))^2}{s\mathfrak{q}(s)} \le \overline{c} < \infty \tag{29}$$

with some $\underline{\Gamma} \in [0,1)$, *where*

$$\mathfrak{q} = \mathfrak{P}_-^{-1} \circ \mathfrak{P}_+. \tag{30}$$

2.

$$0 < \underline{q} = \inf_{s\in(0,\infty)} \frac{\mathfrak{q}(s)}{s\mathfrak{q}'(s)+\mathfrak{q}(s)}. \tag{31}$$

Results for the bi-fluid system with algebraic closure III

Then problem (1–3) admits at least one weak solution in the sense of Definition 3. Moreover, $\mathfrak{a}\varrho_+$ *belongs to the space* $C_{weak}([0,T);L^{\gamma^+}(\Omega))$ *and* $(1-\mathfrak{a})\varrho_-$ *belongs to the space* $C_{weak}([0,T);L^{q_{\gamma^+,\gamma^-}}(\Omega))$*, the vector field* $(\mathfrak{a}\varrho_+ + (1-\mathfrak{a})\varrho_-)\mathbf{u}$ *belongs to* $C_{weak}([0,T);L^r(\Omega;R^3))$ *for some* $r>1$*,* $\mathfrak{P}_\pm(\varrho_\pm)\in L^r((0,T)\times\Omega)$ *for some* $r>1$ *and the function* $\mathfrak{H}$ *in the energy inequality (25) is given by formula*

$$\mathfrak{H}(\mathfrak{a},\varrho_-,\varrho_+) = \mathfrak{a}\varrho_+\int_0^{\mathfrak{a}\varrho_+}\frac{\mathfrak{P}_+\circ\mathfrak{R}\Big(s,\ \frac{(1-\mathfrak{a})\varrho_-}{\mathfrak{a}\varrho_+}\,s\Big)}{s^2}\,\mathrm{d}s,$$

where $\mathfrak{R}(s,z)$ *is the unique solution in* $[\mathfrak{a}\varrho_+,\infty)$ *of equation*

$$\mathfrak{R}\mathfrak{q}(\mathfrak{R}) - \mathfrak{q}(\mathfrak{R})s - \mathfrak{R}z = 0.$$

Examples I

The results of both Theorems remain valid — after well known modifications in the definition of weak solutions in these cases — in the space periodic setting or if we replace the no-slip boundary conditions by the Navier conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega}=0,\ \Big[\mu\Big(\nabla\mathbf{u}+\nabla\mathbf{u}^T-\frac{2}{3}\mathbb{I}\mathrm{div}\mathbf{u}\Big)+(\lambda-\frac{\mu}{3})\mathbb{I}\mathrm{div}\mathbf{u}\Big]\mathbf{n}\times\mathbf{n}|_{\partial\Omega}=\mathbf{0}.$$

Assumptions of Theorem 4 imposed on partial pressure laws are numerous but not so much restrictive: If $\mathfrak{P}_\pm(\varrho)\sim_0 \varrho^{\alpha^\pm}$ and $\mathfrak{P}_\pm(\varrho)\sim_\infty \varrho^{\gamma^\pm}$. Then all hypotheses are satisfied provided $\alpha^+>0$, $\alpha^->\frac{\alpha^+}{\sqrt{\alpha^++1}}$, $\gamma^+\geq 9/5$ and, if $\underline{a}=0$, $0<\gamma^-<\frac{3\gamma^+}{6-2\gamma^+}$ for $\gamma^+<3$, and $\gamma^-+\frac{(\gamma^-)^2-(\gamma^+)^2}{\gamma^-\gamma^+}<G$; if $\underline{a}>0$, then $\gamma^->0$ arbitrary.

Existence of a solution to the bi-fluid system with algebraic closure I

We introduce new unknowns

$$\varrho = \mathfrak{a}\varrho_+, \quad Z = (1-\mathfrak{a})\varrho_- \tag{32}$$

and use

$$\mathfrak{P}_+(\varrho_+) = \mathfrak{P}_-(\varrho_-)$$

to express quantities $(\mathfrak{a}, \varrho_+, \varrho_-)$ in terms of new quantities (ϱ, Z). We get

$$\varrho_+\mathfrak{q}(\varrho_+) - \mathfrak{q}(\varrho_+)\varrho - Z\varrho_+ = 0, \text{ where } \mathfrak{q} = \mathfrak{P}_-^{-1} \circ \mathfrak{P}_+. \tag{33}$$

It admits for any $\varrho \geq 0$, $Z \geq 0$ a unique solution

$$\left\{ \begin{array}{c} 0 < \varrho_+ = \varrho_+(\varrho, Z) \in [\varrho, \infty) \text{ if } \varrho > 0 \text{ or } Z > 0, \\ \varrho_+(0,0) = 0 \end{array} \right\} \tag{34}$$

such that

$$\varrho_+(\varrho, 0) = \varrho, \quad \varrho_+(0, Z) = \mathfrak{q}^{-1}(Z). \tag{35}$$

Existence of a solution to the bi-fluid system with algebraic closure II

We define

$$P(\varrho, Z) := \mathfrak{P}_+(\varrho_+(\varrho, Z)). \tag{36}$$

We now compute

$$0 < \partial_\varrho \varrho_+(\varrho, Z) = \frac{\varrho_+ \mathfrak{q}(\varrho_+)}{\varrho \mathfrak{q}(\varrho_+) + \varrho_+ \mathfrak{q}'(\varrho_+)(\varrho_+ - \varrho)} = \frac{Q'(\varrho_+)}{\mathfrak{q}'(\varrho_+) - \varrho Q''(\varrho_+)} \tag{37}$$

and

$$0 < \partial_Z \varrho_+(\varrho, Z) = \frac{(\varrho_+)^2}{\varrho \mathfrak{q}(\varrho_+) + \varrho_+ \mathfrak{q}'(\varrho_+)(\varrho_+ - \varrho)}, \tag{38}$$

where $Q(s) = \int_0^s \frac{\mathfrak{q}(z)}{z} \, \mathrm{d}z$. Using this we may verify that we reformulated our problem into the academic bi-fluid problem and under the assumptions of our theorem we verify that Hypotheses (H1–H5) are fulfilled.

This finishes the proof of the second theorem.

THANK YOU VERY MUCH

FOR YOUR ATTENTION!