Ex 1: $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^n, \; n \in \mathbb{N}$ $\Big\}$ $f'(x) = n x^{n-1}$, $f''(x) = n(n-1)x^{n-2}$

$n = 2$: $x_0 = 0$ a non-degenerate crit. pts of f
$n \in \mathbb{N} \setminus \{2\}$: $x_0 = 0$ a degenerate —″—

Ex 2: Consider $f_1(x) = x^2$, $f_2(x) = x^3$, $g = ax + b$ for $a, b \in \mathbb{R}$.
$f_1: \mathbb{R} \to \mathbb{R}$
$f_2: \mathbb{R} \to \mathbb{R}$
$g: \mathbb{R} \to \mathbb{R}$

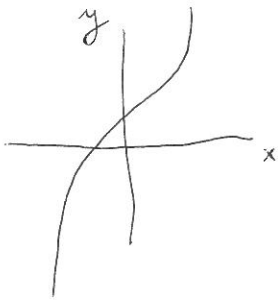
f_1, f_2 have just one and the same crit. pt. $x_0 = 0$.

Let us perturb / deform f_1, f_2: $g_1 := f_1 + g = x^2 + ax + b$,
$g_2 := f_2 + g = x^3 + ax + b$.

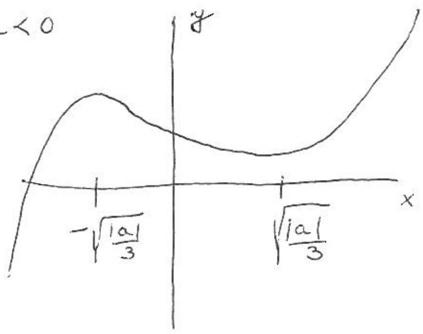
Crit. pts of g_1: $2x + a = 0 \Leftrightarrow x_0 = -\frac{a}{2}$, this pt is non-degenerate since $g_1''(-\frac{a}{2}) = 2 \neq 0$.

Crit. pts of g_2: $3x^2 + a = 0 \Leftrightarrow x_0^{\pm} = \pm\sqrt{-\frac{a}{3}}$. Since $x \in \mathbb{R}$, we have no critical pt for $a > 0$ of g_2, for $a < 0$ we have $g_2''(x_0^{\pm}) = \pm 6\sqrt{\frac{|a|}{3}} \neq 0 \Rightarrow$ $x_0^{\pm}$ are non-degenerate crit. pts

$a > 0$



$a < 0$



Observation: Non-degenerate pts are stable under perturbation / deformation.

Transversality and oriented intersection theory

<u>Def</u>: $X, Y \subseteq Z$ smooth submanifold of (smooth manifold) Z. Then X intersects Y transversally at $x \in X \cap Y$ if $T_x X + T_x Y = T_x Z$. X, Y are transverse if they intersect transversally at all points, write $X \pitchfork Y$. Non-intersecting manifolds are trivially transversal.

Transversality is defined as relative to the ambient manifold Z. E.g., Non $\parallel$ lines in $\mathbb{R}^2$ intersect transversally, but not when embedded in $\mathbb{R}^3$.

<u>Theorem</u>: $X, Y \subseteq Z$ non-empty intersecting (smooth) submanifolds in Z, $X \pitchfork Y$. Then $X \cap Y$ is a submanifold as well, and
$\operatorname{codim}(X \cap Y) = \operatorname{codim} X + \operatorname{codim} Y$ (codim w.r.t. Z).

<u>Pf</u>. $\dim X = k$, $\dim Y = m$, $\dim Z = n$. The aim: $\forall\, p \in X \cap Y$ has a local chart description. The embedding $i: X \hookrightarrow Z$ can be written as $i(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0)$ in a local chart $(x_1, \dots, x_k, 0, \dots, 0) \in U \subseteq Z$, $p = (0, \dots, 0) \in U$.
Consider the map $f: U \to \mathbb{R}^{n-k}$ given by $f(x_1, \dots, x_m) = (x_{m-k+1}, \dots, x_m)$ in the same coordinates; then $f^{-1}(0) = U \cap X$. Similarly, because $Y \subseteq Z$ is an embedded submanifold, there $\exists$ a map $g: U \to \mathbb{R}^{n-m}$ s.t. $g^{-1}(0) = U \cap Y$. By construction, $(df)_z$ and $(dg)_z$ are surjective $\forall z \in U$, i.e. zero is the regular value of both f and g.
By hypothesis, $p \in X \cap Y$ is a regular point for the map $(f, g): U \to \mathbb{R}^{2n-k-m}$ and so there are no critical points for some $\tilde{U}$, $p \in \tilde{U} \subseteq U$. Then $(f, g)|_{\tilde{U}}^{-1}(0, 0) = X \cap Y \cap \tilde{U}$ is a (coordinate chart for) a submanifold of Z, so we have a natural map to parametrize $X \cap Y$. ∎

Even if two manifolds do not intersect transversally, it is possible to deform them to manifolds with transverse intersection:

<u>Def</u>: A deformation of a submanifold $X \subseteq Y$ is a smooth function $i: X \times S \to Y$, where $S \subseteq \mathbb{R}^n$ is an open ball with $0 \in S$, $i_s(x) := i(x, s)$ is an embedding $\forall s \in S$ and $i_0: X \to Y$ is the initial inclusion.

Deformations are very easy to construct:

Lemma: X - smooth compact, $i: X\times S \to Y$ a smooth function s.t. $i_0(x) := i(x,0)$ is the embedding $X \hookrightarrow Y$. Then for $\epsilon > 0$ small enough, i is a deformation of X when restricted to $X \times S^\epsilon$ (S^ϵ .. open ball around 0, radius ϵ)

Pf. X is compact $\Rightarrow$ i_s, $i_s(x) := i(x,s)$ is a family of proper maps (inverse image of compact is compact set), we have to show for all small enough s i_s are immersions and bijections (one to one maps)
$\forall x \in X$, let $U_x \times S^{\epsilon_x} \subseteq X \times S$ such that $d(i_s)_{x'}$ is of full rank ($\forall x' \in U_x$ $\forall s \in S^{\epsilon_x}$). This exists because $d(i_0)_x$ is of full-rank at $\forall x \in X$, and the determinant is a continuous fcn; if there $\exists$ a square submatrix of $d(i_0)_{x'}$ with non-vanishing determinant so does $d(i_s)_{x'}$ (for small enough s, since $i(x,s)$ is smooth in s and x.) X is compact $\Rightarrow$ $\exists$ finite number of charts covering X, and choose the minimum of ϵ_x to find an ϵ s.t. if $s \in S^\epsilon$, i_s is an immersion.
Assume that $\forall \epsilon > 0$ $\exists$ $s \in S^\epsilon$ s.t. i_s is not injective. Define a mapping $F: X\times S \to Y \times S$ by $F(x,s) = (i_s(x), s)$, and consider two pointwise distinct sequences of points in X, $\{x_i\}_i$, $\{y_i\}_i$ s.t. $F(x_i, s_i) = F(y_i, s_i)$ with $\{s_i\}_i$ any sequence $s_i \to 0$. Passing to a subsequences, the compactness of X guarantees $x_i \to x$, $y_i \to y$ (subsequences converge.) Since i_0 is injective and maps to the values both x and y, we have $x = y$.
At $(x,0)$, $dF_{(x,0)}$ is injective since i_0 is injective, so by the Inverse fcn theorem is F injective in a neigh. of $(x,0)$. This contradicts the fact $x_i \neq y_i$ $\forall i$. ∎

The next theorem (without proof) consists of perturbing manifolds on a small open set:

Theorem (ϵ-neigh. theorem): X - compact smooth man., embedded in $\mathbb{R}^m$. Let $X^\epsilon := \{z \in \mathbb{R}^m : |z - x| < \epsilon \text{ for some } x \in X\}$, $|\ \ |$ - Euclid norm. Then there $\exists$ a smooth map $\pi: X^\epsilon \to X$, sending $z \in X^\epsilon$ to the unique closest point to z in X. Moreover, π is a submersion, i.e. it has no critical points.

(38) <u>Theorem</u>: X - compact subman of Y, Y embedded in $\mathbb{R}^n$ with an ϵ - neigh Y^ϵ, and a map $\pi: Y^\epsilon \to Y$. Define a deformation $i: X \times B^n \to Y$ ($B^n = S$ denotes the unit ball in $\mathbb{R}^n$) of X: $i_s(x) := i(x,s) = \pi(x + \epsilon s)$. Let $Z \subseteq Y$ a smooth subman., then for almost all $s \in S$ the manifold X_s defined by the embedding $i_s(X)$ satisfies $X_s \pitchfork Z$.

<u>Pf</u>: We note i is a submersion: a consequence of the fact that π is a submersion (by the previous ϵ-neigh. theorem), and by observing that even for fixed x the map $(x,s) \mapsto x + \epsilon s$ spans all directions of Y^ϵ and so it is a submersion as well; a composition of submersions is a submersion $\Rightarrow$ $\forall$ point in Y is a regular value of $i \Rightarrow i^{-1}(Z)$ is a submanifold of $X \times B^n$.

Consider the projection map $\rho: X \times B^n \to B^n$, $(x,s) \mapsto \rho(x,s) = s$.
When $s \in B^n$ is a regular value of the map $\rho|_{i^{-1}(Z)}$, we have $X_s \pitchfork Z$: then since $i^{-1}(Z)$ is a manifold, Sard's theorem does the work and finishes our proof.

The proof of the last claim: denote $W := i^{-1}(Z)$. The hypothesis of regularity implies (at $s \in B^n$) that $\forall (x,s) \in W$, the map $d\rho_{(x,s)}|_W$ is surjective. Therefore, adding the kernel of $d\rho_{(x,s)}|_W$, which is inside $T_x X \times 0$, to $T_{(x,s)} W$, we get

$$(T_x X \times 0) + T_{(x,s)} W = T_x X \times \underset{\text{"}T_0 B^n \simeq T_s B^n\text{"}}{\mathbb{R}^n},$$

the full tangent space of $X \times B^n$. We notice $T_{(x,s)} W = di^{-1}_{(x,s)}(T_{\pi(x+s\epsilon)}(Z))$: let $j: W \to X$ be the inclusion. Then $i \circ j$ is a submersion, so $d(i \circ j): T_{(x,s)} W \to T_{\pi(x+s\epsilon)} Z$ is surjective, so the assertion follows by the chain rule & $dj_{(x,s)} = Id$. Applying this to, get

$$(T_x X \times 0) + di^{-1}_{(x,s)} T_{\pi(x+s\epsilon)} Z = T_x X \times \mathbb{R}^n$$

$$\Rightarrow di_{(x,s)}(T_x X \times 0) + T_{\pi(x+\epsilon s)} Z = di_{(x,s)}(T_x X \times \mathbb{R}^n)$$

$$\Rightarrow T_{\pi(x+s\epsilon)}(X_s) + T_{\pi(x+\epsilon s)} Z = T_{\pi(x+s\epsilon)} Y,$$

where the last equality follows from the surjectivity of $di_{(x,s)}: T_x X \times \mathbb{R}^n \to T_{\pi(x+\epsilon s)} Y$. This is just the transversality condition, the proof is complete. ▨

Intersection theory on oriented manifolds. On $\mathbb{R}^n$, $\{a_1, .., a_n\} = a$ and $\{b_1, .., b_n\} = b$ two bases. If the determinant of the transition matrix from a to b is positive, a, b are called to have the same orientation; $a \to [a]$, then $[a] = [b]$. If the det is negative, a, b have the opposite orientation ($[a] = -[b]$.) So an equivalence class of oriented bases is the orientation (of $\mathbb{R}^n$)

X - smooth man, orientation on X is a smooth choice of orientations of $T_x X$, $x \in X$. This means $\forall x \in X \; \exists \; \varphi : U \to X$, $x \in \varphi(U)$, an open neigh. such that $d\varphi_x : \mathbb{R}^k \to T_x X$ preserves orientation $\forall x \in U$.

V, W - oriented vector spaces, $V \times W$ is oriented by:

$[v_1, .., v_n]$ orient. of V
$[w_1, .., w_n]$ — " — W $\Big\}$ $[(v_1, 0), .., (v_n, 0), (0, w_1), .., (0, w_m)]$ orient. of $V \times W$ (product orient.)

Direct sums induce product of orientation as well:
$[v_1, .., v_n]$ for V, $[w_1, .., w_m]$ for W, $[v_1, .., v_n, w_1, .., w_m]$ for $V \oplus W$

Back to (smooth, without boundary) manifolds: $X, Z \subseteq Y$

if X and Z intersect transversally, $\Leftarrow$ assume $\dim X + \dim Z = \dim Y$
$\dim(X \pitchfork Z) = 0$ ($\Rightarrow$ discrete set of pts), so $T_x X \oplus T_x Z = T_x Y$.

orientation number of $x \in X \cap Z$ $= 1$ if $T_x Y$ and $T_x X \oplus T_x Z$ have the same orientation)
$= -1$ otherwise;

(orientation number depends on the order of X, Z)

Global counting of orientation numbers:

<u>Def.</u> Let $X \pitchfork Z$. The intersection number of X and Z, $I(X, Z)$, is the sum of orientation numbers of the points in $X \cap Z$.

if $X \pitchfork Z$, $\dim X + \dim Z = \dim Y + 1$, then $X \cap Y$ is an oriented 1-manifold. Because $\forall$ 1-dim manifolds are diff. to circles/segments, the intersection numbers at the boundary of 1-man are zero.

Lemma: Let $i : X \times \langle 0,1\rangle \to Y$, $i \equiv i(s,x) = i_s(x)$ for $x \in X$, be a smooth fcion in X and continuous in $\langle 0,1\rangle \ni s$. Then if $X_0 = i_0(X)$, $X_1 = i_1(X)$, and both $X_0 \pitchfork Z$ and $X_1 \pitchfork Z$, we have $I(X_0, Y) = I(X_1, Y)$.

($\Rightarrow I$ is "homotopy invariant")

Pf: $W := i(X, \langle 0,1\rangle)$ is a (topological) submanifold of Y, $\dim W + \dim Y = \dim Z + 1$. The last theorem implies $\exists$ of deformation W' of W, $W' \pitchfork Y$. Moreover, since $X_0 \pitchfork Z$ and $X_1 \pitchfork Z$, this can be made such that $W' = W$ outside of $X \times \langle \epsilon, 1-\epsilon\rangle$ for some $\epsilon > 0$ (by multiplying the deformation of the last theorem by a bump fcion, which is zero outside of $\langle \epsilon, 1-\epsilon\rangle$.

Since $W' \pitchfork Y$, the intersection $W' \pitchfork Y$ is a 1-manifold, its boundary is $i(X \times \{1\}) - i(X \times \{0\}) = X_1 - X_0$. Because the intersection numbers of the boundary of 1-manifold is always zero, $I(X_1, Y) = I(X_0, Y)$. ∎

If $X \not\pitchfork Z$, we can (by the last theorem) deform Z in Y into some homotopic Z' s.t. $X \pitchfork Z'$. Then we define $I(X,Z) = I(X,Z')$.

Def: $f : X \to Y$ a smooth map (of smooth manifolds.) If $y \in Y$ is a regular value of f, the degree of f a $y \in Y$ is $\deg_y(f) = \sum_{x \in f^{-1}(y)} \operatorname{sign} \det df_x$

Because degree of a function is an intersection number, we have

Lemma: For $y_0, y_1 \in Y$ two regular values of f, $\deg_{y_0} f = \deg_{y_1} f$. Therefore, we define a global degree of f, $\deg f$. Moreover, it is a homotopy invariant (i.e., if $f \underset{\text{homotopic}}{\sim} g$, then $\deg f = \deg g$.)

Def: (Euler characteristic) The Euler charact. $\chi(X)$ of X is $\chi(X) := I(\Delta, \Delta)$, where $\Delta = \{(x,x) \mid x \in X\} \subseteq X \times X$ is a submanifold (the diagonal submanifold)

Poincaré-Hopf theorem

Topology of a smooth manifold $\leftrightarrow$ smooth vector fields on M ($M \to TM$, $x \mapsto T_x M$ smooth in M)

E.g.: $(\mathbb{R}^n, \langle\, ,\rangle)$, $v = v(x)$ a vector field, x_0 a zero of v: $v(x_0)=0$.
index of $v \in C^\infty(\mathbb{R}^n, T\mathbb{R}^n)$, $\mathrm{ind}_x(v)$, at the point x_0, is defined as the degree of the map $x \mapsto \frac{v(x)}{|v(x)|}$ from S^{n-1} to S^{n-1}. In fact,
(x_0 is in the interior of S^{n-1})
this map corresponds to

$$f_* : \underset{\mathbb{Z}}{\underset{\text{\rotatebox{90}{$\cong$}}}{H_{n-1}(S^{n-1}, \mathbb{Z})}} \xrightarrow{\quad} \underset{\mathbb{Z}}{\underset{\text{\rotatebox{90}{$\cong$}}}{H_{n-1}(S^{n-1}, \mathbb{Z})}}$$

$$\mathbb{Z} \xrightarrow{d\cdot} \mathbb{Z}$$

$$\mathrm{ind}_{x_0}(v(x)) = \deg(f_*) = d$$

On M: $\varphi : \mathcal{U} \to M$ a chart around x_0, the zero of $v \in C^\infty(M, TM)$.
Then $\mathrm{ind}_{x_0}(v) := \mathrm{ind}_{\varphi^{-1}(x)}(\varphi^* v)$, where $\varphi^* v = (d\varphi_u^{-1})\, v(\varphi(u))$,
u ... coordinates on $\mathcal{U}$

<u>Theorem</u>: (Poincaré-Hopf) M - compact orient. man., $\vec{v}$ a vector field on M with finitely many zeroes $\{x_0, x_1, \ldots, x_n\}$. Then $\chi(M) = \sum_{j=1}^{n} \mathrm{ind}_{x_j}(v)$.

Let $v \in C^\infty(M, TM)$, $x \mapsto (x, v(x))$; since M is compact, the map $x \mapsto (x, v(x))$ is proper (and it is injective, because the first component of this map is identity) and embeds M into TM ($\Rightarrow$ M is diff to $\{(x, v(x)) \mid x \in M\}$ in TM)

Zeroes of v correspond to the intersection points of v with ~~the~~ zero vector field $v_0 = \{(x,0) \mid x \in M\}$. The zero is non-degenerate if $(dv)_{x_0} : T_{x_0}M \to T_{x_0}M$ is a bijection (? why $(dv)_{x_0}$ maps $T_{x_0}M$ to $T_{x_0}M$?)

<u>Lemma</u>: Let x_0 be a zero of $v \in C^\infty(M, TM)$. Then x_0 is non-degenerate if and only if $M_v \pitchfork M_{v_0}$ ($M_v, M_{v_0} \subseteq TM$ via v, v_0) at $(x_0, 0)$. In this case, $\mathrm{ind}_{x_0}(v)$ is the orientation number of $(x_0, 0)$ in TM of $M_v \pitchfork M_{v_0}$.

<u>Pf</u>: v is non-degenerate at x_0 if and only if $T_{(x_0,0)} M_v + T_{(x_0,0)} M_{v_0} =$
$= T_{(x_0,0)}(TM) = T_{x_0}M \times T_{x_0}M$. The reason is the following observation:

(42) The tangent space of M_v at $(x_0, 0)$ is the graph of $(dv)_{(x_0,0)}$, i.e. $\{(w, (dv)_{x_0}(w)) \mid w \in T_{x_0}M\}$, whereas the tangent space of M_{v_0} is $\{(w, 0) \mid w \in T_{x_0}M\}$; the transversality condition holds iff $(dv)_{x_0}$ is bijective.

As for the second part, the orientation number of $(x_0, 0)$ equals 1 if $(dv)_{x_0}$ preserves orientation, and -1 if reverses: let $[(\alpha_1, (dv)_{x_0}(\alpha_1)), \ldots, (\alpha_n, (dv)_{x_0}(\alpha_n))]$ be a positively oriented basis for TM_v at $(x_0, 0)$, and consider the induced basis for $T_{(x_0,0)}M_{v_0} + T_{(x_0,0)}M_v$ at $(x_0, 0)$ given by

$$[(\alpha_1, 0), \ldots, (\alpha_n, 0), (\alpha_1, (dv)_{x_0}(\alpha_1)), \ldots, (\alpha_n, (dv)_{x_0}(\alpha_n))] =$$

$$= [(\alpha_1, 0), \ldots, (\alpha_n, 0), (0, (dv)_{x_0}(\alpha_1)), \ldots, (dv)_{x_0}(\alpha_n)] = \operatorname{sgn}(\alpha) \cdot \operatorname{sign} dv_{x_0}(\alpha),$$

and the claim follows.

Around a zero x_0 of v, we write $v(x_0 + w) = dv_{x_0}(w) + \epsilon(w)$, $\frac{\epsilon(w)}{|w|} \to 0$ for $w \to 0$. Consider the map

$$F_t(w) = \frac{dv_{x_0}(w) + t\epsilon(w)}{|\quad \text{---}\ " \ \text{---} \quad|_{x_0}}$$

$\leftarrow$ choose any metric at $T_{x_0}M$.

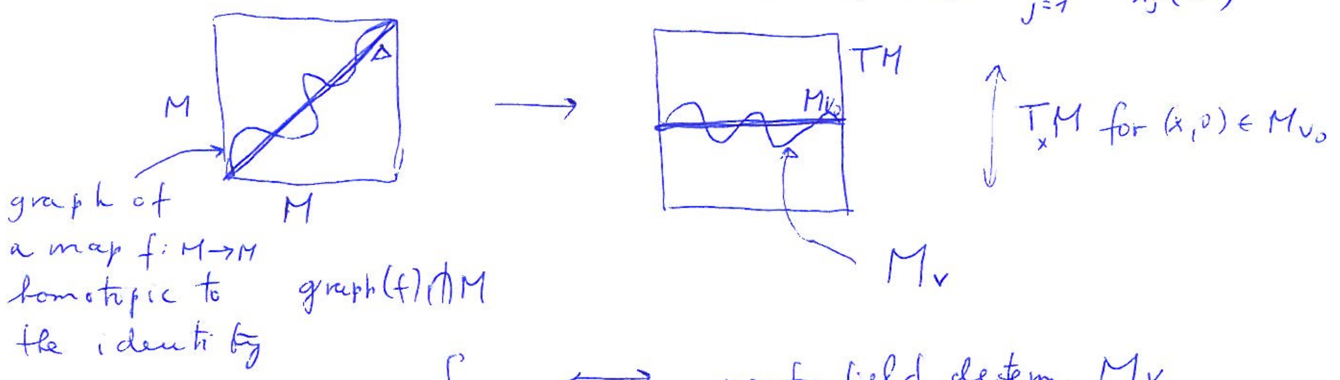
F_t is a smooth map $F_t : S_\epsilon \to S^m$. At $t = 1$, $\deg(F_1) = \operatorname{ind}_{x_0}(v)$. At $t = 0$, $F_0(w) = \frac{dv_{x_0}(w)}{|\ -\!"\!-\ |_{x_0}}$. Since dv_{x_0} is a linear isomorphism of vector spaces isomorphic to $\mathbb{R}^m$, it is either homotopic to the $\mathrm{Id}_{\mathbb{R}^m}$ or the reflection on $\mathbb{R}^m$, so that degree of F_0 is ± 1 (according to the non/preservation of orientation. Since the map F_t is a homotopy and degree is a homotopy invariant, the claim follows. ∎

Lemma: Assume x_0 is zero of v, which is isolated ($\exists\, U \subseteq M$, $x_0 \in U$, containing no other zero of v except x_0.) Then there $\exists$ v_1 such that $v_1 = v$ outside a compact subset of U and v_1 has only non-degenerate zeros in U ($v, v_1 \in C^\infty(M, TM)$.)

Pf: Sard's theorem for $v: M \to TM$: choose $a \in \mathbb{R}^m$ s.t. $-a$ is a regular value of v. Then $v_1(x) := v(x) + a$ only has non-degenerate zeros, since if $v_1(x_0) = 0$ then $v(x_0) = -a \Rightarrow dv(x_0)$ is of full rank and so is $dv_1(x_0)$ (because v_1 and v differ by a constant.) Assume $\rho: U \to \mathbb{R}$ is a compact. supp. fun on U, $\rho = 1$ on a small neigh. of x_0. Then $v_1(x) = v(x) + \rho(x)a$. This is the required vector field. ■

Because v_1 is homotopic to v by $v_t(x) := v(x) + t\rho(x)a$, we can define intersection number of v at x_0 through v_1 on U. By the previous lemma, there v_1 such that M_{v_1} intersects M_{v_0} transversally (we denote v_1 by v lateron.) But any M_v can be smoothly deformed into M_{v_0} by the (smooth) homotopy that multiplies v by a number smoothly varying from 0 to 1. Therefore $I(M_{v_0}, M_{v_0}) = I(M_{v_0}, M_v)$, which corresponds to $\sum_{i=1}^{n} \operatorname{ind}_{x_i}(v)$. The next theorem will show that $I(M_{v_0}, M_{v_0}) = I(\Delta, \Delta)$, completing the proof of Poincaré–Hopf:

$$\chi(M) = I(\Delta, \Delta) = I(M_{v_0}, M_{v_0}) = I(M_{v_0}, M_v) = \sum_{j=1}^{n} \operatorname{ind}_{x_j}(v)$$



$f \longleftrightarrow$ vector field determ. M_v

s.t. $I(\Delta, \text{graph}(f)) = I(M_{v_0}, M_v)$

by understanding diffeomorphic neighborhoods of Δ and M_{v_0}.

<u>Def</u>: $Z \subseteq Y \subseteq \mathbb{R}^n$ smooth embedded manifolds.
The normal bundle to Z in Y is the set
$$N(Z,Y) := \{(z,v),\ z \in Z,\ v \in T_z Y \text{ such that } v \perp T_z Z\}$$
in Eucl. metric on $\mathbb{R}^n$

<u>Theorem</u>: (tubular neigh. theorem) There exists a diffeom. from an open neigh. of Z in $N(Z,Y)$ onto an open neigh. of Z in Y.

<u>Pf</u>: Let $Y^\epsilon \xrightarrow{\pi} Y$, π a projection from the ϵ-neigh. theorem.
The map $h: N(Z,Y) \to \mathbb{R}^m$ is given by $h(z,y) = z + v$. in $\mathbb{R}^n$
Then $W := h^{-1}(Y^\epsilon)$ is an open neigh. of Z in $N(Z,Y)$.
The composition of function/mapping
$$W \xrightarrow{h} Y^\epsilon \xrightarrow{\pi} Y$$
is the identity on Z, so by the inverse function theorem h is diffeomorphism from an open neigh. of Z in $N(Z,Y)$ onto an open neigh. of Z in Y. ∎

The orthogonal complement to $T_{(x,x)}(\Delta)$ in $T_{(x,x)}(M,M)$ is the collection of vectors $\{(-v,v),\ v \in T_x M\}$, as can be seen by taking scalar products. The map
$$TM \to N(\Delta, M\times M)$$
$$(x,v) \mapsto ((x,x),(v,-v))$$
is a diffeomorphism, because it is smooth with smooth inverse. Then the tubular neigh. theorem $\Rightarrow$ $\exists$ a diffeomorphism of a neigh. of M_{v_0} in TM with a neigh. of Δ in $M\times M$ extending the diffeomorphism $M \to \operatorname{diag}(M\times M) = \Delta$. $(x,0) \mapsto (x,x)$. We can deform M_{v_0} inside its neigh. in TM into M'_{v_0} embedded in TM s.t. M_{v_0}, M'_{v_0} are homotopic and $M_{v_0} \pitchfork M'_{v_0}$. The set $\Delta' = \{(x,x'),\ x \in M_{v_0},\ x' \in M'_{v_0}\}$, which
" manifold
is a graph of a function $x \mapsto x'$, intersects Δ when M_{v_0} intersects M'_{v_0}, and with the same orientation (the neigh.

of Δ and M_{V_0} are diffeomorphic. Thus

$$I(M_{V_0}, M_{V_0}) = I(M_{V_0}, M'_{V_0}) = I(\Delta, \Delta') = I(\Delta, \Delta),$$

which completes the proof. ▨

Corollary: (Hairy ball theorem) $\forall$ smooth vector field on S^2 vanishes at some point.

Pf: Because $\chi(S^2) = 2$, $\forall$ smooth vector field must have at least one zero (otherwise the sum of its indices is zero), contradicting Poincaré-Hopf. ▨