Steady compressible Navier–Stokes–Fourier system and related problems: Large data results

Milan Pokorný

DSc. dissertation

*To Terezie, Štěpánka, Amálie,
Kristián and Benjamín*

La matematica é un'arte diabolica, e i matematici, come autori di tutte le eresie, dovrebbero esser scacciati da tutti gli stati.

Fra Tommaso Caccini, December 21st, 1614, Santa Maria Novella di Firenze

Povinná maturita z matematiky nebude. Sláva!

Title from “Reflex”, September 15th, 2019

Preface

The presented DSc. thesis deals with mathematical questions connected with the description of steady flow of compressible heat conducting fluids. The results were achieved in the last ten years in collaboration with three different groups of mathematicians: the group at the Warsaw University, especially with Professor Piotr B. Mucha and his collaborators, the group at the University of Toulon, especially with Professor Antonín Novotný and his students, and the group at the Mathematical Institute of the Czech Academy of Sciences in Prague, especially with Professor Eduard Feireisl, Dr. Šárka Nečasová and their collaborators.

All presented papers deal with the question of the existence of solutions without any assumption on the size of the data or distance to other, more regular solutions. They contain, in the field of steady compressible heat conducting Newtonian single component flow, up to one overview paper (where, however, the author of the thesis is also one of the co-authors), all most important results connected with the existence of solutions. Additionally, the thesis also includes results for steady flows of more complex fluids, where the steady compressible Navier–Stokes–Fourier equations play the central role.

The first part of the thesis is formed by an introduction to the studied problems, together with a short overview of the results presented further. It also contains an overview of further results in closely connected fields of mathematical fluid mechanics, and a list of chosen references. The second part is formed by eight — from my point of view — most important results where the author of the thesis was among the authors.

Prague, January 30th, 2020

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Part I

Introductory material

Chapter 1

Compressible heat conducting Newtonian fluid

We shall briefly introduce the models coming from the continuum mechanics and thermodynamics which we study later. More detailed information can be found e.g. in the monographs [Gurtin 1991], [Gallavotti 2002] or [Lamb 1993] for the case of single component flow, and in [Giovangigli 1999] or [Rajagopal Tao 1995] for the case of multicomponent flow.

1.1 Single component flow

We consider the three fundamental balance laws: the balance of mass, the balance of linear momentum and the balance of total energy. Using the so-called Eulerian description (which is commonly used for equations of fluid dynamics) we have in $(0,T)\times\Omega$

$$\begin{aligned} \frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\ \frac{\partial (\varrho \mathbf{u})}{\partial t} + \operatorname{div}(\varrho \mathbf{u}\otimes \mathbf{u}) - \operatorname{div}\mathbb{T} &= \varrho \mathbf{f}, \\ \frac{\partial (\varrho E)}{\partial t} + \operatorname{div}(\varrho E \mathbf{u}) + \operatorname{div}\mathbf{q} - \operatorname{div}(\mathbb{T}\mathbf{u}) &= \varrho \mathbf{f}\cdot \mathbf{u}. \end{aligned} \tag{1.1}$$

The classical formulation of these equations is actually not what we are going to deal with in this thesis. We shall work with weak or variational entropy solutions. These formulations, stated later in the thesis, can be derived directly from the integral formulation of the balance laws. Therefore we do not need to work with the classical formulation of the balance laws, on the other hand, in the mathematical community of partial differential equations it is quite common to write the classical formulation even though it is not the formulation the authors usually work with. We shall follow this habit.

For simplicity, we assume that the spatial domain $\Omega \subset \mathbb{R}^N$, $N = 2$ or 3, is bounded and fixed. We shall mostly deal with the case $N = 3$, which is physically the most relevant one, however, in some cases we also consider $N = 2$. Above, ϱ: $(0,T) \times \Omega \to \mathbb{R}^+$ is the density of the fluid, $\mathbf{u}$: $(0,T) \times \Omega \to \mathbb{R}^N$ is the velocity, E: $(0,T) \times \Omega \to \mathbb{R}^+$ is the specific total energy, $\mathbb{T}$: $(0,T) \times \Omega \to \mathbb{R}^{N\times N}$ is the stress tensor, $\mathbf{q}$: $(0,T) \times \Omega \to \mathbb{R}^N$ is the heat flux, and the given vector field $\mathbf{f}$: $(0,T) \times \Omega \to \mathbb{R}^N$ denotes the external volume force. Recall that $E = \frac{1}{2}|\mathbf{u}|^2 + e$, where $\frac{1}{2}|\mathbf{u}|^2$ is the specific kinetic energy and e is the specific internal energy. Generally, the balance of the angular momentum should also be taken into account together with (1.1). However, if we do not assume any internal momenta of the continuum, it can be verified that as a consequence of the angular momentum balance the stress tensor $\mathbb{T}$ must be symmetric which we assume in what follows.

We take (as commonly used) for our basic thermodynamic quantities the density ϱ and the thermodynamic temperature ϑ. Therefore all other quantities, i.e., the stress tensor $\mathbb{T}$, the internal energy e and the heat flux $\mathbf{q}$ are given functions of t, x, ϱ, $\mathbf{u}$ and ϑ. However, in what follows, we do not consider processes, where these quantities depend explicitly on the time and space variables. The standard assumptions from the continuum mechanics (as e.g. the material frame indifference) yield that

$$\mathbb{T} = -p(\varrho, \vartheta)\mathbb{I} + \mathbb{S}(\varrho, \mathbb{D}(\mathbf{u}), \vartheta),$$

where $\mathbb{I}$ denotes the unit tensor, the scalar quantity p (a given function of the density and temperature) is the pressure, $\mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the symmetric part of the velocity gradient and the tensor $\mathbb{S}$ is the viscous part of the stress tensor. We mostly consider only linear dependence of the stress tensor on the symmetric part of the velocity gradient. This yields, together with the assumption that the viscosities are density independent (this assumptions is, unfortunately, physically less relevant, but the nowadays available technique is generally not able to deal with problems containing the viscosity both temperature and density dependent)

$$\mathbb{S}(\mathbb{D}(\mathbf{u}), \vartheta) = \mu(\vartheta)\Big(2\mathbb{D}(\mathbf{u}) - \frac{2}{N} \operatorname{div} \mathbf{u}\, \mathbb{I}\Big) + \xi(\vartheta) \operatorname{div} \mathbf{u}\, \mathbb{I}. \tag{1.2}$$

The scalar functions $\mu(\cdot) > 0$ and $\xi(\cdot) \geq 0$ are called the shear and the bulk viscosities. We shall study the situations with $\mu(\vartheta) \sim (1 + \vartheta)^a$ a

Lipschitz continuous function and $\xi(\vartheta) \leq C(1+\vartheta)^a$ a continuous function for $0 \leq a \leq 1$ and $C > 0$. For the pressure, we mostly consider the gas law of the form

$$p(\varrho,\vartheta) = (\gamma - 1)\varrho e(\varrho,\vartheta), \tag{1.3}$$

a generalization of the law for the monoatomic gas, where $\gamma = \frac{5}{3}$. In general, the value $\frac{5}{3}$ is the highest physically interesting value and for all other gases we should take $1 \leq \gamma \leq \frac{5}{3}$, cf. [Elizier et al 1996].

We also sometimes replace assumption (1.3) by

$$p(\varrho,\vartheta) = \varrho^\gamma + \varrho\vartheta, \quad e(\varrho,\vartheta) = \frac{1}{\gamma-1}\varrho^{\gamma-1} + c_v\vartheta, \text{ with } c_v > 0, \tag{1.4}$$

whose physical relevance is discussed in [Feireisl 2004]. The pressure and the specific internal energy from (1.4) are in fact a simplification of (1.3) which still contains the same asymptotic properties and hence also leads to the same main mathematical difficulties as the more general model (1.3).

The heat flux is assumed to fulfil the Fourier law

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla\vartheta) = -\kappa(\vartheta)\nabla\vartheta \tag{1.5}$$

with the heat conductivity $\kappa(\vartheta) \sim (1+\vartheta)^m$ for some $m > 0$.

To get a well posed problem, we must prescribe the initial conditions

$$\varrho(0,x) = \varrho_0(x), \quad (\varrho\mathbf{u})(0,x) = \mathbf{m}_0(x), \quad \vartheta(0,x) = \vartheta_0(x) \tag{1.6}$$

in Ω and the boundary conditions on $\partial\Omega$. The problem of the correct choice of the boundary conditions is far from being trivial. We restrict ourselves to the following simple cases. For the heat flux, we take

$$-\mathbf{q}\cdot\mathbf{n} + L(\vartheta)(\vartheta - \Theta_0) = 0 \tag{1.7}$$

and for the velocity we consider either the homogeneous Dirichlet boundary conditions

$$\mathbf{u} = \mathbf{0} \tag{1.8}$$

or the (partial) slip boundary conditions (sometimes also called the Navier boundary conditions)

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad (\mathbb{S}\mathbf{n})\times\mathbf{n} + \alpha\mathbf{u}\times\mathbf{n} = \mathbf{0}. \tag{1.9}$$

Above, $\mathbf{n}$ denotes the external normal vector to $\partial\Omega$, Θ_0: $(0,T)\times\partial\Omega \to \mathbb{R}^+$ is the external temperature, $L(\vartheta) \sim (1+\vartheta)^l$, a continuous function, characterizes the thermal insulation of the boundary, and $\alpha \geq 0$ is the friction coefficient which is for simplicity assumed to be constant. Since in what follows we consider only the steady or time-periodic problems, we cannot assume the boundary to be at the same time thermally (i.e. zero heat flux) and mechanically insulated as the set of such solutions would be quite trivial, cf. [Feireisl Pražák 2010].

The Second law of thermodynamics implies the existence of a differentiable function $s(\varrho, \vartheta)$ called the specific entropy which is (up to an additive constant) given by the Gibbs relation

$$\frac{1}{\vartheta}\Big(\mathbf{D}e(\varrho,\vartheta) + p(\varrho,\vartheta)\mathbf{D}\Big(\frac{1}{\varrho}\Big)\Big) = \mathbf{D}s(\varrho,\vartheta).$$

Due to (1.3) and (1.1), it is not difficult to verify, at least formally, that the specific entropy obeys the entropy equation

$$\frac{\partial(\varrho s)}{\partial t} + \operatorname{div}(\varrho s\mathbf{u}) + \operatorname{div}\Big(\frac{\mathbf{q}}{\vartheta}\Big) = \frac{\mathbb{S} : \nabla\mathbf{u}}{\vartheta} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2}. \tag{1.10}$$

On this level, equation (1.10) is fully equivalent with the total energy equality $(1.1)_3$ and can replace it. Another equivalent formulation is the internal energy balance in the form

$$\frac{\partial(\varrho e)}{\partial t} + \operatorname{div}(\varrho e\mathbf{u}) + \operatorname{div}\mathbf{q} + p\operatorname{div}\mathbf{u} = \mathbb{S} : \nabla\mathbf{u}. \tag{1.11}$$

It can be deduced easily from the total energy balance $(1.1)_3$ subtracting the kinetic energy balance, i.e. $(1.1)_2$ multiplied by $\mathbf{u}$. Indeed, at the level of classical solutions such computations are possible; later on, on the level of weak solutions, these formulations may not be equivalent.

It is also easy to verify that the functions p and e are compatible with the existence of entropy if and only if they satisfy the Maxwell relation

$$\frac{\partial e(\varrho,\vartheta)}{\partial\varrho} = \frac{1}{\varrho^2}\Big(p(\varrho,\vartheta) - \vartheta\frac{\partial p(\varrho,\vartheta)}{\partial\vartheta}\Big). \tag{1.12}$$

Note that the choice (1.4) fulfils it. Assuming relation (1.3), if the pressure function $p \in C^1((0,\infty)^2)$, then it has necessarily the form

$$p(\varrho,\vartheta) = \vartheta^{\frac{\gamma}{\gamma-1}} P\Big(\frac{\rho}{\vartheta^{\frac{1}{\gamma-1}}}\Big), \tag{1.13}$$

where $P \in C^1((0,\infty))$.

We shall assume that

$$\begin{gathered} P(\cdot) \in C^1([0,\infty)) \cap C^2((0,\infty)), \\ P(0) = 0, \quad P'(0) = p_0 > 0, \quad P'(Z) > 0, \quad Z > 0, \\ \lim_{Z\to\infty} \frac{P(Z)}{Z^\gamma} = p_\infty > 0, \\ 0 < \frac{1}{\gamma-1} \frac{\gamma P(Z) - ZP'(Z)}{Z} \le c_7 < \infty, \quad Z > 0. \end{gathered} \tag{1.14}$$

For more details about (1.3) and about physical motivation for assumptions (1.14) see e.g. [Feireisl Novotný 2009, Sections 1.4.2 and 3.2].

We shall need several elementary properties of the functions $p(\varrho,\vartheta)$, $e(\varrho,\vartheta)$ and the entropy $s(\varrho,\vartheta)$ satisfying (1.3) together with (1.12). They follow more or less directly from assumptions (1.14) above. We shall only list them referring to [Feireisl Novotný 2009] for more details. Therein, the case $\gamma = \frac{5}{3}$ is considered, however, the computations for general $\gamma > 1$ are exactly the same.

We have for K a fixed constant

$$\begin{array}{rcl} c_1\varrho\vartheta & \le & p(\varrho,\vartheta) \le c_2\varrho\vartheta, \text{ for } \varrho \le K\vartheta^{\frac{1}{\gamma-1}}, \\ c_3\varrho^\gamma & \le & p(\varrho,\vartheta) \le c_4 \begin{cases} \vartheta^{\frac{\gamma}{\gamma-1}}, & \text{for } \varrho \le K\vartheta^{\frac{1}{\gamma-1}}, \\ \varrho^\gamma, & \text{for } \varrho > K\vartheta^{\frac{1}{\gamma-1}}. \end{cases} \end{array} \tag{1.15}$$

Further

$$\begin{gathered} \frac{\partial p(\varrho,\vartheta)}{\partial\varrho} > 0 \qquad \text{in } (0,\infty)^2, \\ p = d\varrho^\gamma + p_m(\varrho,\vartheta), \quad d > 0, \qquad \text{with} \quad \frac{\partial p_m(\varrho,\vartheta)}{\partial\varrho} > 0 \qquad \text{in } (0,\infty)^2. \end{gathered} \tag{1.16}$$

For the specific internal energy defined by (1.3) it follows

$$\left.\begin{aligned} \frac{1}{\gamma-1} p_\infty \varrho^{\gamma-1} \le e(\varrho,\vartheta) \le c_5(\varrho^{\gamma-1} + \vartheta), \\ \frac{\partial e(\varrho,\vartheta)}{\partial\varrho}\varrho \le c_6(\varrho^{\gamma-1} + \vartheta) \end{aligned}\right\} \text{ in } (0,\infty)^2. \tag{1.17}$$

Moreover, for the specific entropy $s(\varrho,\vartheta)$ defined by the Gibbs law we have

$$\begin{gathered} \frac{\partial s(\varrho,\vartheta)}{\partial\varrho} = \frac{1}{\vartheta}\Big(-\frac{p(\varrho,\vartheta)}{\varrho^2} + \frac{\partial e(\varrho,\vartheta)}{\partial\varrho}\Big) = -\frac{1}{\varrho^2}\frac{\partial p(\varrho,\vartheta)}{\partial\vartheta}, \\ \frac{\partial s(\varrho,\vartheta)}{\partial\vartheta} = \frac{1}{\vartheta}\frac{\partial e(\varrho,\vartheta)}{\partial\vartheta} = \frac{1}{\gamma-1}\frac{\vartheta^{\frac{1}{\gamma-1}}}{\varrho}\Big(\gamma P\Big(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\Big) - \frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}P'\Big(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\Big)\Big) > 0. \end{gathered} \tag{1.18}$$

We also have for suitable choice of the additive constant in the definition of the specific entropy

$$
\begin{array}{llll}
|s(\varrho,\vartheta)| & \leq & c_7(1+|\ln\varrho|+|\ln\vartheta|) & \text{in} \quad (0,\infty)^2, \\
|s(\varrho,\vartheta)| & \leq & c_8(1+|\ln\varrho|) & \text{in} \quad (0,\infty)\times(1,\infty), \\
s(\varrho,\vartheta) & \geq & c_9 > 0 & \text{in} \quad (0,1)\times(1,\infty), \\
s(\varrho,\vartheta) & \geq & c_{10}(1+\ln\vartheta) & \text{in} \quad (0,1)\times(0,1).
\end{array}
\tag{1.19}
$$

Since, later on, we deal only with steady or time-periodic solutions to (1.1), let us now recall the most important and interesting results in the evolutionary case. The first global in time results for system $(1.1)_{1-2}$ together with the internal energy balance (1.11) go back to the papers [Matsumura Nishida 1979] or [Matsumura Nishida 1980]. However, these results require smallness of the data. Similar results can be found e.g. in [Valli Zajączkowski 1986], [Salvi Straškraba 1993] or, in a more recent paper [Mucha Zajączkowski 2002]. In this situation it is possible to obtain either classical or strong solutions. Actually, there is no significant difference in the difficulty for the compressible Navier–Stokes or for the compressible Navier–Stokes–Fourier system for such kind of results.

The first global in time existence result without any assumption on the size of the of the data appeared in [Lions 1998], however, only for $\gamma \geq \frac{9}{5}$. The improvement to $\gamma > \frac{3}{2}$ ($\gamma > 1$ if $N = 2$) can be found in [Feireisl et al 2001] and is based on the estimates of the *oscillation defect measure*. Note that in the book [Feireisl et al 2016], the existence proof is based on a numerical method, mixed finite element and finite volume method. All these results consider only the compressible Navier–Stokes equations, i.e. system $(1.1)_{1-2}$.

The first treatment of global in time solutions for large data in the heat conducting case appeared in the book [Feireisl 2004]. This approach was based on the internal energy formulation, however, the equality was replaced by the inequality together with the total energy balance (inequality) "in global", i.e. integrated only over Ω (the test function identically equal to 1). Another approach, based on the entropy inequality, appeared for the first time in [Feireisl Novotný 2005]. More detailed existence proof can be found in [Feireisl Novotný 2009]. Finally, there is one more possible formulation, based on the relative entropy inequality (see [Feireisl et al 2012a], [Feireisl Novotný 2012]); the proof of existence of such solutions can be found in [Feireisl Novotný 2005].

In [Plotnikov Weigant 2015b], the existence proof was in two space dimensions extended to the border case $\gamma = 1$; in three space dimensions, the border case $\gamma = \frac{3}{2}$ remains open, however, the compactness of the convective term for a suitable approximation was proved in the overview paper [Plotnikov Weigant 2018].

Finally, let us mention the case of density dependent viscosities. The first result, in two space dimensions, appeared in [Vaigant Kazhikhov 1995]. In three space dimensions, it was observed in [Bresch et al 2007] that if the viscosities fulfill a certain relation (from physics, however, not clearly supported), then it is possible to deduce improved density estimates. In combination with the result from [Mellet Vasseur 2007] it was recently proved that it is possible to construct a suitable approximation which satisfies at the same time the Bresch–Desjardins and the Mellet–Vasseur estimates, allowing to prove existence of solution in a very specific situation (see the independent papers [Vasseur Yu 2016] and [Li Xin 2016]).

1.2 Multicomponent flow

In this part, we follow the approach from monograph [Giovangigli 1999]. We describe the whole mixture using just one velocity field (barycentric), one stress tensor and one temperature and we describe the separate constituent using the partial densities ϱ_k or rather the mass fractions $Y_k = \frac{\varrho_k}{\varrho}$. Hence $\sum_{k=1}^L Y_k = 1$, where L is the number of constituents. We study the following system of equations

$$\begin{aligned} \frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\ \frac{\partial (\varrho \mathbf{u})}{\partial t} + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p - \operatorname{div} \mathbb{S} &= \varrho \mathbf{f}, \\ \frac{\partial (\varrho E)}{\partial t} + \operatorname{div}(\varrho E \mathbf{u}) + \operatorname{div} \mathbf{Q} - \operatorname{div}(\mathbb{S}\mathbf{u}) + \operatorname{div}(p\mathbf{u}) &= \varrho \mathbf{f} \cdot \mathbf{u}, \\ \frac{\partial (\varrho Y_k)}{\partial t} + \operatorname{div}(\varrho Y_k \mathbf{u}) + \operatorname{div} \mathbf{F}_k &= m_k \omega_k, \quad k = 1, 2, \ldots, L. \end{aligned} \tag{1.20}$$

Most of the quantities above were explained and defined in the previous section, we briefly explain the meaning of the others and then specify more precisely their form. Above, $\mathbf{Q} = \mathbf{q} + \sum_{k=1}^L h_k \mathbf{F}_k$ is the heat flux, where $\mathbf{q}$ has the same form as for the single component flow, $\{\mathbf{F}_k\}_{k=1}^L$ are the multicomponent fluxes and will be specified below, and h_k are the partial enthalpies. Further, $\{m_k\}_{k=1}^L$ denote the molar masses and due to mathematical reasons (for the steady problem, we have significant troubles to consider them different for each constituent) they are assumed to be equal; hence without loss of generality, $m_k = 1$, $k = 1, 2, \ldots, L$. The terms ω_k describe the source terms for the k-th constituent due to chemical reactions. The compatibility condition $\sum_{k=1}^{L} Y_k = 1$ dictates $\sum_{k=1}^{L} \mathbf{F}_k = \mathbf{0}$ and $\sum_{k=1}^{L} \omega_k = 0$, i.e. the sum of $(1.20)_4$ yields $(1.20)_1$.

The system is completed by the boundary conditions on $\partial\Omega$ (for simplicity, we assume the Dirichlet boundary conditions for the velocity); below $\mathbf{n}$ denotes the exterior normal to $\partial\Omega$

$$\begin{aligned} \mathbf{u} &= \mathbf{0}, \\ \mathbf{F}_k \cdot \mathbf{n} &= 0, \\ -\mathbf{Q} \cdot \mathbf{n} + L(\vartheta - \Theta_0) &= 0, \end{aligned} \tag{1.21}$$

and the initial conditions

$$\begin{aligned} &\mathbf{u}(0,x) = \mathbf{u}_0, \quad (\varrho\mathbf{u})(0,x) = \mathbf{m}_0(x), \\ &\vartheta(0,x) = \vartheta_0(x), \quad Y_k(0,x) = Y_k^0(x), \ k = 1, 2, \ldots, L. \end{aligned}$$

The temperature ϑ enters the game in the same way as in the single component flow: we choose the density, the mass fractions and the temperature as the basic thermodynamic quantities and assume all other thermodynamic functions to be given functions of these quantities.

We consider the pressure law

$$p(\varrho, \vartheta) = p_c(\varrho) + p_m(\varrho, \vartheta), \tag{1.22}$$

with p_m obeying the Boyle law (here the fact that the molar masses are the same plays an important role)

$$p_m(\varrho, \vartheta) = \sum_{k=1}^{L} \varrho Y_k \vartheta = \varrho\vartheta, \tag{1.23}$$

and the so-called “cold” pressure

$$p_c(\varrho) = \varrho^\gamma, \quad \gamma > 1. \tag{1.24}$$

The corresponding form of the specific total energy is

$$E(\varrho, \mathbf{u}, \vartheta, Y_1, \ldots, Y_L) = \frac{1}{2}|\mathbf{u}|^2 + e(\varrho, \vartheta, Y_1, \ldots, Y_L), \tag{1.25}$$

where the specific internal energy takes the form

$$e(\varrho, \vartheta, Y_1, \ldots, Y_L) = e_c(\varrho) + e_m(\vartheta, Y_1, \ldots, Y_L) \tag{1.26}$$

with

$$e_c(\varrho) = \frac{1}{\gamma - 1}\varrho^{\gamma-1}, \qquad e_m(\vartheta, Y_1, \dots, Y_L) = \sum_{k=1}^{L} Y_k e_k = \vartheta \sum_{k=1}^{L} c_{vk} Y_k. \quad (1.27)$$

Above, $\{c_{vk}\}_{k=1}^L$ are the constant-volume specific heat coefficients. The constant-pressure specific heat coefficients, denoted by $\{c_{pk}\}_{k=1}^L$, are related (under the assumption on the equality of molar masses) to $\{c_{vk}\}_{k=1}^L$ in the following way

$$c_{pk} = c_{vk} + 1, \quad k = 1, 2, \dots, L, \quad (1.28)$$

and both c_{vk} and c_{pk} are assumed to be constant (but possibly different for each constituent).

The specific entropy

$$s = \sum_{k=1}^{L} Y_k s_k \quad (1.29)$$

with s_k the specific entropy of the k-th constituent. The Gibbs formula for the multicomponent flow has the form

$$\vartheta \mathbf{D} s = \mathbf{D} e + \pi \mathbf{D}\left(\frac{1}{\varrho}\right) - \sum_{k=1}^{n} g_k \mathbf{D} Y_k, \quad (1.30)$$

with the Gibbs functions

$$g_k = h_k - \vartheta s_k, \quad (1.31)$$

where $s_k = s_k(\varrho, \vartheta, Y_k)$, and $h_k = h_k(\vartheta)$ denotes the specific enthalpy of the k-th species with the following exact forms connected with our choice of the pressure law (1.23)–(1.25)

$$h_k(\vartheta) = c_{pk}\vartheta, \quad s_k(\varrho, \vartheta, Y_k) = c_{vk} \log \vartheta - \log \varrho - \log Y_k. \quad (1.32)$$

The cold pressure and the cold energy correspond to isentropic processes, therefore using (1.29) it is not difficult to derive an equation for the specific entropy s

$$\operatorname{div}(\varrho s \mathbf{u}) + \operatorname{div}\left(\frac{\mathbf{Q}}{\vartheta} - \sum_{k=1}^{n} \frac{g_k}{\vartheta}\mathbf{F}_k\right) = \sigma, \quad (1.33)$$

where σ is the entropy production rate

$$\sigma = \frac{\mathbb{S} : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{Q} \cdot \nabla \vartheta}{\vartheta^2} - \sum_{k=1}^{L} \mathbf{F}_k \cdot \nabla\left(\frac{g_k}{\vartheta}\right) - \frac{\sum_{k=1}^{L} g_k \omega_k}{\vartheta}. \quad (1.34)$$

The viscous stress tensor is assumed to have the same form as above, i.e.

$$\mathbb{S} = \mathbb{S}(\mathbb{D}(\mathbf{u}), \vartheta) = \mu(\vartheta)\left[\nabla\mathbf{u} + \nabla^T\mathbf{u} - \frac{2}{3}\operatorname{div}\mathbf{u}\,\mathbb{I}\right] + \xi(\vartheta)\operatorname{div}\mathbf{u}\,\mathbb{I}, \tag{1.35}$$

with the viscosities $\mu(\cdot)$ globally Lipschitz continuous and $\xi(\cdot)$ continuous on $\mathbb{R}^+$,

$$\mu(\vartheta) \sim (1+\vartheta), \qquad 0 \le \xi(\vartheta) \le (1+\vartheta).$$

The Fourier part of the heat flux has the form

$$\mathbf{q} = -\kappa(\vartheta)\nabla\vartheta, \tag{1.36}$$

where $\kappa = \kappa(\vartheta) \sim (1+\vartheta^m)$, continuous on $\mathbb{R}^+$, is the thermal conductivity coefficient.

For the diffusion flux, we assume

$$\mathbf{F}_k = -Y_k \sum_{l=1}^{L} D_{kl}\nabla Y_l, \tag{1.37}$$

where $D_{kl} = D_{kl}(\vartheta, Y_1, \dots, Y_L)$, $k, l = 1, \dots, L$ are the multicomponent diffusion coefficients. We aim at working with generally non-diagonal matrix $\mathbb{D}$ which leads to mathematical difficulties, therefore sometimes relation (1.37) is replaced by the Fick law

$$\mathbf{F}_k = -D_k\nabla Y_k, \quad k = 1, 2, \dots, L.$$

We consider

$$\begin{aligned} &\mathbb{D} = \mathbb{D}^T, \quad N(\mathbb{D}) = \mathbb{R}\vec{Y}, \quad R(\mathbb{D}) = \vec{Y}^\perp, \\ &\mathbb{D} \quad \text{is positive semidefinite over } \mathbb{R}^L, \end{aligned} \tag{1.38}$$

where we assumed that $\vec{Y} = (Y_1, \dots, Y_L)^T > 0$ and $N(\mathbb{D})$ denotes the nullspace of matrix $\mathbb{D}$, $R(\mathbb{D})$ its range, $\vec{U} = (1, \dots, 1)^T$ and $\vec{U}^\perp$ denotes the orthogonal complement of $\mathbb{R}\vec{U}$. Furthermore, we assume that the matrix $\mathbb{D}$ is homogeneous of a non-negative order with respect to $Y_1, \dots, Y_L$ and that D_{ij} are differentiable functions of $\vartheta, Y_1, \dots, Y_L$ for any $i, j \in \{1, \dots, L\}$ such that

$$|D_{ij}(\vartheta, \vec{Y})| \le C(\vec{Y})(1+\vartheta^b)$$

for some $b \ge 0$.

The species production rates

$$\omega_k = \omega_k(\varrho, \vartheta, Y_1, \dots, Y_L)$$

are smooth bounded functions of their variables such that

$$\omega_k(\varrho, \vartheta, Y_1, \ldots, Y_L) \geq 0 \quad \text{whenever} \quad Y_k = 0. \tag{1.39}$$

We assume even a stronger restriction, namely that $\omega_k \geq -CY_k^r$ for some positive C, r. The source term is sometimes modeled as function of ϱ_k instead of ϱ, hence the term $\omega_k(\vartheta, Y_1, \ldots, Y_L)$ is replaced by $\varrho\omega_k(\vartheta, Y_1, \ldots, Y_L)$. Next, in accordance with the second law of thermodynamics we assume that

$$-\sum_{k=1}^{L} g_k \omega_k \geq 0, \tag{1.40}$$

where g_k are specified in (1.31). Note that thanks to this inequality and properties of D_{kl}, together with (1.35) and (1.36), the entropy production rate defined in (1.34) is non-negative. Similarly as for the single component flow, we may replace $(1.20)_3$ by the internal energy balance (since we do not use such formulation here, we do not write it explicitly) or with the entropy equation (1.33)–(1.34) (which we shall use later).

In what follows, we restrict ourselves again to the steady case. Therefore we recall now the main results for the evolutionary system. The first global in time solution (for small data only) can be found in the book [Giovangigli 1999]. The first large data global in time solution appeared in [Feireisl et al 2008]; the diffusion matrix was diagonal, i.e. the Fick law was assumed. The non-diagonal diffusion matrix however, with a special form) was considered in [Mucha et al 2015]. The paper is based on the total energy formulation. Due to technical reasons, the used fluid model was the compressible Navier–Stokes–Fouries system with density dependent viscosities fulfilling the Bresch–Desjardins relation and with singular cold pressure. The weak compactness of solutions with entropy inequality formulation was studied in [Zatorska 2015], in the isothermal case in [Zatorska 2012b]. See also [Xi Xie 2016], where the authors achieved similar results under less restrictive assumptions, however, for two species only. In [Zatorska Mucha 2015] the authors studied the evolutionary problem using time discretization. More general situation, with however slightly different fluid model, was considered in [Dreyer et al 2016] and [Druet 2016].

Chapter 2

Mathematical theory for steady single component flow

In this chapter, we restrict ourselves to the steady solutions of (1.1). We therefore consider

$$
\begin{aligned}
\operatorname{div}(\varrho \mathbf{u}) &= 0, \\
\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbb{T} &= \varrho \mathbf{f}, \\
\operatorname{div}(\varrho E \mathbf{u}) + \operatorname{div} \mathbf{q} - \operatorname{div}(\mathbb{T}\mathbf{u}) &= \varrho \mathbf{f} \cdot \mathbf{u},
\end{aligned}
\tag{2.1}
$$

together with the Newton (or Robin) type boundary conditions for the heat flux

$$
-\mathbf{q} \cdot \mathbf{n} + L(\vartheta)(\vartheta - \Theta_0) = 0 \tag{2.2}
$$

and either the homogeneous Dirichlet boundary conditions

$$
\mathbf{u} = \mathbf{0} \tag{2.3}
$$

or the (partial) slip boundary conditions (sometimes also called the Navier boundary conditions)

$$
\mathbf{u} \cdot \mathbf{n} = 0, \qquad (\mathbb{S}\mathbf{n}) \times \mathbf{n} + \alpha \mathbf{u} \times \mathbf{n} = \mathbf{0} \tag{2.4}
$$

on $\partial\Omega$. Indeed, on the level of smooth solutions, we may replace $(2.1)_3$ by either the internal energy balance

$$
\operatorname{div}(\varrho e \mathbf{u}) + \operatorname{div} \mathbf{q} = \mathbb{T} : \nabla \mathbf{u} \tag{2.5}
$$

or by the entropy equation

$$
\operatorname{div}(\varrho s \mathbf{u}) + \operatorname{div}\Big(\frac{\mathbf{q}}{\vartheta}\Big) = \frac{\mathbb{S} : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2}. \tag{2.6}
$$

Moreover, we have to prescribe the total mass of the fluid

$$\int_\Omega \varrho \,\mathrm{d}x = M > 0. \tag{2.7}$$

Other assumptions are the same as in Section 1.1 (either (1.4) or (1.3) with (1.12)–(1.19), and (1.2) with (1.5)).

2.1 Definitions of solutions for different formulations

The case of small data (i.e. strong or classical solutions) was for the first time considered in papers [Padula 1981], [Padula 1982] or [Valli 1983] in the L^2-setting and in [Beirão da Veiga 1987] in the L^p-setting. Then, a series of papers studying different aspects of the solutions (not only their existence, but also the decay of solutions near infinity which is expected to be different in two and three space dimensions) appeared. Since we do not deal here with this type of problems, we only refer to the overview paper [Kreml et al 2018] and to the references therein.

Our aim is to prove existence of solutions without any restriction on the size of the data and keep the regularity assumptions on the data as general as possible. This leads us naturally to the notion of weak solution (or, as explained below, variational entropy solution). Before dealing with the formulations allowing very low exponent γ, we introduce a definition based on the internal energy balance, where we can obtain relatively regular solutions for a certain range of γ. We consider the Navier boundary conditions (2.2) for the velocity, assume the viscosities to be constant (i.e., we take $a = 0$ below (1.2)) and use the pressure law (1.4).

In what follows, we use standard notation for the functions spaces (Lebesgue, Sobolev or spaces of continuous or continuously differentiable functions). We denote

$$W^{1,p}_{\mathbf{n}}(\Omega;\mathbb{R}^3) = \{\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^3); \mathbf{u}\cdot\mathbf{n} = 0 \text{ in the sense of traces}\}.$$

Similarly the space $C^1_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^3)$ contains all differentiable functions in $\overline{\Omega}$ with zero normal trace at $\partial\Omega$. Then we have

Definition 1 (Weak solution for internal energy formulation.) *The triple $(\varrho, \mathbf{u}, \vartheta)$ is called a weak solution to system (2.1)$_{1-2}$, (2.2), (2.4), (2.5) and (2.7) if $\varrho \in L^{\frac{6\gamma}{5}}(\Omega)$, $\mathbf{u} \in W^{1,2}_{\mathbf{n}}(\Omega;\mathbb{R}^3)$, $\vartheta \in W^{1,r}(\Omega)\cap L^{3m}(\Omega)\cap L^{l+1}(\partial\Omega)$,*

$r > 1$ with $\varrho|\mathbf{u}|^2 \in L^{\frac{6}{5}}(\Omega)$, $\varrho\mathbf{u}\vartheta \in L^1(\Omega;\mathbb{R}^3)$, $\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta) : \mathbb{D}(\mathbf{u}) \in L^1(\Omega)$, $\vartheta^m\nabla\vartheta \in L^1(\Omega;\mathbb{R}^3)$. Moreover, the continuity equation is satisfied in the weak sense

$$\int_\Omega \varrho\mathbf{u}\cdot\nabla\psi \,\mathrm{d}x = 0 \qquad \forall\psi \in C^1(\overline{\Omega}), \tag{2.8}$$

the momentum equation holds in the weak sense

$$\begin{aligned}\int_\Omega \big(-\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} - p(\varrho,\vartheta)\operatorname{div}\boldsymbol{\varphi} + \mathbb{S}(\mathbb{D}(\mathbf{u})) : \nabla\boldsymbol{\varphi}\big) \,\mathrm{d}x \\ +\alpha\int_{\partial\Omega} \mathbf{u}\cdot\boldsymbol{\varphi} \,\mathrm{d}S = \int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi} \,\mathrm{d}x \quad \forall\boldsymbol{\varphi} \in C^1_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^3),\end{aligned} \tag{2.9}$$

and the internal energy balance holds in the weak sense

$$\begin{aligned}\int_\Omega \Big(\kappa(\vartheta)\nabla\vartheta - \varrho\vartheta\mathbf{u}\Big)\cdot\nabla\psi \,\mathrm{d}x + \int_{\partial\Omega} L(\vartheta)(\vartheta-\Theta_0)\psi \,\mathrm{d}S \\ = \int_\Omega \Big(\mathbb{S}(\mathbb{D}(\mathbf{u})) : \nabla\mathbf{u} + \varrho\vartheta\operatorname{div}\mathbf{u}\Big)\psi \,\mathrm{d}x \quad \forall\psi \in C^1(\overline{\Omega}).\end{aligned} \tag{2.10}$$

Note that we used the fact that in the weak formulation of the internal energy balance, the cold pressure terms are cancelled with the cold energy terms. This is, at least formally, true always, but it requires certain integrability of the density. Since we deal with this definition only with $\gamma > 3$ later on, these terms cancel even for weak solutions. Note that the existence of weak solutions which satisfy the internal energy balance can be obtained only for the Navier boundary conditions.

Next we consider either the total energy balance formulation (which leads to the weak formulation). The definitions for the Dirichlet and Navier boundary conditions slightly differ, therefore we present both. Note that we consider (2.1)–(2.3) (the Dirichlet boundary conditions) or (2.1)–(2.2) and (2.4) (the slip boundary conditions). In both cases, we consider either (1.4) or (1.3) with (1.12)–(1.19) and as above, we must prescribe the total mass (2.7).

Definition 2 (Total energy formulation for Dirichlet b.c.) *The triple $(\varrho,\mathbf{u},\vartheta)$ is called a weak solution to system (2.1)–(2.3) and (2.7), if $\varrho \in L^{\frac{6\gamma}{5}}(\Omega)$, $\int_\Omega \varrho \,\mathrm{d}x = M$, $\mathbf{u} \in W^{1,2}_0(\Omega;\mathbb{R}^3)$, $\vartheta \in W^{1,r}(\Omega)\cap L^{3m}(\Omega)\cap L^{l+1}(\partial\Omega)$, $r > 1$ with $\varrho|\mathbf{u}|^2 \in L^{\frac{6}{5}}(\Omega)$, $\varrho\mathbf{u}\vartheta \in L^1(\Omega;\mathbb{R}^3)$, $\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta)\mathbf{u} \in L^1(\Omega;\mathbb{R}^3)$, $\vartheta^m\nabla\vartheta \in L^1(\Omega;\mathbb{R}^3)$, and*

$$\int_\Omega \varrho\mathbf{u}\cdot\nabla\psi \,\mathrm{d}x = 0 \qquad \forall\psi \in C^1(\overline{\Omega}), \tag{2.11}$$

$$\begin{aligned}\int_\Omega \big(-\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} - p(\varrho,\vartheta)\operatorname{div}\boldsymbol{\varphi} + \mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta) : \nabla\boldsymbol{\varphi}\big) \,\mathrm{d}x \\ = \int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi} \,\mathrm{d}x \quad \forall\boldsymbol{\varphi}\in C_0^1(\Omega;\mathbb{R}^3),\end{aligned} \tag{2.12}$$

$$\begin{aligned}\int_\Omega -\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\Big)\mathbf{u}\cdot\nabla\psi \,\mathrm{d}x = \int_\Omega \big(\varrho\mathbf{f}\cdot\mathbf{u}\psi + p(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\big) \,\mathrm{d}x \\ - \int_\Omega \big(\big(\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta)\mathbf{u}\big)\cdot\nabla\psi + \kappa(\cdot,\vartheta)\nabla\vartheta\cdot\nabla\psi\big) \,\mathrm{d}x \\ - \int_{\partial\Omega} L(\vartheta)(\vartheta-\Theta_0)\psi \,\mathrm{d}S \qquad \forall\psi\in C^1(\overline{\Omega}).\end{aligned} \tag{2.13}$$

Definition 3 (Total energy formulation for Navier b.c.) *The triple $(\varrho,\mathbf{u},\vartheta)$ is called a weak solution to system (2.1)–(2.2), (2.4) and (2.7), if $\varrho\in L^{\frac{6\gamma}{5}}(\Omega)$, $\int_\Omega \varrho \,\mathrm{d}x = M$, $\mathbf{u}\in W^{1,2}_{\mathbf{n}}(\Omega;\mathbb{R}^3)$, $\vartheta\in W^{1,r}(\Omega)\cap L^{3m}(\Omega)\cap L^{l+1}(\partial\Omega)$, $r>1$ with $\varrho|\mathbf{u}|^2\in L^{\frac{6}{5}}(\Omega)$, $\varrho\mathbf{u}\vartheta\in L^1(\Omega;\mathbb{R}^3)$, $\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta)\mathbf{u}\in L^1(\Omega;\mathbb{R}^3)$, $\vartheta^m\nabla\vartheta\in L^1(\Omega;\mathbb{R}^3)$. Moreover, the continuity equation is satisfied in the sense as in (2.8), and*

$$\begin{aligned}\int_\Omega \big(-\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} - p(\varrho,\vartheta)\operatorname{div}\boldsymbol{\varphi} + \mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta) : \nabla\boldsymbol{\varphi}\big) \,\mathrm{d}x \\ +\alpha\int_{\partial\Omega}\mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}S = \int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi} \,\mathrm{d}x \quad \forall\boldsymbol{\varphi}\in C_{\mathbf{n}}^1(\overline{\Omega};\mathbb{R}^3),\end{aligned} \tag{2.14}$$

$$\begin{aligned}\int_\Omega -\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\Big)\mathbf{u}\cdot\nabla\psi \,\mathrm{d}x = \int_\Omega \big(\varrho\mathbf{f}\cdot\mathbf{u}\psi + p(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\big) \,\mathrm{d}x \\ - \int_\Omega \big(\big(\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta)\mathbf{u}\big)\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta\cdot\nabla\psi\big) \,\mathrm{d}x \\ - \int_{\partial\Omega} L(\vartheta)(\vartheta-\Theta_0)\psi \,\mathrm{d}S - \alpha\int_{\partial\Omega}|\mathbf{u}|^2\psi\,\mathrm{d}S \ \ \forall\psi\in C^1(\overline{\Omega}).\end{aligned} \tag{2.15}$$

Another definition concerns the formulation with the entropy equation. The main problem is that due to mathematical reasons it is difficult to expect that it is possible to obtain equality in the entropy formulation. However, it is enough to prove inequality and in order to keep the weak–strong compatibility (sufficiently smooth solution of this formulation is in fact classical solution to the original formulation), it is necessary to extract at least a part of the information from the total energy balance. Again, formulations for both boundary conditions may include either (1.4) or (1.3) with (1.12)–(1.19).

Definition 4 (Variational entropy solution for Dirichlet b.c.) *The triple $(\varrho, \mathbf{u}, \vartheta)$ is called a variational entropy solution to system (2.1)–(2.3) and (2.7), if $\varrho \in L^\gamma(\Omega)$, $\int_\Omega \varrho \, \mathrm{d}x = M$, $\mathbf{u} \in W_0^{1,2}(\Omega; \mathbb{R}^3)$, $\vartheta \in W^{1,r}(\Omega) \cap L^{3m}(\Omega) \cap L^{l+1}(\partial\Omega)$, $r > 1$, with $\varrho\mathbf{u} \in L^{\frac{6}{5}}(\Omega; \mathbb{R}^3)$, $\varrho\vartheta \in L^1(\Omega)$, and $\vartheta^{-1}\mathbb{S}(\mathbb{D}(\mathbf{u}), \vartheta)\mathbf{u} \in L^1(\Omega; \mathbb{R}^3)$, $L(\vartheta), \frac{L(\vartheta)}{\vartheta} \in L^1(\partial\Omega)$, $\kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2} \in L^1(\Omega)$ and $\kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta} \in L^1(\Omega; \mathbb{R}^3)$. Moreover, equalities (2.11) and (2.12) are satisfied in the same sense as in Definition 2, and we have the entropy inequality*

$$\begin{aligned}
&\int_\Omega \Big(\frac{\mathbb{S}(\mathbb{D}(\mathbf{u}), \vartheta) : \nabla\mathbf{u}}{\vartheta} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\Big)\psi \, \mathrm{d}x + \int_{\partial\Omega} \frac{L(\vartheta)}{\vartheta}\Theta_0\psi \, \mathrm{d}S \\
&\le \int_{\partial\Omega} L(\vartheta)\psi \, \mathrm{d}S + \int_\Omega \Big(\kappa(\vartheta)\frac{\nabla\vartheta \cdot \nabla\psi}{\vartheta} - \varrho s(\varrho, \vartheta)\mathbf{u} \cdot \nabla\psi\Big) \, \mathrm{d}x
\end{aligned} \tag{2.16}$$

for all non-negative $\psi \in C^1(\overline{\Omega})$, together with the global total energy balance

$$\int_{\partial\Omega} L(\vartheta)(\vartheta - \Theta_0) \, \mathrm{d}S = \int_\Omega \varrho\mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x. \tag{2.17}$$

Similarly as above we have

Definition 5 (Variational entropy solution for Navier b.c.) *The triple $(\varrho, \mathbf{u}, \vartheta)$ is called a variational entropy solution to system (2.1)–(2.2), (2.4) and (2.7), if $\varrho \in L^\gamma(\Omega)$, $\int_\Omega \varrho \, \mathrm{d}x = M$, $\mathbf{u} \in W_{\mathbf{n}}^{1,2}(\Omega; \mathbb{R}^3)$, $\vartheta \in W^{1,r}(\Omega) \cap L^{3m}(\Omega) \cap L^{l+1}(\partial\Omega)$, $r > 1$, with $\varrho\mathbf{u} \in L^{\frac{6}{5}}(\Omega; \mathbb{R}^3)$, $\varrho\vartheta \in L^1(\Omega)$, $\vartheta^{-1}\mathbb{S}(\mathbb{D}(\mathbf{u}), \vartheta)\mathbf{u} \in L^1(\Omega; \mathbb{R}^3)$, $L(\vartheta), \frac{L(\vartheta)}{\vartheta} \in L^1(\partial\Omega)$, $\kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2} \in L^1(\Omega)$ and $\kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta} \in L^1(\Omega; \mathbb{R}^3)$. Moreover, equalities (2.11) and (2.14) are satisfied in the same sense as in Definition 3, we have the entropy inequality (2.16) in the same sense as in Definition 4, together with the global total energy balance*

$$\alpha \int_{\partial\Omega} |\mathbf{u}|^2 \, \mathrm{d}S + \int_{\partial\Omega} L(\vartheta)(\vartheta - \Theta_0) \, \mathrm{d}S = \int_\Omega \varrho\mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x. \tag{2.18}$$

We will also need the notion of the renormalized solution to the continuity equation

Definition 6 (Renormalized solution to continuity equation.) *Let $\mathbf{u} \in W_{loc}^{1,2}(\mathbb{R}^3; \mathbb{R}^3)$ and $\varrho \in L_{loc}^{\frac{6}{5}}(\mathbb{R}^3)$ solve*

$$\operatorname{div}(\varrho\mathbf{u}) = 0 \ \textit{in} \ \mathcal{D}'(\mathbb{R}^3).$$

Then the pair $(\varrho, \mathbf{u})$ *is called a renormalized solution to the continuity equation, if*

$$\operatorname{div}(b(\varrho)\mathbf{u}) + \big(\varrho b'(\varrho) - b(\varrho)\big) \operatorname{div} \mathbf{u} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3) \tag{2.19}$$

for all $b \in C^1([0,\infty)) \cap W^{1,\infty}((0,\infty))$ *with* $zb'(z) \in L^\infty((0,\infty))$.

Before going into details concerning the existence proofs in different situations for the heat conducting fluid, let us recall results dealing with steady compressible Navier–Stokes equations. The first existence proof appeared in [Lions 1998]. The method based on the Bogovskii-type estimates and the Friedrich lemma allowed to deal with $\gamma \geq \frac{5}{3}$. Later improvements of the a priori estimates of the density, combined with Feireisl's ideas from the evolutionary situation, based on ideas from [Plotnikov Sokolowski 2005], improved in [Březina Novotný 2008] allowed finally to get existence of solutions in three space dimensions for $\gamma > \frac{4}{3}$ (see [Frehse et al 2009]) and in two space dimensions for $\gamma = 1$ (see [Frehse et al 2010]). Later on, in [Jiang Zhou 2011], at least for the space periodic boundary conditions, the authors established existence in three space dimensions for any $\gamma > 1$. The existence of solutions for any $\gamma > 1$ was finally achieved also for the Navier boundary conditions (see [Jesslé Novotný 2013]) and for the Dirichlet boundary conditions (see [Plotnikov Weigant 2015a]), where in the latter a different method, based on the Radon transform estimates was used. Let us also mention the paper [Łasica 2014], where the author obtained existence of a solution for a pressure law singular at zero density which has density bounded strictly away from zero. Finally, note that the papers dealing with potential pressure estimates up to the boundary contained a small gap which was removed in [Mucha et al 2018].

Note that we assumed above that there is no flow through the boundary, i.e. $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Indeed, this condition is quite restrictive as it excludes, e.g., the flow through a channel and other important applications. It is well known that such a problem is not easy even in the case when the flow is incompressible (i.e., the density is constant) due difficulties to control the convective term. Indeed, if the density is unbounded, the problem is for steady compressible Navier–Stokes equations totally open. Therefore only small data results (for smooth solutions) are know in this case, see e.g. [Piasecki 2010], [Piasecki Pokorný 2014] or [Zhou 2018]. On the other hand, the existence of weak solutions for large data was recently established in [Feireisl Novotný 2018] for the hard sphere pressure. It means that the pressure is assumed to be unbounded provided the density approaches a certain positive value ϱ_0. This implies that the density is bounded by ϱ_0

and it is possible to control the convective term. Indeed, the whole proof is technically complicated. We will not deal here with results of this type.

2.2 Existence of a solution for internal energy formulation

We first describe the result from Chapter 5. It deals with the internal energy formulation and with the situation when it is possible to obtain solutions with bounded density and almost Lipschitz continuous velocity and temperature. The result comes from [Mucha Pokorný 2009].

Before 2009, except for small data results, the only result dealing with steady compressible Navier–Stokes–Fourier system appeared in [Lions 1998]; however, P.L. Lions treated the case when $p(\varrho, \vartheta) \sim \varrho\vartheta$ and to overcome the lack of estimates for the density he assumed a priori that the density is bounded in $L^q(\Omega)$ for sufficiently large q. Such a result is indeed not satisfactory.

Therefore, the first aim was to obtain a priori estimates (for pressure with the cold pressure part) assuming a priori only the L^1-bound corresponding to the given total mass. Some results for the steady compressible Navier–Stokes equations were available from [Lions 1998] (for $\gamma \geq \frac{5}{3}$), but they were not enough to deal with the heat equation.

The first approach was based on the previous results of both authors, see [Mucha Pokorný 2006] and [Pokorný Mucha 2008]. The novelty of these papers consists in the special approximation scheme for the compressible flow which allowed to construct approximate solutions with bounded density where it was possible to show that if the parameters of the approximation are suitably chosen, the L^∞ bound of the density is actually independent of the parameters and hence it is possible to construct solutions to the compressible Navier–Stokes equations (in two space dimensions for $\gamma > 1$ and in three space dimensions for $\gamma > 3$) which have the density bounded. Note that for large data in the context of weak solution, due to a counterexample of P.L. Lions (see [Lions 1998]) such a regularity is the best one can expect if it is not possible to exclude the existence of vacuum regions. For more ideas in the case of isentropic flow, see [Novotný 1996] and [Lions 1998]. See also [Łasica 2014] where the author constructs smooth solution under the assumption that the pressure becomes singular for small densities.

For the Navier–Stokes–Fourier system, the result reads as follows

Theorem 1 (Internal energy formulation.) [Mucha Pokorný 2009] *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^3$ which is not axially symmetric if $\alpha = 0$. Let the viscosities be constant. Let $\mathbf{f} \in L^\infty(\Omega;\mathbb{R}^3)$ and*

$$\gamma > 3, \qquad m = l + 1 > \frac{3\gamma - 1}{3\gamma - 7}.$$

Then there exists a weak solution to our problem $(2.1)_{1-2}$, (2.2), (2.4) and (2.5) in the sense of Definition 1 such that

$$\varrho \in L^\infty(\Omega), \qquad \mathbf{u} \in W^{1,q}(\Omega;\mathbb{R}^3), \quad \vartheta \in W^{1,q}(\Omega) \ \text{ for all } 1 \le q < \infty,$$

and $\varrho \ge 0$, $\vartheta > 0$ a.e. in Ω.

A similar result in two space dimensions can be found in the paper [Pecharová Pokorný 2010], for $\gamma > 2$ and $m = l + 1 > \frac{\gamma-1}{\gamma-2}$. Let us briefly explain the main ideas of the proof. For $k \gg 1$ we define

$$K(t) = \begin{cases} 1 & \text{for} \quad t < k-1 \\ \in [0,1] & \text{for} \quad k-1 \le t \le k \\ 0 & \text{for} \quad t > k; \end{cases} \tag{2.20}$$

moreover, we assume that $K'(t) < 0$ for $t \in (k-1,k)$. Take $\varepsilon > 0$ and $K(\cdot)$ as above. The approximate problem reads

$$\begin{aligned} \varepsilon\varrho + \operatorname{div}(K(\varrho)\varrho\mathbf{u}) - \varepsilon\Delta\varrho &= \varepsilon h K(\varrho) \\ \frac{1}{2}\operatorname{div}(K(\varrho)\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{2}K(\varrho)\varrho\mathbf{u}\cdot\nabla\mathbf{u} - \operatorname{div}\mathbb{S}(\mathbb{D}(\mathbf{u})) + \nabla P(\varrho,\vartheta) &= \varrho K(\varrho)\mathbf{f} \\ -\operatorname{div}\Big(\kappa(\vartheta)\frac{\varepsilon+\vartheta}{\vartheta}\nabla\vartheta\Big) + \operatorname{div}\Big(\mathbf{u}\int_0^\rho K(t)\mathrm{d}t\Big)\vartheta + \operatorname{div}\Big(K(\varrho)\varrho\mathbf{u}\Big)\vartheta & \\ +K(\varrho)\varrho\mathbf{u}\cdot\nabla\vartheta - \vartheta K(\varrho)\mathbf{u}\cdot\nabla\varrho &= \mathbb{S}((\mathbb{D}(\mathbf{u})) : \nabla\mathbf{u} \end{aligned} \tag{2.21}$$

in Ω, where

$$P(\varrho,\vartheta) = \int_0^\varrho \gamma t^{\gamma-1}K(t)\mathrm{d}t + \vartheta\int_0^\rho K(t)\mathrm{d}t = P_b(\varrho) + \vartheta\int_0^\varrho K(t)\mathrm{d}t,$$

and $h = \frac{M}{|\Omega|}$.

We also modify the boundary conditions on $\partial\Omega$

$$(1+\vartheta^m)(\varepsilon+\vartheta)\frac{1}{\vartheta}\frac{\partial\vartheta}{\partial\mathbf{n}} + L(\vartheta)(\vartheta - \Theta_0) + \varepsilon\ln\vartheta = 0,$$

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad \boldsymbol{\tau}_k\cdot(\mathbb{S}(\mathbb{D}(\mathbf{u}))\mathbf{n}) + \alpha\mathbf{u}\cdot\boldsymbol{\tau}_k = 0, \qquad k = 1,2,$$

$$\frac{\partial\varrho}{\partial\mathbf{n}} = 0.$$

The shape of the function K ensures that the approximate density will be bounded by the positive number k from above and by zero from below. So the aim is to verify that it is possible to prove estimates for approximate problem (2.20)–(2.21) which ensure that one can improve the bound for the density in such a way that

$$\lim_{\varepsilon\to 0^+} \left|\left\{x \in \Omega; \varrho_\varepsilon(x) > k-3\right\}\right| = 0.$$

This problem is connected with obtaining higher integrability of the velocity and the temperature. Here, the choice of the slip boundary conditions plays an important role. Using the Helmoltz decomposition

$$\mathbf{u} = \nabla\phi + \operatorname{rot}\mathbf{A},$$

the regularity of the vorticity $\boldsymbol{\omega}$ (note that $\operatorname{rot}\boldsymbol{\omega} = \operatorname{rot}\operatorname{rot}\mathbf{A}$) up to the boundary is possible to show for the slip boundary conditions, but not for e.g. the Dirichlet boundary conditions for the velocity. Namely, the Navier boundary conditions for the velocity imply the following boundary conditions for $\boldsymbol{\omega}_\varepsilon$ on $\partial\Omega$

$$\begin{aligned}\boldsymbol{\omega}_\varepsilon\cdot\boldsymbol{\tau}_1 &= -(2\chi_2-\alpha/\mu)\mathbf{u}_\varepsilon\cdot\boldsymbol{\tau}_2,\\ \boldsymbol{\omega}_\varepsilon\cdot\boldsymbol{\tau}_2 &= (2\chi_1-\alpha/\mu)\mathbf{u}_\varepsilon\cdot\boldsymbol{\tau}_1,\\ \operatorname{div}\boldsymbol{\omega}_\varepsilon &= 0,\end{aligned}$$

where χ_k are the curvatures associated with the directions $\boldsymbol{\tau}_k$.

Another difficulty consists in obtaining estimates of the temperature, but central problem for the limit passage with $\varepsilon\to 0^+$ is to justify the strong convergence of the sequence of densities, since no estimates of derivatives of the density are available.

However, for

$$G_\varepsilon = -\Big(\frac{4}{3}\mu+\xi\Big)\Delta\phi_\varepsilon + P(\varrho_\varepsilon,\vartheta_\varepsilon) = -\Big(\frac{4}{3}\mu+\nu\Big)\operatorname{div}\mathbf{u}_\varepsilon + P(\varrho_\varepsilon,\vartheta_\varepsilon)$$

and its limit version

$$G = -\Big(\frac{4}{3}\mu+\nu\Big)\operatorname{div}\mathbf{u} + \overline{P(\varrho,\vartheta)},$$

where $\overline{P(\varrho,\vartheta)}$ denotes the weak limit of $P(\varrho_\varepsilon,\vartheta_\varepsilon)$, we can show that G_ε converges strongly to G in $L^2(\Omega)$ which finally implies not only the strong convergence of the density, but also the strong convergence of the velocity gradient in $L^2(\Omega)$; exactly this information is sufficient to pass to the limit in the unpleasant term $\mathbb{S}(\mathbb{D}(\mathbf{u})) : \nabla \mathbf{u}$. More details can be found in the paper [Mucha Pokorný 2009] which is contained in Chapter 5.

In [Mucha Pokorný 2010] the authors extended the existence result for larger interval of γ's ($\gamma > \frac{7}{3}$) and Dirichlet boundary conditions. However, for $\gamma \leq 3$ even for the slip boundary conditions and for the Dirichlet boundary conditions in general, we lose the possibility to prove that the density is bounded. Hence we are not able to verify the strong convergence of the velocity gradient which results into the necessity of using the total energy formulation.

The approach described above inspired some other authors to study similar problems, see e.g. papers [Muzereau et al 2010], [Muzereau et al 2011], [Zatorska 2012a], [Meng 2017] or [Amirat Hamdache 2019]. On the other hand, the result in [Yan 2016] contains a serious gap, the result does not hold for $\gamma > \frac{4}{3}$, but only for $\gamma > 3$.

2.3 Weak and variational entropy solution

In this section we shall explain the main ideas connected with results in Chapters 6, 7 and 8, i.e. with results from papers [Novotný Pokorný 2011a], [Novotný Pokorný 2011b] and [Jesslé et al. 2014]. The main disadvantage of the results from the previous section ([Mucha Pokorný 2009]) is that the estimate of the velocity gradient is deduced from the momentum equation which means that it depends on the density. The main novelty of the aforementioned series of papers considered in this chapter is that the estimate of the velocity is deduced from the entropy inequality. It is then independent of any other unknown quantities. Together with the total energy balance integrated over Ω we get an estimate of the temperature which, however, depends on the density. Hence we must deduce estimates of the density (which may depend on the previously obtained velocity estimates without any restriction, and on the estimate of the temperature in such a way that we may close the estimates). It can be obtained either directly, using the Bogovskii-type estimates or indirectly, using the potential estimates. This technique will be described below, in Subsection 2.3.1. All these estimates are in fact performed for a certain approximate problem and we must pass to the limit in the equations. The most difficult part is to get the strong convergence for the density sequence, since the a priori estimates provide only L^p-estimates for a certain $p > \gamma$, i.e. the concentrations of the sequence of densities are excluded and we must fight only with possible oscillations. The technique will be explained in Subsection 2.3.2.

We present the following results

Theorem 2 (Dirichlet boundary conditions.) [Novotný Pokorný 2011a] *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^3$, $\mathbf{f} \in L^\infty(\Omega;\mathbb{R}^3)$, $\Theta_0 \geq K_0 > 0$ a.e. at $\partial\Omega$, $\Theta_0 \in L^1(\partial\Omega)$. Let $\gamma > \frac{3}{2}$, $m > \max\left\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\right\}$, $l = 0$. Then there exists a variational entropy solution to (2.1)–(2.3), (2.6)–(2.7) in the sense of Definition 4. Moreover, $\varrho \geq 0$, $\vartheta > 0$ a.e. in Ω and $(\varrho, \mathbf{u})$ is a renormalized solution to the continuity equation in the sense of Definition 6.*

In addition, if $m > 1$ and $\gamma > \frac{5}{3}$, then the solution is a weak solution in the sense of Definition 2.

Theorem 3 (Dirichlet boundary conditions.) [Novotný Pokorný 2011b] *Let Ω be a C^2 be a bounded domain in $\mathbb{R}^3$, $\mathbf{f} \in L^\infty(\Omega;\mathbb{R}^3)$, $\Theta_0 \geq K_0 > 0$ a.e. at $\partial\Omega$, $\Theta_0 \in L^1(\partial\Omega)$. Let $\gamma > 1$, $m > \max\left\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\right\}$, $l = 0$. Then there exists a variational entropy solution to (2.1)–(2.3), (2.6)–(2.7) in the sense of Definition 4. Moreover, $\varrho \geq 0$, $\vartheta > 0$ a.e. in Ω and $(\varrho, \mathbf{u})$ is a renormalized solution to the continuity equation in the sense of Definition 6.*

In addition, if $m > \max\{1, \frac{2\gamma}{3(3\gamma-4)}\}$ and $\gamma > \frac{4}{3}$, then the solution is a weak solution in the sense of Definition 2.

Theorem 4 (Navier boundary conditions.) [Jesslé et al 2014] *Let Ω be a C^2 bounded domain in $\mathbb{R}^3$, $\mathbf{f} \in L^\infty(\Omega;\mathbb{R}^3)$, $\Theta_0 \geq K_0 > 0$ a.e. at $\partial\Omega$, $\Theta_0 \in L^1(\partial\Omega)$. Let $\gamma > 1$, $m > \max\left\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\right\}$, $l = 0$. Then there exists a variational entropy solution to (2.1)–(2.2), (2.4) and (2.6)–(2.7) in the sense of Definition 5. Moreover, $\varrho \geq 0$, $\vartheta > 0$ a.e. in Ω and $(\varrho, \mathbf{u})$ is a renormalized solution to the continuity in the sense of Definition 6.*

In addition, if $m > 1$ and $\gamma > \frac{5}{4}$, then the solution is a weak solution in the sense of Definition 3.

Remark 2.3.1 (i) Note that the results of Theorem 2 hold also for the Navier boundary conditions, just the proof in [Novotný Pokorný 2011a] was performed for the Dirichlet ones.
(ii) In fact, the paper [Novotný Pokorný 2011b] contains a weaker result than Theorem 3. However, as explained in [Mucha et al 2018], to obtain Theorem 3, it is enough to modify slightly at one step the proof for the limit passages.
(iii) It is worth mentioning that the result of Theorem 4 is stronger than

the result of Theorem 3 in the sense that the weak solution exists for larger interval of γ. We shall point out the moment where the Navier boundary conditions give better results than the Dirichlet ones.

2.3.1 A priori estimates

In this subsection we try to illustrate the main idea of obtaining the a priori estimates which allow to prove existence of solutions. In the context of steady solutions, the procedure described below appeared for the first time in [Novotný Pokorný 2011a]; this paper also contains a carefully described approximation procedure used to prove existence of solutions.

For simplicity, we assume in what follows the Dirichlet boundary conditions. Basically identically (with a few small changes of technical character if $\alpha > 0$) it can be used also for the Navier boundary conditions. Moreover, we use (1.4) instead of (1.3) considered in the original paper.

We start with the entropy inequality (2.16) with the test function $\psi \equiv 1$; for the approximate system we must be able to deduce it from the approximate internal energy balance which requires a certain regularity of the solutions to the approximate problem. We have

$$\int_\Omega \Big(\kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2} + \frac{1}{\vartheta}\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta) : \nabla\mathbf{u}\Big)\,\mathrm{d}x + \int_{\partial\Omega}\frac{L\Theta_0}{\vartheta}\,\mathrm{d}S \le \int_{\partial\Omega} L\,\mathrm{d}S. \tag{2.22}$$

Next we also employ the global total energy equality from (2.17) and get

$$\int_{\partial\Omega} L\vartheta\,\mathrm{d}S = \int_\Omega \varrho\mathbf{u}\cdot\mathbf{f}\,\mathrm{d}x + \int_{\partial\Omega} L\Theta_0\,\mathrm{d}S. \tag{2.23}$$

Using the Korn inequality we have from (2.22)

$$\|\mathbf{u}\|_{1,2}^2 + \|\nabla(\vartheta^{m/2})\|_2^2 + \|\ln\vartheta\|_{1,2}^2 \le C,$$

while (2.22) and (2.23) together with the Sobolev embedding theorem yield

$$\|\vartheta\|_{3m} \le C(1 + \|\mathbf{u}\|_6\|\varrho\|_{\frac{6}{5}}\|\mathbf{f}\|_\infty) \le C(1 + \|\varrho\|_{\frac{6}{5}}).$$

It remains to estimate the density. In order to simplify the situation as much as possible at this moment, we use the estimates based on the application of the Bogovskii operator. Let us recall that it is a solution operator of the problem

$$\begin{aligned} \operatorname{div}\mathbf{v} &= f \quad \text{in } \Omega, \\ \mathbf{v} &= \mathbf{0} \quad \text{on } \partial\Omega \end{aligned} \tag{2.24}$$

for f in $L^p(\Omega)$ with $\int_\Omega f \, \mathrm{d}x = 0$, $1 < p < \infty$, such that

$$\|\mathbf{v}\|_{1,p} \le C\|f\|_p.$$

For the proof of its existence see e.g. [Novotný Straškraba 2004]. We use as a test function in (2.12) the solution to

$$\begin{aligned} \operatorname{div} \boldsymbol{\varphi} &= \varrho^\Theta - \frac{1}{|\Omega|}\int_\Omega \varrho^\Theta \, \mathrm{d}x \quad \text{in } \Omega \\ \boldsymbol{\varphi} &= \mathbf{0} \quad \text{on } \partial\Omega \end{aligned}$$

for $\Theta > 0$. It yields

$$\int_\Omega p(\varrho,\vartheta)\varrho^\Theta \, \mathrm{d}x = -\int_\Omega \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} \, \mathrm{d}x + \int_\Omega \mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta) : \nabla\boldsymbol{\varphi} \, \mathrm{d}x$$
$$-\int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi} \, \mathrm{d}x + \frac{1}{|\Omega|}\int_\Omega p(\varrho,\vartheta) \, \mathrm{d}x \int_\Omega \varrho^\Theta \, \mathrm{d}x = \sum_{i=1}^4 I_i.$$

Recalling that the density is bounded in $L^1(\Omega)$ (the prescribed total mass) and using the properties of the Bogovskii operator above it is not difficult to check that the most restrictive terms are I_1 and I_2 leading to bounds

$$\Theta \le \min\Big\{2\gamma - 3, \frac{3m-2}{3m+2}\gamma\Big\}, \qquad \gamma > \frac{3}{2},\ m > \frac{2}{3}. \tag{2.25}$$

Hence under assumption (2.25) we have

$$\|\mathbf{u}\|_{1,2} + \|\nabla(\vartheta^{m/2})\|_2 + \|\ln\vartheta\|_{1,2} + \|\vartheta\|_{3m} + \|\varrho\|_{\gamma+\Theta} \le C.$$

Therefore we see that all quantities in the weak formulation integrable (i.e., in particular, the terms $\varrho|\mathbf{u}|^3$ and $\mathbb{S}(\vartheta,\mathbb{D}(\mathbf{u}))\mathbf{u}$ are integrable in $L^r(\Omega)$ for some $r > 1$) if

$$\gamma > \frac{5}{3}, \qquad m \ge 1,$$

while all terms in the variational entropy formulation are bounded if

$$\gamma > \frac{3}{2}, \qquad m > \frac{2}{3}.$$

The above described method has one important limitation: it cannot deal with exponents $\gamma \le \frac{3}{2}$. The idea how to proceed in this case goes back to papers [Plotnikov Sokolowski 2005] or [Březina Novotný 2008] and it was used to prove existence of weak solutions to the compressible Navier–Stokes equations for $\gamma > \frac{4}{3}$ in [Frehse et al 2009]. Another approach which gives existence of weak solutions for the compressible Navier–Stokes system even for any $\gamma > 1$ from [Plotnikov Sokolowski 2005] does not seem to be so efficient in the case of the heat conducting fluid.

The idea from [Frehse et al 2009] has been applied to the Dirichlet boundary problem for heat conducting case in [Novotný Pokorný 2011b]. It consists in reading certain integrability properties of the density and kinetic energy from the pressure term using as test function

$$\boldsymbol{\varphi}(x; x_0) \sim \frac{(x - x_0)}{|x - x_0|^a} \tag{2.26}$$

for some $0 < a < 1$. Then

$$\operatorname{div} \boldsymbol{\varphi}(x; x_0) \sim \frac{1}{|x - x_0|}$$

which provides estimates of the type

$$\int_\Omega \frac{\varrho^\gamma}{|x - x_0|} \, \mathrm{d}x$$

at least locally around x_0 (far from x_0 such estimates do not say anything new) for x_0 far from the boundary $\partial\Omega$. More difficulties appear when $x_0 \in \partial\Omega$. Here, the main difference between the slip and the Dirichlet boundary conditions appears. The Navier boundary conditions require that only the normal projection of the test function vanishes at $\partial\Omega$ while the Dirichlet boundary conditions require the whole test function to vanish there. Hence it is slightly less demanding to construct such suitable test function for the slip boundary conditions as was observed in [Jesslé Novotný 2013]. The method from this paper was applied to the case of heat conducting fluids in [Jesslé et al. 2014].

Note, however, that all the papers mentioned above contained a small gap in the proof: they did not consider the case when $x_0 \notin \partial\Omega$, but x_0 is close to $\partial\Omega$. The problem is that the test function from (2.26) must be multiplied by a suitable cut-off function to vanish on $\partial\Omega$ and it is not possible to control the derivatives of the cut-off function for x_0 approaching $\partial\Omega$. A slightly nontrivial construction of the test function for this situation was introduced in the overview paper [Mucha et al 2018] and the gap from all papers was removed.

The details of the construction and the procedure how to obtain replace the potential pressure estimates by suitable L^p-estimates of the pressure and the kinetic energy are performed in the papers [Novotný Pokorný 2011b], [Jesslé et al. 2014] (the cases x_0 far from boundary and on the boundary) and in [Mucha et al 2018] (including the case $x_0 \notin \partial\Omega$, but close to it). Since the computations are quite technical, let us only conclude here that for the Dirichlet boundary conditions we end up with

Lemma 1 *Let $\gamma > 1$, $m > \frac{2}{3}$ and $m > \frac{2}{9}\frac{\gamma}{\gamma-1}$. Then there exists $s > 1$ such that ϱ is bounded in $L^{s\gamma}(\Omega)$ and $p(\varrho, \vartheta)$, $\varrho|\mathbf{u}|$ and $\varrho|\mathbf{u}|^2$ are bounded in $L^s(\Omega)$. Moreover, if $\gamma > \frac{4}{3}$, and*

$$\begin{aligned} m > 1 \qquad &\textit{for} \qquad \gamma > \frac{12}{7}, \\ m > \frac{2\gamma}{3(3\gamma-4)} \qquad &\textit{for} \qquad \gamma \in \Big(\frac{4}{3}, \frac{12}{7}\Big], \end{aligned} \tag{2.27}$$

we can take $s > \frac{6}{5}$.

For the Navier boundary conditions, we have a better result

Lemma 2 *Let $\gamma > 1$, $m > \frac{2}{3}$ and $m > \frac{2}{4\gamma-3}$. Then there exists $s > 1$ such that ϱ is bounded in $L^{s\gamma}(\Omega)$ and $p(\varrho, \vartheta)$, $\varrho|\mathbf{u}|$ and $\varrho|\mathbf{u}|^2$ are bounded in $L^s(\Omega)$. Moreover, if $\gamma > \frac{5}{4}$, and*

$$\begin{aligned} m > 1 \qquad &\textit{for} \qquad \gamma > \frac{5}{3}, \\ m > \frac{2\gamma+10}{17\gamma-15} \qquad &\textit{for} \qquad \gamma \in \Big(\frac{5}{4}, \frac{5}{3}\Big], \end{aligned}$$

we can take $s > \frac{6}{5}$.

2.3.2 Compensated compactness for the density

To avoid technicalities and keep the idea as easy as possible, we present here the main ideas of proving weak compactness of solutions to our problem with Navier boundary conditions assuming the a priori bounds from Subsection 2.3.1. We denote our sequence of solutions by $(\varrho_\delta, \mathbf{u}_\delta, \vartheta_\delta)$. Recalling the a priori estimates from the previous subsection, we have

$$\begin{aligned}
\mathbf{u}_\delta &\rightharpoonup \mathbf{u} \quad \text{in } W^{1,2}(\Omega;\mathbb{R}^3),\\
\mathbf{u}_\delta &\to \mathbf{u} \quad \text{in } L^q(\Omega;\mathbb{R}^3),\ q<6\\
\mathbf{u}_\delta &\to \mathbf{u} \quad \text{in } L^r(\partial\Omega;\mathbb{R}^3),\ r<4\\
\varrho_\delta &\rightharpoonup \varrho \quad \text{in } L^{s\gamma}(\Omega),\\
\vartheta_\delta &\rightharpoonup \vartheta \quad \text{in } W^{1,p}(\Omega),\ p=\min\{2,\tfrac{3m}{m+1}\},\\
\vartheta_\delta &\to \vartheta \quad \text{in } L^q(\Omega),\ q<3m,\\
\vartheta_\delta &\to \vartheta \quad \text{in } L^r(\partial\Omega),\ r<2m,\\
p(\varrho_\delta,\vartheta_\delta) &\to \overline{p(\varrho,\vartheta)} \quad \text{in } L^r(\Omega),\ \text{for some } r>1,\\
e(\varrho_\delta,\vartheta_\delta) &\to \overline{e(\varrho,\vartheta)} \quad \text{in } L^r(\Omega),\ \text{for some } r>1,\\
s(\varrho_\delta,\vartheta_\delta) &\to \overline{s(\varrho,\vartheta)} \quad \text{in } L^r(\Omega),\ \text{for some } r>1.
\end{aligned}$$

We can now pass to the limit in the weak formulation of the continuity equation, momentum equation, entropy inequality and global total energy balance to get

$$\int_\Omega \varrho\mathbf{u}\cdot\nabla\psi\ \mathrm{d}x = 0$$

for all $\psi \in C^1(\overline{\Omega})$,

$$\begin{aligned}
\int_\Omega \Big(-\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} + \mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta) : \nabla\boldsymbol{\varphi} - \overline{p(\varrho,\vartheta)}\operatorname{div}\boldsymbol{\varphi}\Big)\ \mathrm{d}x \\
+\,\alpha\int_{\partial\Omega}\mathbf{u}\cdot\boldsymbol{\varphi}\ \mathrm{d}S = \int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\ \mathrm{d}x
\end{aligned} \tag{2.28}$$

for all $\boldsymbol{\varphi} \in C^1_{\mathbf{n}}(\overline{\Omega};\mathbb{R}^3)$,

$$\begin{aligned}
&\int_\Omega \Big(\vartheta^{-1}\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta) : \nabla\mathbf{u} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\Big)\psi\ \mathrm{d}x\\
&\le \int_\Omega \Big(\kappa(\vartheta)\frac{\nabla\vartheta\cdot\nabla\psi}{\vartheta} - \overline{\varrho s(\varrho,\vartheta)}\mathbf{u}\cdot\nabla\psi\Big)\ \mathrm{d}x + \int_{\partial\Omega}\frac{L}{\vartheta}(\vartheta-\Theta_0)\psi\ \mathrm{d}S
\end{aligned}$$

for all non-negative $\psi \in C^1(\overline{\Omega})$,

$$\int_{\partial\Omega}(L(\vartheta-\Theta_0)+\alpha|\mathbf{u}|^2)\ \mathrm{d}S = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x.$$

However, to pass to the limit in the total energy balance, we need that $\varrho_\delta|\mathbf{u}_\delta|^2 \rightharpoonup \varrho|\mathbf{u}|^2$ in some $L^q(\Omega)$, $q>\frac{6}{5}$, $\vartheta_\delta \to \vartheta$ in some $L^r(\Omega)$, $r>3$. This is true for $s>\frac{6}{5}$ and $m>1$.

Hence, assuming $\gamma > \frac{5}{4}$, $m > \max\{1, \frac{2\gamma+10}{17\gamma-15}\}$ we also get the total energy balance

$$\begin{aligned}\int_\Omega \Big(\big(-\frac{1}{2}\varrho|\mathbf{u}|^2 - \overline{\varrho e(\varrho,\vartheta)}\big)\mathbf{u}\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta\cdot\nabla\psi\Big)\,\mathrm{d}x \\ + \int_{\partial\Omega}\big(L(\vartheta-\Theta_0) + \alpha|\mathbf{u}|^2\big)\psi\,\mathrm{d}S = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\psi\,\mathrm{d}x \\ + \int_\Omega \big(-\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta)\mathbf{u} + \overline{p(\varrho,\vartheta)\mathbf{u}}\big)\cdot\nabla\psi\,\mathrm{d}x\end{aligned}$$

for all $\psi \in C^1(\overline{\Omega})$.

To finish the proof, we need to verify that $\varrho_\delta \to \varrho$ in some $L^r(\Omega)$, $r \geq 1$. The proof is based on three main ingrediences: the effective viscous flux identity, the oscillation defect measure estimate and the verification of the validity of renormalized continuity equation. We start with the effective viscous flux identity.

We use in the momentum equation as a test function

$$\boldsymbol{\varphi} = \zeta\nabla\Delta^{-1}(1_\Omega T_k(\varrho_\delta)), \quad k \in \mathbb{N},$$

with $\zeta \in C_c^\infty(\Omega)$,

$$T_k(z) = kT\Big(\frac{z}{k}\Big), \qquad T(z) = \begin{cases} z \text{ for } 0 \leq z \leq 1, \\ \text{concave on } (0,\infty), \\ 2 \text{ for } z \geq 3. \end{cases} \tag{2.29}$$

In its limit version (2.28) we use

$$\boldsymbol{\varphi} = \zeta\nabla\Delta^{-1}(1_\Omega\overline{T_k(\varrho)}), \quad k \in \mathbb{N},$$

where $\overline{T_k(\varrho)}$ is the weak limit of $T_k(\varrho_\delta)$ as $\delta \to 0^+$ (the corresponding chosen subsequence). After technical, but standard computation and using certain commutator relations to pass to the limit in the stress tensor and the convective term (cf. [Novotný Pokorný 2011b] or [Feireisl Novotný 2009] for the evolutionary case) we get *the effective viscous flux* identity

$$\begin{aligned}\overline{p(\varrho,\vartheta)T_k(\varrho)} - \Big(\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)\Big)\overline{T_k(\varrho)\operatorname{div}\mathbf{u}} \\ = \overline{p(\varrho,\vartheta)}\;\overline{T_k(\varrho)} - \Big(\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)\Big)\overline{T_k(\varrho)}\operatorname{div}\mathbf{u}.\end{aligned} \tag{2.30}$$

Our aim is to show that the renormalized continuity equation is fulfilled. However, for γ close to 1 ϱ is generally not in $L^2(\Omega)$ and we cannot justify it directly using the Friedrich commutator lemma. Following the idea originally due to E. Feireisl, we introduce the *oscillation defect measure*

$$\mathbf{osc}_q[\varrho_\delta \to \varrho](Q) = \sup_{k>1} \Big(\limsup_{\delta\to 0^+} \int_Q |T_k(\varrho_\delta) - T_k(\varrho)|^q \, \mathrm{d}x \Big). \tag{2.31}$$

We have (see [Feireisl Novotný 2009, Lemma 3.8] in the evolutionary case; in the steady case the proof is the same)

Lemma 3 *Let $\Omega \subset \mathbb{R}^3$ be open and let*

$$\begin{array}{ll} \varrho_\delta \rightharpoonup \varrho & \text{in } L^1(\Omega), \\ \mathbf{u}_\delta \rightharpoonup \mathbf{u} & \text{in } L^r(\Omega;\mathbb{R}^3), \\ \nabla\mathbf{u}_\delta \rightharpoonup \nabla\mathbf{u} & \text{in } L^r(\Omega;\mathbb{R}^{3\times 3}), \quad r>1. \end{array}$$

Let

$$\mathbf{osc}_q[\varrho_\delta \to \varrho](\Omega) < \infty$$

for $\frac{1}{q} < 1 - \frac{1}{r}$, where $(\varrho_\delta, \mathbf{u}_\delta)$ solve the renormalized continuity equation. Then the limit functions solve (2.19) for all $b \in C^1([0,\infty)) \cap W^{1,\infty}((0,\infty))$, $zb' \in L^\infty((0,\infty))$.

We can verify

Lemma 4 *Let $(\varrho_\delta, \mathbf{u}_\delta, \vartheta_\delta)$ be as above and let $m > \max\{\frac{2}{3(\gamma-1)}, \frac{2}{3}\}$. Then there exists $q > 2$ such that* (2.31) *holds true. Moreover,*

$$\begin{gathered} \limsup_{\delta\to 0^+} \int_\Omega \frac{1}{\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)} |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1} \, \mathrm{d}x \\ \le \int_\Omega \frac{1}{\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)} \Big(\overline{p(\varrho,\vartheta) T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\; \overline{T_k(\varrho)} \Big) \, \mathrm{d}x. \end{gathered} \tag{2.32}$$

As $(\varrho_\delta, \mathbf{u}_\delta)$ and $(\varrho, \mathbf{u})$ verify the renormalized continuity equation, we have the identities

$$\int_\Omega T_k(\varrho) \operatorname{div} \mathbf{u} \, \mathrm{d}x = 0,$$

and

$$\int_\Omega T_k(\varrho_\delta) \operatorname{div} \mathbf{u}_\delta \, \mathrm{d}x = 0, \qquad \text{i.e.} \int_\Omega \overline{T_k(\varrho) \operatorname{div} \mathbf{u}} \, \mathrm{d}x = 0.$$

Hence, employing the effective viscous flux identity (2.30),

$$\begin{gathered} \int_\Omega \frac{1}{\frac{4}{3}\mu(\vartheta) + \xi(\vartheta)} \Big(\overline{p(\varrho,\vartheta) T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\; \overline{T_k(\varrho)} \Big) \, \mathrm{d}x \\ = \int_\Omega \big(T_k(\varrho) - \overline{T_k(\varrho)} \big) \operatorname{div} \mathbf{u} \, \mathrm{d}x. \end{gathered} \tag{2.33}$$

We easily have $\lim_{k\to\infty}\|T_k(\varrho)-\varrho\|_1 = \lim_{k\to\infty}\|\overline{T_k(\varrho)}-\varrho\|_1 = 0$. Thus, (2.32) and (2.33) yield

$$\lim_{k\to\infty}\int_\Omega \frac{1}{\frac{4}{3}\mu(\vartheta)+\xi(\vartheta)}\Big(\overline{p(\varrho,\vartheta)T_k(\varrho)}-\overline{p(\varrho,\vartheta)}\;\overline{T_k(\varrho)}\Big)\,\mathrm{d}x = 0.$$

Using once more (2.32) we get

$$\lim_{k\to\infty}\limsup_{\delta\to 0^+}\int_\Omega \frac{1}{\frac{4}{3}\mu(\vartheta)+\xi(\vartheta)}|T_k(\varrho_\delta)-T_k(\varrho)|^{\gamma+1}\,\mathrm{d}x = 0,$$

which implies

$$\lim_{k\to\infty}\limsup_{\delta\to 0^+}\int_\Omega |T_k(\varrho_\delta)-T_k(\varrho)|^q\,\mathrm{d}x = 0$$

with q as in Lemma 4.

As

$$\|\varrho_\delta-\varrho\|_1 \le \|\varrho_\delta - T_k(\varrho_\delta)\|_1 + \|T_k(\varrho_\delta)-T_k(\varrho)\|_1 + \|T_k(\varrho)-\varrho\|_1,$$

we have

$$\varrho_\delta \to \varrho \qquad \text{in } L^1(\Omega);$$

whence

$$\varrho_\delta \to \varrho \quad \text{in } L^p(\Omega) \quad \forall 1\le p< s\gamma.$$

To finish the proof of Theorem 4, note that the condition $m > \frac{2}{3(\gamma-1)}$ is the most restrictive one. For Theorem 3 we also easily check that $\frac{2}{3(\gamma-1)} > \frac{2}{9(\gamma-1)}$ and that for the weak solutions, both $m>1$ and $m>\frac{2\gamma}{3(3\gamma-4)}$ must be taken into account.

Note finally that the paper [Zhong 2015] contains a serious gap. Therefore the result that for the Dirichlet boundary conditions one gets existence of a weak solution under the same assumptions as for the Navier boundary conditions is not proved. It remains as an open problem whether an improvement in this direction is possible.

2.4 Two dimensional flow

We consider our system of equations (2.1) with the boundary conditions (2.2)–(2.3) and the given total mass (2.7) in a bounded domain $\Omega\subset\mathbb{R}^2$. We assume the viscous part of the stress tensor in the form (1.2) ($N=2$) and the heat flux in the form (1.5). Moreover, we take $L = const$ in (2.2). We assume for $\gamma > 1$ the pressure law in the form (1.4) or, formally for $\gamma = 1$, we take

$$p = p(\varrho, \vartheta) = \varrho\vartheta + \frac{\varrho^2}{\varrho + 1} \ln^\alpha(1 + \varrho) \tag{2.34}$$

with $\alpha > 0$. The corresponding specific internal energy fulfils the Maxwell relation (1.12)

$$e = e(\varrho, \vartheta) = \frac{\ln^{\alpha+1}(1 + \varrho)}{\alpha + 1} + c_v\vartheta, \qquad c_v = const > 0, \tag{2.35}$$

and the specific entropy is

$$s(\varrho, \vartheta) = \ln \frac{\vartheta^{c_v}}{\varrho} + s_0. \tag{2.36}$$

We consider weak solutions to the problem above defined similarly as in Definition 2 with the corresponding modifications for the pressure law (2.34). This problem was studied in [Novotný Pokorný 2011c] for both (1.4) and (2.34) (see Chapter 9). The improvement for the pressure law (2.34) can be found in [Pokorný 2011]. The corresponding results read as follows

Theorem 5 (2D flow.) [Novotný Pokorný 2011] & [Pokorný 2011] *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^2$, $\mathbf{f} \in L^\infty(\Omega; \mathbb{R}^2)$, $\Theta_0 \geq K_0 > 0$ a.e. on $\partial\Omega$, $\Theta_0 \in L^1(\partial\Omega)$, $L > 0$.*
(i) Let $\gamma > 1$, $m > 0$. Then there exists a weak solution to our problem with the pressure law (1.4).
(ii) Let $\alpha > 1$ and $\alpha \geq \frac{1}{m}$, $m > 0$. Then there exists a weak solution to our problem with the pressure law (2.34).
Moreover, $(\varrho, \mathbf{u})$ extended by zero outside of Ω is a renormalized solution to the continuity equation.

As the proof for $\gamma > 1$ is easy, we only refer to [Novotný Pokorný 2011c] and consider the pressure law (2.34). Here, we need to work with a class of Orlicz spaces.

Let Φ be the Young function. We denote by $L_\Phi(\Omega)$ the set of all measurable functions u such that the Luxemburg norm

$$\|u\|_\Phi = \inf \Big\{ k > 0; \int_\Omega \Phi\Big(\frac{1}{k}|u(x)|\Big) dx \leq 1 \Big\} < +\infty.$$

For $\alpha \geq 0$ and $\beta \geq 1$ we denote by $L_{z^\beta \ln^\alpha(1+z)}(\Omega)$ the Orlicz spaces generated by $\Phi(z) = z^\beta \ln^\alpha(1+z)$. The complementary function to $z \ln^\alpha(1+z)$ behaves as $e^{z^{1/\alpha}}$. We denote by $L_{e(1/\alpha)}(\Omega)$ the corresponding sets of functions. We need an analogous result for the Bogovskii operator as in the L^p-spaces. Such a result can be obtained for Orlicz spaces such that the Young function Φ satisfies the global Δ_2-condition and for certain $\gamma \in (0,1)$ the function Φ^γ is quasiconvex, see [Vodák 2002]. Hence, especially for $\alpha \geq 0$ and $\beta > 1$ we have (provided $\int_\Omega f \,\mathrm{d}x = 0$) the existence of a solution to (2.24) such that

$$\||\nabla \boldsymbol{\varphi}|\|_{z^\beta \ln^\alpha(1+z)} \leq C \|f\|_{z^\beta \ln^\alpha(1+z)}.$$

Similarly as for the three-dimensional problem, we present only the main idea of the proof, i.e. the weak compactness of the sequence of solutions to our problem. As above, it is not difficult to deduce from the entropy inequality and the total energy balance integrated over Ω the following bounds

$$\|\mathbf{u}_\delta\|_{1,2} + \|\nabla \vartheta_\delta^{\frac{m}{2}}\|_2 + \|\nabla \ln \vartheta_\delta\|_2 + \|\vartheta_\delta^{-1}\|_{1,\partial\Omega} \leq C,$$

and

$$\|\vartheta_\delta^{\frac{m}{2}}\|_{1,2}^{\frac{2}{m}} \leq C\Big(1 + \|\vartheta_\delta\|_{1,\partial\Omega} + \|\nabla \vartheta_\delta^{\frac{m}{2}}\|_2^{\frac{2}{m}}\Big) \leq C\Big(1 + \Big|\int_\Omega \varrho_\delta \mathbf{f}\cdot \mathbf{u}_\delta \,\mathrm{d}x\Big|\Big)$$

with C independent of δ.

Indeed, it is more difficult to prove the estimates for the density. We use the method of the Bogovskii operator, i.e. we take in (2.24) $f = \varrho_\delta^s - \frac{1}{|\Omega|}\int_\Omega \varrho_\delta^s \,\mathrm{d}x$ for some $0 < s < 1$ (this is the method from [Pokorný 2011]; in [Novotný Pokorný 2011c], $s = 1$) and use the corresponding $\boldsymbol{\varphi}$ as test function in the momentum equation $(2.1)_2$. It yields the estimate

$$\int_\Omega \varrho_\delta^{1+s} \ln^\alpha(1+\varrho_\delta) \,\mathrm{d}x \leq C(s),$$

where $C(s) \to +\infty$ for $s \to 1^-$ if $\frac{1}{m} \leq \alpha < \frac{2}{m}$ for $m \leq 2$ and $\alpha > 1$.

Remark 2.4.1 In [Novotný Pokorný 2011c], where $s = 1$, the sequence of densities is bounded in $L^2(\Omega)$ which yields immediately that the limit pair $(\varrho, \mathbf{u})$ is a renormalized solution to the continuity equation. However, this method also requires additional restriction on α and m, namely $\alpha \geq \max\{1, \frac{2}{m}\}$. Note that above, we were able to get the estimates for any $m > 0$ and $\alpha > 1$; nevertheless, a certain restriction on α in terms of m appears later, when proving the strong convergence of the density.

We can now pass to the limit in the weak formulation of the approximate system (note that we still do not know whether the density converges strongly). The main task is to get strong convergence of the density which is based on the effective viscous flux identity and validity of the renormalized continuity equation; this is connected with the boundedness of the oscillation defect measure. As the proof is similar to the three-dimensional solutions, we shall only comment on steps which are different here.

First of all, we may verify the effective viscous flux identity in the form

$$\begin{aligned}&\overline{p(\varrho,\vartheta)T_k(\varrho)} - \big(\mu(\vartheta)+\xi(\vartheta)\big)\overline{T_k(\varrho)\operatorname{div}\mathbf{u}}\\ &= \overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)} - \big(\mu(\vartheta)+\xi(\vartheta)\big)\overline{T_k(\varrho)}\operatorname{div}\mathbf{u},\end{aligned}$$

where $T_k(\cdot)$ is defined in (2.29).

Next, we introduce the oscillation defect measure defined in a more general context of the Orlicz spaces

$$\mathbf{osc}_\Phi[T_k(\varrho_\delta) - T_k(\varrho)] = \sup_{k\in\mathbb{N}} \limsup_{\delta\to 0^+} \|T_k(\varrho_\delta) - T_k(\varrho)\|_\Phi.$$

In what follows, we show that there exists $\sigma > 0$ such that

$$\mathbf{osc}_{z^2\ln^\sigma(1+z)}[T_k(\varrho_\delta) - T_k(\varrho)] < +\infty; \tag{2.37}$$

further we verify that this fact implies the renormalized continuity equation to be satisfied. Note that to show the latter we cannot use the approach from the book [Feireisl Novotný 2009] (or [Novotný Pokorný 2011a]). However, we have

Lemma 5 *Under the assumptions of Theorem 5 (particularly, for $\alpha > 1$ and $\alpha \geq \frac{1}{m}$) we have (2.37).*

Next we can show

Lemma 6 *Under the assumptions of Theorem 5, the pair $(\varrho, \mathbf{u})$ is a renormalized solution to the continuity equation.*

The last step, i.e. that the validity of the renormalized continuity equation, the effective viscous flux identity, and estimates above imply the strong convergence of the density can be shown similarly as in three space dimensions, thus we skip it. More details can also be found in [Pokorný 2011]. Note that a similar problem in the evolutionary case was studied in [Erban 2003] for the isentropic and in [Skříšovský 2019] for the heat conducting flow.

2.5 Compressible fluid flow with radiation

We now present the result from Chapter 10, where a steady flow of a radiative gas has been considered. We are not going into details of its modelling, more information can be found e.g. in [Kreml et al. 2013] and references therein. We consider the following system of equations in a bounded $\Omega \subset \mathbb{R}^3$

$$\begin{aligned} \operatorname{div}(\varrho\mathbf{u}) &= 0, \\ \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbb{S} + \nabla p &= \varrho\mathbf{f} - \mathbf{s}_F, \\ \operatorname{div}(\varrho E\mathbf{u}) &= \varrho\mathbf{f}\cdot\mathbf{u} - \operatorname{div}(p\mathbf{u}) + \operatorname{div}(\mathbb{S}\mathbf{u}) - \operatorname{div}\mathbf{q} - s_E, \\ \lambda I + \boldsymbol{\omega}\cdot\nabla_x I &= S, \end{aligned} \tag{2.38}$$

where the last equation describes the transport of radiative intensity denoted by I. The right-hand side S is a given function of I, $\boldsymbol{\omega}$ and $\mathbf{u}$. The quantity $\mathbf{s}_F$ denotes the radiative flux and s_E is the radiative energy. The viscous part of the stress tensor is taken in the form (1.2) with the temperature dependent viscosities

$$\mu(\vartheta) \sim (1+\vartheta)^a, \qquad 0 \le \xi(\vartheta) \le C(1+\vartheta)^a$$

for $0 \le a \le 1$. The pressure is considered in the form (1.4) and the heat flux fulfills (1.5), L is a bounded function ($l = 0$). The system is considered together with the homogeneous Dirichlet boundary conditions for the velocity (2.3) and the Newton boundary condition for the heat flux (2.2). The existence of a solution for the evolutionary problem was proved in [Ducomet et al 2011]. Further results, dealing with different aspects (1D problem, large time behaviour, existence of solutions in unbounded domains etc.) were obtained in the series of papers [Ducomet Nečasová 2010], [Ducomet Nečasová 2012] or [Ducomet Nečasová 2014].

For the steady problem, we also prescribe the total mass of the fluid (2.7). The main result reads as follows

Theorem 6 (Steady radiative flow.) [Kreml et al 2013] *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^3$, $\mathbf{f} \in L^\infty(\Omega;\mathbb{R}^3)$, $\Theta_0 \ge K_0 > 0$ a.e. at $\partial\Omega$,*

$\Theta_0 \in L^1(\partial\Omega)$, $M > 0$. *Moreover, let*

$$
\begin{aligned}
& a \in (0,1], \\
& \gamma > \max\left\{\frac{3}{2}, 1 + \frac{1-a}{6a} + \frac{1}{2}\sqrt{\frac{4(1-a)}{3a} + \frac{(1-a)^2}{9a^2}}\right\}, \\
& m > \max\left\{1-a, \frac{1+a}{3}, \frac{\gamma(1-a)}{2\gamma-3}, \frac{\gamma(1-a)^2}{3(\gamma-1)^2a - \gamma(1-a)}, \right. \\
& \qquad \left. \frac{1-a}{6(\gamma-1)a-1}, \frac{1+a+\gamma(1-a)}{3(\gamma-1)}\right\}.
\end{aligned}
\tag{2.39}
$$

Then there exists a variational entropy solution to our system. Moreover, the pair $(\varrho, \mathbf{u})$ *is a renormalized solution to the continuity equation.*

If additionally

$$
\begin{aligned}
& \gamma > \max\left\{\frac{5}{3}, \frac{2+a}{3a}\right\}, \\
& m > \max\left\{1, \frac{(3\gamma-1)(1-a)}{3\gamma-5}, \frac{(3\gamma-1)(1-a)+2}{3(\gamma-1)}, \right. \\
& \qquad \left. \frac{(1-a)(\gamma(2-3a)+a)}{a(6\gamma^2-9\gamma+5)-2\gamma}\right\},
\end{aligned}
\tag{2.40}
$$

then this solution is a weak solution.

Remark 2.5.1 For special values of a we get from formulas (2.39) and (2.40) the following restrictions.

For $a = 1$:

$$\gamma > \frac{3}{2} \quad \text{and} \quad m > \max\left\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\right\}$$

for the variational entropy solutions, and additionally

$$\gamma > \frac{5}{3} \quad \text{and} \quad m > 1$$

for the weak solutions.

For $a = \frac{1}{2}$ (physically the most relevant case):

$$\gamma > \frac{7+\sqrt{13}}{6} \quad \text{and} \quad m > \max\left\{\frac{1}{2}, \frac{\gamma}{4\gamma-6}, \frac{\gamma}{6\gamma^2-14\gamma+6}\right\}$$

for the variational entropy solutions, and additionally

$$m > \max\left\{1, \frac{\gamma+1}{2(\gamma-1)}, \frac{3\gamma-1}{6\gamma-10}\right\}$$

for the weak solutions.

The proof is similar to the case without radiation with two additional difficulties. One is connected with radiation, especially with compactness properties of the transport equation and we are not going to comment on this issue here, the other one is connected with the fact that for $a < 1$ we lose the nice structure of the a priori estimates coming from the entropy inequality and the situation becomes more complex. More precisely, the entropy inequality provides us only

$$\|\mathbf{u}\|_{1,p}^p \le C\|\vartheta\|_{3m}^{\frac{3m(2-p)}{2}},$$

where $p = \frac{6m}{3m+1-a}$ (i.e. $p = 2$ if $a = 1$). This complicates technically the situation, on the other hand, the values of a below 1 are physically more realistic. More details can be found in [Kreml et al. 2013] in Chapter 10.

2.6 Time-periodic solution

We could have observed that in a sense, the results for heat conducting fluids are easier to obtain due to the entropy (in)equality. Let us demonstrate this also on the time-periodic problem which inherits the properties of both the evolutionary and the steady problems and therefore it is in fact more difficult than both these problems.

We consider $(1.1)_{1-2}$ together with (1.10), with the Dirichlet boundary conditions (1.8) for the velocity and the Newton boundary conditions (1.8) for the temperature with $L(\vartheta) = d = const.$ The initial conditions (1.6) are replaced by the fact that all functions are time-periodic with the period $T_{per} > 0$. We consider the Fourier law (1.5) and the pressure law (1.3) and its consequences for $\gamma = \frac{5}{3}$, i.e. the monoatomic gas (some extensions were considered in [Axmann Pokorný 2015]). Note, however, that we must assume in the pressure additionally a radiation term (the term can be justified from physics and, in mathematical treatment, plays an important role), i.e.

$$\begin{aligned} p(\varrho,\vartheta) &= p_0(\varrho,\vartheta) + \frac{a}{3}\vartheta^4, \\ e(\varrho,\vartheta) &= e_0(\varrho,\vartheta) + \frac{a}{\varrho}\vartheta^4, \\ s(\varrho,\vartheta) &= s_0(\varrho,\vartheta) + \frac{4a}{3\varrho}\vartheta^3, \end{aligned} \tag{2.41}$$

where p_0, e_0 and s_0 fulfill (1.3), (1.12)–(1.19) with $\gamma = \frac{5}{3}$. We also prescribe the total mass (2.7).

When dealing with time-periodic problems, it is convenient to consider all quantities defined on a time "sphere"

$$\mathcal{S}^1 = [0, T_{per}]|_{\{0, T_{per}\}}.$$

Definition 7 (Time-periodic solution.) *We say that a triple* $\{\varrho, \mathbf{u}, \vartheta\}$ *is a time-periodic weak solution to the Navier–Stokes–Fourier system (1.1)–(1.3),* (2.41)*, (1.5), (1.7), (1.8), (1.10) and* (2.7) *if the following holds:*

- *the solution belongs to the class* $\varrho \geq 0$, $\vartheta > 0$ *a.e.,*

$$\varrho \in L^\infty(\mathcal{S}^1; L^{5/3}(\Omega)),\ \vartheta \in L^\infty(\mathcal{S}^1; L^4(\Omega)),\ \mathbf{u} \in L^2(\mathcal{S}^1; W_0^{1,2}(\Omega; \mathbb{R}^3)),$$

$$\vartheta^{3/2}, \ln \vartheta \in L^2(\mathcal{S}^1; W^{1,2}(\Omega))$$

- *equation of continuity (1.1)*$_1$ *is satisfied in the sense of renormalized solutions,*

$$\int_{\mathcal{S}^1}\int_\Omega \Big(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla\varphi + \big(b(\varrho) - b'(\varrho)\varrho\big)\operatorname{div}\mathbf{u}\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t = 0$$

for any $b \in C^\infty[0,\infty)$, $b' \in C_c^\infty[0,\infty)$, *and any test function* $\varphi \in C^\infty(\mathcal{S}^1 \times \overline{\Omega})$

- *momentum equation (1.1)*$_2$ *holds in the sense of distributions:*

$$\begin{aligned}\int_{\mathcal{S}^1}\int_\Omega &\big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} + p(\varrho,\vartheta)\operatorname{div}\boldsymbol{\varphi}\big)\ \mathrm{d}x\ \mathrm{d}t \\ &= \int_{\mathcal{S}^1}\int_\Omega \big(\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta) : \nabla\boldsymbol{\varphi} - \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\big)\ \mathrm{d}x\ \mathrm{d}t\end{aligned} \tag{2.42}$$

for any $\boldsymbol{\varphi} \in C_c^\infty(\mathcal{S}^1 \times \Omega; \mathbb{R}^3)$

- *entropy equation (1.10) with the boundary condition (1.7) are satisfied in the sense of the integral identity*

$$\begin{aligned}\int_{\mathcal{S}^1}\int_\Omega &\Big(\varrho s(\varrho,\vartheta)\partial_t\psi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi + \frac{\mathbf{q}(\vartheta,\nabla\vartheta)}{\vartheta}\cdot\nabla\psi\Big)\ \mathrm{d}x\ \mathrm{d}t \\ &= \int_{\mathcal{S}^1}\int_{\partial\Omega} \frac{d}{\vartheta}(\vartheta - \Theta_0)\psi\ \mathrm{d}S\ \mathrm{d}t - < \sigma; \psi >\end{aligned} \tag{2.43}$$

for any $\psi \in C^\infty(\mathcal{S}^1 \times \overline{\Omega})$, *where* $\sigma \in \mathcal{M}^+(\mathcal{S}^1 \times \overline{\Omega})$ *is a non-negative measure satisfying*

$$\sigma \geq \frac{1}{\vartheta}\Big(\mathbb{S}(\mathbb{D}(\mathbf{u}),\vartheta) : \nabla\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta}\Big) \tag{2.44}$$

- *the total energy balance*

$$\begin{aligned}&\int_{\mathcal{S}^1}\Big(\partial_t\psi\int_\Omega\Big(\frac{1}{2}\varrho|\mathbf{u}|^2+\varrho e(\varrho,\vartheta)\Big)\ \mathrm{d}x\Big)\ \mathrm{d}t\\&=\int_{\mathcal{S}^1}\psi\Big(\int_{\partial\Omega}d(\vartheta-\Theta_0)\ \mathrm{d}S-\int_\Omega\varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x\Big)\ \mathrm{d}t\end{aligned} \tag{2.45}$$

holds for any $\psi\in C^\infty(\mathcal{S}^1)$.

It is not difficult to see that the entropy production inequality (2.43) reduces to (1.10) as soon as the solution is smooth enough.

Our aim is to show the following result:

Theorem 7 (Time-periodic solution.) [Feireisl et al 2012c] *Let* $\Omega\subset\mathbb{R}^3$ *be a bounded domain with a boundary of class* $C^{2+\nu}$. *Suppose that the thermodynamic functions* p, e, *and* s *satisfy hypotheses (2.41), (1.3) and (1.12)–(1.19). Let* $\mathbf{f}\in L^\infty(\mathcal{S}^1\times\Omega;\mathbb{R}^3)$.

Then for any $M>0$ *the Navier–Stokes–Fourier system possesses at least one time-periodic-solution* $\{\varrho,\mathbf{u},\vartheta\}$ *in the sense specified above such that*

$$\int_\Omega\varrho(t,\cdot)\ \mathrm{d}x=M\quad\text{for all } t\in\mathcal{S}^1.$$

Before we sketch the idea of the proof, we briefly explain how to obtain the a priori bounds. We have

Lemma 7 *Let* $(\varrho,\mathbf{u},\vartheta)$ *be a sufficiently smooth solution to our problem. Then*

$$\begin{aligned}&\sup_{t\in\mathcal{S}^1}\int_\Omega(\varrho\mathbf{u}^2+\varrho^{5/3}+\vartheta^4)\ \mathrm{d}x\\&+\int_{\mathcal{S}^1}\int_\Omega\Big(|\nabla\mathbf{u}|^2+(1+\vartheta^3)\frac{|\nabla\vartheta|^2}{\vartheta^2}+\varrho^{5/3+1/9}\Big)\ \mathrm{d}x\,\mathrm{d}t\le C,\end{aligned}$$

where C *depends only on the data of the problem.*

The proof of this lemma is naturally split into two parts dealing with bounds resulting from the entropy inequality and energy estimates, and improvement of integrability of the density.
Entropy and energy estimates: To begin, observe that the total mass of the fluid is a constant of motion, meaning

$$\int_\Omega\varrho(t,\cdot)\ \mathrm{d}x=M,\quad\text{in particular, }\varrho\in L^\infty(\mathcal{S}^1;L^1(\Omega)).$$

Next step is to take in the entropy inequality (2.43) test function $\psi \equiv 1$ to obtain

$$\int_{\mathcal{S}^1}\int_\Omega \Big(\frac{1}{2}\frac{\mu(\vartheta)}{\vartheta}\Big|2\mathbb{D}(\mathbf{u}) - \frac{2}{3}\operatorname{div}\mathbf{u}\,\mathbb{I}\Big|^2 + \frac{\kappa(\vartheta)|\nabla\vartheta|^2}{\vartheta^2}\Big)\ \mathrm{d}x\ \mathrm{d}t \\ + \int_{\mathcal{S}^1}\int_{\partial\Omega}\frac{d}{\vartheta}\Theta_0\ \mathrm{dS}\ \mathrm{d}t \le \int_{\mathcal{S}^1}\int_{\partial\Omega} d\ \mathrm{dS}\ \mathrm{d}t \le C.$$

Consequently, we deduce that

$$\mathbf{u} \in L^2(\mathcal{S}^1; W_0^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\nabla\vartheta^{3/2} \in L^2(\mathcal{S}^1\times\Omega;\mathbb{R}^3), \quad \text{and} \quad \nabla\ln\vartheta \in L^2(\mathcal{S}^1\times\Omega;\mathbb{R}^3).$$

Next, we use in the total energy balance (2.45) $\psi \equiv 1$ to obtain

$$\int_{\mathcal{S}^1}\int_{\partial\Omega} d(\vartheta - \Theta_0)\ \mathrm{d}S\ \mathrm{d}t = \int_{\mathcal{S}^1}\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t,$$

where

$$\Big|\int_{\mathcal{S}^1}\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t\Big| \le c\big(1 + \|\varrho\|_{L^2(\mathcal{S}^1;L^{6/5}(\Omega))}\big).$$

Consequently,

$$\|\vartheta\|_{L^1(\mathcal{S}^1;L^6(\Omega))} \le c\big(1 + \|\varrho\|_{L^2(\mathcal{S}^1;L^{6/5}(\Omega))}\big)$$

which, employing the standard interpolation theorem yields

$$\|\vartheta\|_{L^1(\mathcal{S}^1;L^6(\Omega))} \le c\Big(1 + \Big(\int_{\mathcal{S}^1}\Big(\int_\Omega \varrho^{5/3}\ \mathrm{d}x\Big)^{1/2}\ \mathrm{d}t\Big)^{1/2}\Big).$$

Next observe that, by virtue of hypotheses on the pressure function, there exist two positive constants c_1, c_2 such that

$$c_1(\varrho^{5/3} + \vartheta^4) \le \varrho e(\varrho,\vartheta) \le c_2(\varrho\vartheta + \varrho^{5/3} + \vartheta^4).$$

Denoting $E(t) = \int_\Omega \big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\big)\ \mathrm{d}x$ the total energy, the energy balance yields that $E(t) \le E(s) + c\Big(1 + \int_{\mathcal{S}^1} E(z)\ \mathrm{d}z\Big)$ for any $t \le s$. The mean value theorem implies

$$\sup_{t\in\mathcal{S}^1} E(t) \le c\Big(1 + \int_{\mathcal{S}^1} E(s)\ \mathrm{d}s\Big).$$

As

$$\int_{\mathcal{S}^1}\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2\ \mathrm{d}x\,\mathrm{d}t \le c\|\varrho\|_{L^\infty(\mathcal{S}^1;L^{3/2}(\Omega))},$$

we have

$$\sup_{t\in\mathcal{S}^1} E(t) \le c\Big(1+\int_{\mathcal{S}^1}\int_\Omega \varrho e(\varrho,\vartheta)\ \mathrm{d}x\,\mathrm{d}t\Big)$$
$$\le c\Big(1+\int_{\mathcal{S}^1}\int_\Omega \varrho^{5/3}\ \mathrm{d}x\ \mathrm{d}t + \int_{\mathcal{S}^1}\int_\Omega \vartheta^4\ \mathrm{d}x\,\mathrm{d}t\Big).$$

We write $\|\vartheta\|^4_{L^4(\Omega)} \le \|\vartheta\|_{L^6(\Omega)}\|\vartheta\|^3_{L^4(\Omega)} \le c\|\vartheta\|_{L^6(\Omega)}\sup_{t\in\mathcal{S}^1} E^{3/4}(t)$, hence,

$$\sup_{t\in\mathcal{S}^1} E(t) \le c\Big[1+\int_{\mathcal{S}^1}\int_\Omega \varrho^{5/3}\ \mathrm{d}x\,\mathrm{d}t + \Big(\int_{\mathcal{S}^1}\|\vartheta\|_{L^6(\Omega)}\,\mathrm{d}t\Big)^4\Big].$$

Thus we conclude that

$$\sup_{t\in\mathcal{S}^1} E(t) \le c\left(1+\int_{\mathcal{S}^1}\int_\Omega \varrho^{5/3}\ \mathrm{d}x\,\mathrm{d}t\right). \tag{2.46}$$

Pressure estimates: Having established the crucial relation (2.46), the remaining a priori bounds can be derived in the same way as for the isentropic case, see [Feireisl et al. 1999], namely using the Bogovskii operator. We use in the momentum equation (2.42) the function

$$\mathcal{B}\Big[\varrho^\alpha - \{\varrho^\alpha\}_\Omega\Big] \text{ for a certain (small) } \alpha > 0,$$

i.e. the solution to (2.27) for $f = \varrho^\alpha - \{\varrho^\alpha\}_\Omega$, where

$$\{g\}_\Omega = \frac{1}{|\Omega|}\int_\Omega g\ \mathrm{d}x.$$

Estimating the corresponding terms on the right-hand side we conclude

$$\int_{\mathcal{S}^1}\int_\Omega \varrho^{\frac{5}{3}+\alpha}\ \mathrm{d}x\ \mathrm{d}t \le C.$$

The most restrictive term is, as usually, the convective one, yielding the estimate for $\alpha = \frac{1}{9}$. Combining (2.46) with the estimates obtained in this section, we conclude that

$$\sup_{t\in\mathcal{S}^1} E(t) \le c.$$

The approximation itself is even more complex than in the steady case. Unlike many other methods, where the periodic solutions are constructed by means of a fixed point argument, we construct the solution directly, using several regularizing parameters. The proof is contained in Chapter 11. Note, however, that it is possible to obtain even stronger results, see [Axmann Pokorný 2015].

Chapter 3

Mathematical theory for steady multicomponent flow

The first result for the model considered in Section 1.2 can be found in [Zatorska 2011]; the author works with only three species inside a dilutant. For the species, the Fick law is considered. The author was also able to consider the non-diagonal diffusion matrix for the dilutant at least in some cases.

The first result with general non-diagonal diffusion matrix was proved in [Giovangigli et al 2015]. In this paper, a significant simplification was used, namely that the molar masses of all components are the same. The results from this paper were extended in [Piasecki Pokorný 2017] which is contained in Chapter 12. In [Piasecki Pokorný 2018] the results were extended for the Navier boundary conditions for the velocity.

3.1 Weak and variational entropy solutions

Let us recall the system of equations we consider

$$
\begin{aligned}
\operatorname{div}(\varrho \mathbf{u}) &= 0,\\
\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbb{S} + \nabla \pi &= \varrho \mathbf{f},\\
\operatorname{div}(\varrho E \mathbf{u}) + \operatorname{div}(\pi \mathbf{u}) + \operatorname{div} \mathbf{Q} - \operatorname{div}(\mathbb{S}\mathbf{u}) &= \varrho \mathbf{f} \cdot \mathbf{u},\\
\operatorname{div}(\varrho Y_k \mathbf{u}) + \operatorname{div} \mathbf{F}_k &= \omega_k, \quad k \in \{1, \dots, L\}
\end{aligned}
\tag{3.1}
$$

with the boundary conditions

$$
\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \tag{3.2}
$$

together with

$$\mathbf{F}_k \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{3.3}$$

and the Newton boundary condition for the heat flux

$$-\mathbf{Q} \cdot \mathbf{n} + L(\vartheta - \vartheta_0) = 0. \tag{3.4}$$

The total mass of the mixture is prescribed,

$$\int_\Omega \varrho \ \mathrm{d}x = M > 0. \tag{3.5}$$

More details concerning the model were given in (1.22)–(1.40). We consider the following two definitions of solutions.

Definition 8 (Multicomponent flow; weak solution.) *We say the set of functions* $(\varrho, \mathbf{u}, \vartheta, \vec{Y})$ *is a weak solution to problem (3.1)–(3.5) with assumptions stated above, provided*

- $\varrho \geq 0$ *a.e. in* Ω, $\varrho \in L^{6\gamma/5}(\Omega)$, $\int_\Omega \varrho \ \mathrm{d}x = M$
- $\mathbf{u} \in W_0^{1,2}(\Omega)$, $\varrho|\mathbf{u}|$ *and* $\varrho|\mathbf{u}|^2 \in L^{\frac{6}{5}}(\Omega)$
- $\vartheta \in W^{1,2}(\Omega) \cap L^{3m}(\Omega)$, $\varrho\vartheta, \varrho\vartheta|\mathbf{u}|, \mathbb{S}\mathbf{u}, \kappa|\nabla\vartheta| \in L^1(\Omega)$
- $\vec{Y} \in W^{1,2}(\Omega)$, $Y_k \geq 0$ *a.e. in* Ω, $\sum_{k=1}^L Y_k = 1$ *a.e. in* Ω, $\mathbf{F}_k \cdot \mathbf{n}|_{\partial\Omega} = 0$

and the following integral equalities hold
• *the weak formulation of the continuity equation*

$$\int_\Omega \varrho\mathbf{u} \cdot \nabla\psi \ \mathrm{d}x = 0 \tag{3.6}$$

holds for any test function $\psi \in C^\infty(\overline{\Omega})$
• *the weak formulation of the momentum equation*

$$-\int_\Omega \big(\varrho\,(\mathbf{u} \otimes \mathbf{u}) : \nabla\boldsymbol{\varphi} - \mathbb{S} : \nabla\boldsymbol{\varphi}\big) \ \mathrm{d}x - \int_\Omega \pi \operatorname{div}\boldsymbol{\varphi} \ \mathrm{d}x = \int_\Omega \varrho\mathbf{f} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \tag{3.7}$$

holds for any test function $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$
• *the weak formulation of the species equations*

$$-\int_\Omega Y_k\varrho\mathbf{u} \cdot \nabla\psi \ \mathrm{d}x - \int_\Omega \mathbf{F}_k \cdot \nabla\psi \ \mathrm{d}x = \int_\Omega \omega_k\psi \ \mathrm{d}x \tag{3.8}$$

holds for any test function $\psi \in C^\infty(\overline{\Omega})$ *and for all* $k = 1, \dots, L$

- *the weak formulation of the total energy balance*

$$
\begin{aligned}
& - \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right) \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x + \int_\Omega \kappa \nabla \vartheta \cdot \nabla \psi \, \mathrm{d}x \\
& - \int_\Omega \left(\sum_{k=1}^L h_k \mathbf{F}_k \right) \cdot \nabla \psi \, \mathrm{d}x = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \psi \, \mathrm{d}x - \int_\Omega (\mathbb{S}\mathbf{u}) \cdot \nabla \psi \, \mathrm{d}x \\
& + \int_\Omega \pi \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x - \int_{\partial\Omega} L(\vartheta - \vartheta_0) \psi \, \mathrm{d}S
\end{aligned}
\tag{3.9}
$$

holds for any test function $\psi \in C^\infty(\overline{\Omega})$.

The admissible range of γ in the pressure law for which we are able to show existence of weak solutions in the above sense is limited mostly by the terms $\varrho|\mathbf{u}|^2\mathbf{u}$ and $\mathbb{S}\mathbf{u}$ in the weak formulation of total energy balance. Therefore, similarly as in the single component flow, we replace the total energy balance $(3.1)_3$ by the entropy inequality specified in Definition 8 below. Note also that for the Navier boundary conditions for the velocity it is possible to obtain the existence of both weak and variational entropy solutions (see below) under less restrictive assumptions on γ, cf. [Piasecki Pokorný 2018].

Definition 9 (Multicomponent flow; variational entropy solution.) *We say the set of functions* $(\varrho, \mathbf{u}, \vartheta, \vec{Y})$ *is a variational entropy solution to problem (3.1–3.5) with assumptions stated above, provided*

- $\varrho \geq 0$ *a.e. in* Ω, $\varrho \in L^{s\gamma}(\Omega)$ *for some* $s > 1$, $\int_\Omega \varrho \, \mathrm{d}x = M$

- $\mathbf{u} \in W_0^{1,2}(\Omega)$, $\varrho\mathbf{u} \in L^{\frac{6}{5}}(\Omega)$

- $\vartheta \in W^{1,r}(\Omega) \cap L^{3m}(\Omega)$, $r > 1$, $\varrho\vartheta, \mathbb{S} : \frac{\nabla \mathbf{u}}{\vartheta}, \kappa \frac{|\nabla\vartheta|^2}{\vartheta^2}, \kappa \frac{\nabla\vartheta}{\vartheta} \in L^1(\Omega)$, $\frac{1}{\vartheta} \in L^1(\partial\Omega)$

- $\vec{Y} \in W^{1,2}(\Omega)$, $Y_k \geq 0$ *a.e. in* Ω, $\sum_{k=1}^L Y_k = 1$ *a.e. in* Ω, $\mathbf{F}_k \cdot \mathbf{n}|_{\partial\Omega} = 0$

satisfy equations (3.6)–(3.8)*, the following entropy inequality*

$$\int_\Omega \frac{\mathbb{S}:\nabla\mathbf{u}}{\vartheta}\psi \ \mathrm{d}x + \int \kappa\frac{|\nabla\vartheta|^2}{\vartheta^2}\psi \ \mathrm{d}x - \int_\Omega \sum_{k=1}^L \omega_k(c_{pk} - c_{vk}\log\vartheta + \log Y_k)\psi \ \mathrm{d}x$$
$$+ \int_\Omega \psi \sum_{k,l=1}^n D_{kl}\nabla Y_k\cdot\nabla Y_l \ \mathrm{d}x + \int_{\partial\Omega} \frac{L}{\vartheta}\vartheta_0\psi \, \mathrm{d}S \le \int \frac{\kappa\nabla\vartheta\cdot\nabla\psi}{\vartheta} \ \mathrm{d}x$$
$$- \int_\Omega \varrho s\mathbf{u}\cdot\nabla\psi \ \mathrm{d}x - \int_\Omega \log\vartheta\Big(\sum_{k=1}^L \mathbf{F}_k c_{vk}\Big)\cdot\nabla\psi \ \mathrm{d}x$$
$$+ \int_\Omega \Big(\sum_{k=1}^L \mathbf{F}_k \log Y_k\Big)\cdot\nabla\psi \ \mathrm{d}x + \int_{\partial\Omega} L\psi \, \mathrm{d}S \quad (3.10)$$

for all non-negative $\psi \in C^\infty(\overline{\Omega})$ *and the global total energy balance (i.e.* (3.9) *with* $\psi \equiv 1$*)*

$$\int_{\partial\Omega} L(\vartheta - \vartheta_0) \, \mathrm{d}S = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u} \ \mathrm{d}x. \quad (3.11)$$

Note, however, that (3.10) does not contain all terms from the formally deduced entropy identity, some of them are missing. These terms are formally equal to zero due to assumptions that ω_k and $\mathbf{F}_k$ sum up to zero. We removed them from the formulation of the entropy inequality due to the fact that we cannot exclude the situation that $\varrho = 0$ in some large portions of Ω (with positive Lebesgue measure), thus $\log\varrho$ is not well defined there. However, the variational entropy solution has still the property that any sufficiently smooth variational entropy solution in the sense above is a classical solution to our problem, provided the density is strictly positive in Ω.

We are now in position to formulate our main result.

Theorem 8 (Multicomponent flow.) [Piasecki Pokorný 2017] *Let* $\gamma > 1$, $M > 0$, $m > \max\{\frac{2}{3}, \frac{2}{3(\gamma-1)}\}$, $b < \frac{3m}{2}$. *Let* $\Omega \in C^2$. *Then there exists at least one variational entropy solution to our problem above. Moreover,* $(\varrho, \mathbf{u})$ *is the renormalized solution to the continuity equation.*

In addition, if $m > \max\{1, \frac{2\gamma}{3(3\gamma-4)}\}$, $\gamma > \frac{4}{3}$, $b < \frac{3m-2}{2}$, *then the solution is a weak solution in the sense above.*

3.2 Existence of a solution

The method of the proof of existence is based on [Giovangigli et al 2015], however, with one important difference. Immediately after we prove existence of the approximate problem, we need to establish the correct form of the entropy inequality. Therefore the solution to the approximate problem must be sufficiently regular. Moreover, we cannot guarantee at the very beginning that $\sum_{k=1}^{L} Y_k = 1$ which means that we cannot read estimates of ∇Y_k from the diffusion matrix and must obtain them from another extra terms in the approximation. However, after several limit passages, we get the equality $\sum_{k=1}^{L} Y_k = 1$.

In the last limit passage we need to prove extra estimates of the pressure. We proceed as in the case of steady compressible Navier–Stokes(–Fourier) system, therefore we can get existence of weak solutions for $\gamma > \frac{4}{3}$ (note that we consider the Dirichlet boundary conditions) and existence of variational entropy solutions for any $\gamma > 1$. The last limit passage is basically the same as for the system without chemical reactions. The details of the proof can be found in Chapter 12. Note also that in [Piasecki Pokorný 2018] similar results (in fact, even slightly better ones) were shown for the Navier boundary conditions.

In [Guo et al 2018] the authors removed the assumption on the that the molar masses are the same. They were able to obtain existence of a weak solution, however, for quite large γ (> 2) and they do not work with the entropy inequality. In [Zatorska 2011] the author proved existence for the isothermal case, however, for $\gamma > \frac{7}{3}$ only. Finally, let me mention the series of papers [Kucher et al 2012], [Mamontov Prokudin 2013] and [Mamontov Prokudin 2014], where similar problems were studied for two species without chemical reactions, however, with different velocities for each species.

Chapter 4

Conclusion

The presented thesis contains mostly the existence results for equations describing steady flow of heat conducting compressible viscous Newtonian fluid, i.e. for the steady compressible Navier–Stokes–Fourier system under different boundary conditions. It deals with existence of solutions for large data, i.e., we do not try to construct solutions which are close to some known regular solutions. This fact leads to the necessity of considering the weak solutions and their generalizations instead of the classical or strong ones.

The formulation of the problem as a system of balance laws allows several formulations which are equivalent on the level of classical or strong solutions: the balance of mass (the continuity equation) and the balance of the linear momentum can be combined with the internal energy balance, total energy balance or the entropy balance. These three possible formulations are not any more equivalent on the level of weak solutions. It is, however, important to recall that all three types of solutions possess the property of the weak-strong compatibility. In the thesis, it is demonstrated that in different situations (properties of viscosities, different values of physical constants and different boundary conditions for the velocity) existence of solutions for different formulations can be obtained.

Based on similar situation in the evolutionary problems, it is demonstrated that the entropy inequality is an extremely effective tool in this type of problems. It provides useful estimates which are stronger than estimates coming from the energy inequality, and, in addition, the solution based on the entropy inequality (together with a partial information from the total energy balance) exists for the largest set of parameters (the value of the adiabatic constant and the speed growth of the heat conductivity with respect to the temperature).

This observation, together with tools used in the mathematical fluid mechanics and thermodynamics for evolutionary problems (density estimates based on the Bogovskii operator, effective viscous flux identity, renormalized solution to the continuity equation and oscillation defect measure estimates) and tools specific for steady problems (potential estimates of the density up to the boundary, possibility to use total energy balance in the weak formulation) enabled to understand relatively well the problems of existence of solutions for steady systems describing flow of heat conducting compressible Newtonian fluid.

This technique also helped to study closely related problems like existence of time periodic solutions for heat conducting compressible fluids with physically realistic parameters (including at least the monatomic gas model) or obtain results for more complex systems as chemically reacting gaseous mixtures or flow of gases with radiation. The thesis also includes a very specific result dealing with formulation of the problem with the internal energy balance which was actually the first real large data existence result for steady equations of compressible heat conducting fluids. All the presented results inspired other scientists who used the therein developed tools to study similar problems.

The thesis is divided into two parts. In the first, introductory one, after a short description of the studied problems, the known existence results are formulated in dependence on the parameters of the problem. Furthermore, the main ideas of the existence proofs as well as the necessary tools used therein are briefly explained. Due to the complexity of the problem, all the proofs are long and technically complicated. The second part then contains eight selected most important papers from the perspective of the author of the thesis. They were mostly published in high-ranked journals from the field of partial differential equations or mathematical fluid mechanics and were obtained in collaboration with different leading experts in the field of mathematical fluid mechanics and thermodynamics.

Indeed, especially in the mathematical theory for models of complex fluids, many important questions and problems remained unsolved or even untouched. Dealing with them can bring development of new tools and techniques which may lead to improvement of results for the "simpler" problems, but for sure, will also open new perspective and enable to study problems which, nowadays, we even do not dare to dream about.

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