①

Continuity, connectedness, compactness

$\mathbb{R}^k = \{(x_1, \dots, x_k) \mid x_i \in \mathbb{R},\ 1 \le i \le k\}$, $k \in \mathbb{N}_0$, $\mathbb{R}^0 := \{0\}$.

Euclidean metric/norm: $d(x, x') := \left(\sum_{i=1}^{k} (x_i - x_i')^2\right)^{1/2} = \|x - x'\|$.

$x = (x_1, \dots, x_k)$, $x' = (x_1', \dots, x_k')$

$X, Y \subseteq \mathbb{R}^k$ subspaces; $f: X \to Y$ is continuous at $x \in X$ if $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \epsilon$ $(*)$

Ball at $x \in X$ of radius δ: $D_X(x, \delta) = \{x' \in X \mid d(x, x') < \delta\}$

$(*) \Leftrightarrow f(D_X(x, \delta)) \subseteq D_Y(f(x), \epsilon)$.

$f: X \to Y$ is continuous $\Leftrightarrow$ f is cont. at $\forall x \in X$.

<u>Ex</u>: a/ $f: X \to \mathbb{R}^k$ is continuous iff $f_i: X \to \mathbb{R} \subseteq \mathbb{R}^k$, $x \mapsto x_i$ are continuous; $\forall i = 1, \dots, k$

b/ The multiplication map $m: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous;

c/ The inversion $x \to \frac{1}{x}$ is continuous map on $\mathbb{R}^* = \mathbb{R} \setminus \{0\} \subseteq \mathbb{R}$.

d/ The inclusion $i: X \hookrightarrow Y$ of a subspace, $i(x) = x$ $\forall x \in X$, is continuous.

<u>Def</u>: a/ $x \in X$, $V \subseteq X$ neigh. of x if $x \in V$ and $\exists \delta > 0$ s.t. $D_X(x, \delta) \subseteq V$.

b/ $U \subseteq X$ is open (in X) if U is a neighborhood of $\forall$ of its points.

<u>Ex</u>: $X = \mathbb{Z} \subseteq \mathbb{R}$; then $D_{\mathbb{Z}}(n, \frac{1}{2}) = \{n\}$, $\forall$ sets $\{n\}$ are open in $\mathbb{Z}$.

<u>Th</u>: X, Y spaces, $f: X \to Y$. Then

a/ f is continuous at $x \in X$ $\Leftrightarrow$ $\forall$ neigh. N of $f(x)$, $f^{-1}(N)$ is a neigh. of x. (local characterization.)

b/ f is cont. if and only if $\forall V \subseteq Y$ open it follows $f^{-1}(V) \subset X$ is open (global characterization.)

<u>Pf</u>: a/ $\Rightarrow$ N a neigh. of $f(x)$, by def. $\exists \epsilon > 0$ s.t. $D_Y(f(x), \epsilon) \subseteq N$;

f is cont. at x $\Rightarrow$ (for $\epsilon > 0$) $\exists \delta > 0$ s.t.

$f(D_X(x, \delta)) \subseteq D_Y(f(x), \epsilon) \subseteq N$. Thus

$f^{-1}(N) \supset f^{-1}(f(D_X(x, \delta))) \supset D_X(x, \delta)$, so $f^{-1}(N)$ is a neigh. of x.

$\Leftarrow$ and b/ are proved analogously.

Remark: $\emptyset$ and X are open. Intersections of fin. many and union of ~~[illegible]~~ arbitrary number of open sets are open. This is a specialized case of top. spaces (we shall work with metrizable top. spaces.)

Def: a/ For $Y \subseteq X$, define the interior of Y in X by $\mathrm{int}_X(Y) := \bigcup_{\substack{U \subseteq Y \\ U \text{ open in } Y}} U$.

The boundary of Y in X: $B_X(Y) : X \setminus \{\mathrm{int}_X(Y) \cup \mathrm{int}_X(X \setminus Y)\}$

b/ $F \subseteq X$ is closed $\Leftrightarrow$ $X \setminus F$ is open.

c/ The closure of Y in X: $cl_X(Y) := \bigcap_{\substack{F \supset Y \\ F \subset X \text{ is closed}}} F$.

We note $cl_X(Y) = \mathrm{int}_X(Y) \cup B_X(Y)$.

In $\mathbb{R}^k$, there is part. useful system of open sets: $a \in \mathbb{Q}^k$, $q \in \mathbb{Q}$, the ball $D(a,q) = \{x \in \mathbb{R}^k \mid d(a,x) < q\}$ (bijective with $\mathbb{Q}^k \times \mathbb{Q} = \mathbb{Q}^{k+1}$, thus countable.) $\forall$ open $W \subseteq \mathbb{R}^k$ is a union of subsets a/ $D(a,q)$, or $cl_{\mathbb{R}^k}(D(a,q))$, for suitable subsets of $\mathbb{Q}^{k+1}$ (so countable, in part.)

Let $X \subseteq \mathbb{R}^k$ and $(U_\alpha)_{\alpha \in J}$ be a family of open sets in $\mathbb{R}^k$, $\bigcup_{\alpha \in J} U_\alpha \supset X$. Then there exists a countable family $(V_i)_{i \in \mathbb{N}}$ with a/ $\bigcup V_i \supset X$, and b/ $\forall i \ \exists \alpha \in J$ s.t. $V_i \subset U_\alpha$.

lemma (pasting) $X = X_1 \cup X_2$, $X_i \subseteq X$ closed for $i = 1, 2$. Let $f : X \to Y$ be ~~[illegible]~~ s.t. $f|_{X_i}$ are continuous for $i = 1, 2$. Then f is continuous.

Def: A continuous bijection $f : X \to Y$ is called homeomorphism if its inverse $f^{-1} : Y \to X$ is also continuous. We write $X \approx Y$ if there exists a homeomorphism between X and Y.

Ex: $T := \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| + |x_2| = 1\} \subseteq \mathbb{R}^2$

$S^1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$

$T \approx S^1$ $\quad f : S^1 \to T \quad f(x_1, x_2) = \left(\frac{x_1}{|x_1| + |x_2|}, \frac{x_2}{|x_1| + |x_2|} \right)$

$g : T \to S^1 \quad g(x_1, x_2) = \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \right)$

$f \circ g = Id_T$, $g \circ f = Id_{S^1}$

Extend to $\mathbb{R}^2 \setminus \{0\}$, see both fctions are continuous $\Rightarrow$ f, $g = f^{-1}$ ~~[illegible]~~ is homeom.

Ex: $f : [0,1) \to S^1$, $\theta \mapsto e^{2\pi i\theta} = (\cos(2\pi\theta), \sin(2\pi\theta))$
in $\mathbb{R}$

f is continuous bijection, but $[0,1) \not\approx S^1$, because (for example) deletion of a pt in $[0,1)$ separates $[0,1)$ on two connected components while deletion of any point in S^1 does not (see the relation between connectedness and homeomorphism later on.)

$\left.\begin{array}{l} X \subseteq \mathbb{R}^k \\ Y \subseteq \mathbb{R}^l \end{array}\right\}$ product $X \times Y \subseteq \mathbb{R}^{k+l}$; the projections $\begin{array}{l} p_X : X \times Y \to X \\ p_Y : X \times Y \to Y \end{array}$ are continuous.

Ex: $(0,1) \not\approx [0,1)$. On the other hand, $(0,1) \times [0,1) \approx [0,1] \times [0,1)$.

The explicit homeomorphism $f : (0,1) \times [0,1) \to [0,1] \times [0,1)$ is constructed as follows:

 

$$f(x_1,x_2) = \begin{cases} \left(\frac{x_2}{2}, 1 - 3x_1\right) & \text{for } 3x_1 \leq 1 - x_2, \\ \left(x_1 + (1-2x_1)\frac{2x_2 - 1}{2x_2 + 1}, x_2\right) & \text{for } 1 - x_2 \leq 3x_1 \leq 2 + x_2, \\ \left(1 - \frac{x_2}{3}, 3x_1 - 2\right) & \text{for } 3x_1 \geq 2 + x_2. \end{cases}$$

f is continuous (by pasting) and easy to see homeomorphism.

Def: $X \subseteq \mathbb{R}^k$ is compact if $\forall$ open covering $(U_\alpha)_{\alpha \in J}$ (of X), i.e. $U_\alpha \subset \mathbb{R}^k$ open and $\bigcup_\alpha U_\alpha \supset X$, $\exists$ finitely many $\alpha_1, \dots, \alpha_r \in J$ s.t. $U_{\alpha_1} \cup \dots \cup U_{\alpha_r} \supseteq X$.
Then $(U_{\alpha_i})_{i=1,\dots,r}$ is a subcovering (of X.)

(Compactness is an intrinsic property, does not depend on embedding of X in $\mathbb{R}^l$)

Several elementary properties related to compactness:

If $Y \subseteq X$ is compact then Y is closed in X.

If X is compact and $A \subseteq X$ is closed, then A is compact as well.

If X is compact and $f : X \to Y$ is continuous, then $f(X)$ is compact.

(Pf: $U_\alpha, \alpha \in J$ covering of $f(X) \subseteq Y$ by open subsets; then $f^{-1}(Y) \subseteq X$ is open because f is continuous, $\bigcup_{\alpha, \alpha \in J} f^{-1}(U_\alpha) = X$. By compact., $X = f^{-1}(U_{\alpha_1}) \cup \dots \cup f^{-1}(U_{\alpha_r})$ for some $r \in \mathbb{N}$, $\alpha_1, \dots, \alpha_r \in J$.
Then $\bigcup_{i=1}^{r} U_{\alpha_i} \supset f(X)$, so $f(X)$ is compact.)

If X is compact and $f : X \to Y$ is a cont. bijection then f is a homeomorphism.

(4) Theorem (Heine - Borel) : A subset of $\mathbb{R}^k$ is compact iff it is closed and bounded.

Def: A pair (U, V) (open subsets of X) is called a separation of X iff $U \neq \emptyset$, $V \neq \emptyset$, $U \cap V = \emptyset$ and $X = U \cup V$. A space is connected if it has no separation (or, the only open and closed subsets of X are $\emptyset$ and X.

Lemma: A space X is ~~connected~~ disconnected iff $\exists$ a cont. map onto : $f : X \to \{0,1\}$.

Pf: (U, V) is a separation (X is disconnected), define $f : X \to \{0,1\}$ by $f|_U = 0$ and $f|_V = 1$. Conversely, define a separation of X by $U = f^{-1}(0)$, $V = f^{-1}(1)$.

Rem: $f : X \to Y$ cont., X connected $\Rightarrow$ Y is connected. Let (U, V) be a separation of X, $W \subset X$ connected. Then $W \subseteq U$ or $W \subseteq V$.

For $(C_\alpha)_{\alpha \in J}$ a family of connected spaces (in a $\mathbb{R}^k$), $\bigcap_{\alpha \in J} C_\alpha \neq \emptyset$, then $C := \bigcup_{\alpha \in J} C_\alpha$ is connected.

Ex: The union of lines in $\mathbb{R}^n$ (passing through the origin) is a connected top. space.

A map $f : X \to Y$ is locally constant if $\forall x \in X$ $\exists$ $U \subseteq X$, $x \in U$ neighbor. s.t. $f|_U$ is a constant map. Locally constant $\Rightarrow$ continuous. For $f : X \to Y$, X connected and Y discrete : if f is locally constant, then f is constant.

Def: A cont. map $\gamma : \langle 0,1 \rangle \to X$, a path in X from x to y.

$$0 \mapsto x$$
$$1 \mapsto y$$

X is path connected if $\forall$ $x, y \in X$ $\exists$ a path from x to y in X.

Products of path connected spaces are path connected. If $f : X \to Y$ is continuous and X path connected, then $f(X)$ is path connected.

Lemma: A path connected space X is connected.

Pf: X - path connected but not connected, so $\exists f : X \to \{0,1\}$ contin. and onto. Let $x, y \in X$ with $f(x) = 0$, $f(y) = 1$ and $\gamma : \langle 0,1 \rangle \to X$ be a path from x to y. Then $f \circ \gamma$ is path in $\{0, 1\}$ from 0 to 1. $\forall$ path in $\{0,1\}$ is constant, because $\langle 0,1 \rangle$ is connected (and so locally constant $\Rightarrow$ constant)

A contradiction.

5 **Ex:** $X = A \cup B$, $A = \{(x,y) \in \mathbb{R}^2 \mid x=0,\ y \in \langle -1, 1 \rangle\}$,
$B = \{(x,y) \in \mathbb{R}^2 \mid y = \sin \frac{1}{x},\ x \in (0, 1\rangle\}$.

X is connected, but not path-connected.

If $\forall x \in X$ has a path connected neigh. (& X is connected) $\Rightarrow$ X is path connected

Smooth manifolds and maps

$U \subseteq \mathbb{R}^k$, $V \subseteq \mathbb{R}^\ell$ open ; a map $f: U \to V$ is smooth (or C^∞) if all (partial derivatives) $\dfrac{\partial^j}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_j}} : U \to \mathbb{R}^\ell$ for all $1 \le i_1, \dots, i_j \le k$ and $\forall j \in \mathbb{N}$ $\Big\}$ exist and are continuous

A neigh $\supseteq$ an open disk $\Rightarrow$ $\dfrac{\partial f}{\partial x_i}(x_1, x_2, \dots, x_k) = \lim\limits_{h \to 0} \dfrac{f(x_1, \dots, x_i + h, \dots, x_k) - f(x_1, \dots, x_k)}{h}$

Jacobian for $f = (f_1, \dots, f_\ell)$: $\left\{ \dfrac{\partial f_i}{\partial x_j}(x) \right\}_{\substack{1 \le i \le \ell \\ 1 \le j \le k}}$ – represent matrix of $Df = df$ at $x \in \mathbb{R}^k$
$df: \mathbb{R}^k \to \mathbb{R}^\ell$ a linear map

Def: $X \subseteq \mathbb{R}^k$, $Y \subseteq \mathbb{R}^\ell$, $x \in X$. Then $f: X \to Y$ is smooth at x if $\exists\, U \subseteq \mathbb{R}^k$ open, $x \in U$, and a smooth map $F: U \to \mathbb{R}^\ell$ s.t. $F|_{U \cap X} = f|_{U \cap X}$ (F = a smooth local extension of f at $x \in X$)
f is smooth ~~at~~ if f is smooth at x $\forall x \in X$.

(The smoothness of F is defined on U by $\uparrow$, local exten. of smoothness at x extends to smooth extension ~~on~~ some open neigh. of x)

Examples:

1/ $X \subseteq \mathbb{R}^k$, id_X is smooth because it extends to $id_{\mathbb{R}^k}$.

2/ f is smooth at $x \in X \Rightarrow f$ is cont. at x (smooth local ext. F is cont., and the restr. of continuous map is cont.)

3/ Assume: f is smooth at x, g is smooth at $f(x)$. Then $g \circ f$ is smooth at x: let $F: U \to \mathbb{R}^\ell$ smooth local ext. of f, $G: V \to \mathbb{R}^m$ smooth local ext. of g at $f(x)$, $f(x) \in V$, $V \subseteq \mathbb{R}^\ell$ open. F is cont. & V open $\Rightarrow$ $F^{-1}(V) \subseteq U$ is open in U $\Rightarrow$ $\mathbb{R}^k \supset F^{-1}(V) \xrightarrow{G \circ F} \mathbb{R}^m$ is smooth, and
— " — $\mathbb{R}^k$
because $G \circ F|_{F^{-1}(V) \cap X} = (g \circ f)|_{F^{-1}(V) \cap X}$, $G \circ F$ defines a smooth local ext. of $g \circ f$ at x.

Def: $f: X \to Y$ is a diffeomorphism if f is smooth homeomorphism and f^{-1} (which is cont.) is smooth. The notation is $X \cong Y$.

Examples:

1/ $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is a smooth homeomorphism; $f^{-1}(x) = \sqrt[3]{x}$ and so f^{-1} is not smooth.

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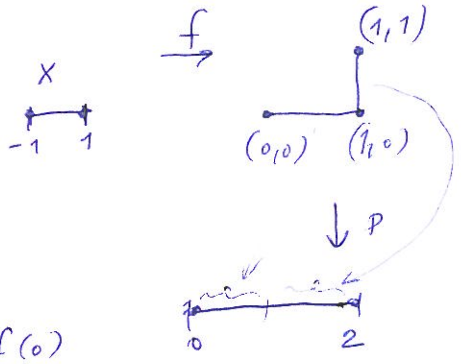
2/ $X = \langle -1, 1\rangle$, $Y = \{ (x,y) \in \mathbb{R}^2 \mid y = 0 \ \& \ 0 \le x \le 1$ or $x = 1 \ \& \ 0 \le y \le 1 \}$

$X \approx Y$ is easy to see. Let $f = (f_1, f_2) : X \to Y \subseteq \mathbb{R}^2$ is a diffeom, then $f(x) = (1,0)$ for some $x \in (-1,1)$ (for x a boundary point $X \setminus x$ is connected while $Y \setminus f(x)$ is not connected.) Assume (w.l.o.g.) that $f(0) = (1,0)$.

Then $f'(0) \neq 0$, because: let G be a smooth local extens. of f^{-1} at $(1,0)$, so by the chain rule (later reminded) $\left(\frac{\partial G}{\partial x_1}(1,0), \frac{\partial G}{\partial x_2}(1,0)\right)\begin{pmatrix} f_1'(0) \\ f_2'(0)\end{pmatrix} = 1$

it follows $(G \circ f)'(0) = G'(1,0) f'(0) = 1$ (since $G \circ f = Id$ near $0 = x \in X$.)



Let $p : Y \to \langle 0, 2\rangle$ be defined by

$p(x,y) = x$ for $y = 0$, $0 \le x \le 1$

$\qquad\quad y + 1$ for $x = 1$, $0 \le y \le 1$,

it is a homeomorphism. Assume w.l.o.g. $p \circ f$ is increasing. Consider $f'(0) := \lim_{h \to 0} \frac{f(h) - f(0)}{h}$,

so $\lim_{h \to 0, h < 0} \frac{f(h) - f(0)}{h}$ is a pos. mult. of e_1,

$\lim_{h \to 0, h > 0}$ — " — — " — neg. mult. of e_2

} e_1, e_2 standard bans of $\mathbb{R}^2$

This implies f is not diff.; however, there exist smooth maps $\langle -1, 1\rangle \to Y$, but no diffeom.

3/ $a, b > 0$, ellipse $E_{a,b} := \{ (x_1, x_2) \mid \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = 1 \}$ is diff. to $S^1 = E_{1,1}$: $f : S^1 \to E_{a,b}$ defined by $(x_1, x_2) \mapsto (a x_1, b x_2)$ is a diff. with the inverse $(x_1, x_2) \mapsto \left(\frac{x_1}{a}, \frac{x_2}{b}\right)$.

Goal of Diff. top.: classification of subsets of Euclidean space up to diffeomorphism.

Def. A space $M \subseteq \mathbb{R}^k$ is called a smooth manifold ($\dim M = m$) if $\forall x \in M$ $\exists W \subseteq \mathbb{R}^k$ s.t. $W \cap M$ is diff. to an open subset of $\mathbb{R}^m$.

$\varphi : \mathbb{R}^m \supseteq U \to W \cap M$ ... parametrization of $W \cap M \subset M$,

φ^{-1}: ... coordinate system — " — with chart $W \cap M$

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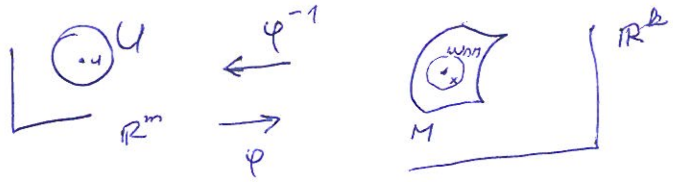
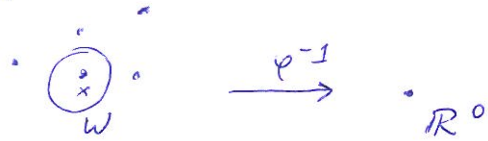


(We should talk about smooth submanifold of $\mathbb{R}^k$, and we shall see $k \geq m$ later on.)

Ex: 1/ $m=0$, W of $x \in M$ $(x \in W)$: $\mathbb{R}^2$

discrete subsets of Eucl. space

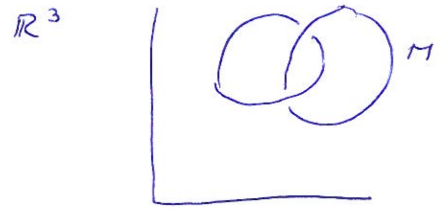


$W \cap M \simeq \mathbb{R}^0 \Rightarrow$
$(W \cap_M \simeq x)$

2/ $U \subseteq \mathbb{R}^m$ open is smooth man. of dim m, parametr. is id_U.

3/ M smooth man. of dim $= m$, $N \subseteq M$ open $\Rightarrow$ N is a smooth man., $\dim N = m$: $\varphi: U \to M$ parametr. at $x \in N \subseteq M$, then $\Phi := \varphi|_{\varphi^{-1}(N)}$ is a parametr. for N at x.

A 1-dim compact smooth man. $M \subseteq \mathbb{R}^3$ is called a link. If it is connected, it is a knot.



Rem: $f: M \to \mathbb{R}^l$ smooth at $x \in M$ in our sense $\Leftrightarrow$ $\exists$ a parametrization $\varphi: U \to M$ s.t. $f \circ \varphi: \mathbb{R}^m \supseteq U \to \mathbb{R}^l$ is smooth.

Ex: $S^{n-1} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 = 1\} \subseteq \mathbb{R}^n$ is a smooth (sub)manifold of dimension $(n-1)$.
For $y = (y_1, \dots, y_n) \in S^{n-1}$ $\exists\, 1 \leq i \leq n$ with $y_i \neq 0$; if $y_i > 0$, $W_i := \{x \in \mathbb{R}^n \mid x_i > 0\}$. $W_i \subseteq \mathbb{R}^n$ is open, $y \in W_i \cap S^{n-1} = \{x \in S^{n-1} \mid x_i > 0\}$.
Define $\psi_i: W_i \cap S^{n-1} \to D^{n-1} := \{x \in \mathbb{R}^{n-1} \mid \sum_{i=1}^{n-1} x_i^2 < 1\}$

$$(x_1, \dots, x_n) \mapsto \psi_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

$\psi_i(W_i \cap S^{n-1}) \subseteq D^{n-1}$, because $\sum_{i=1}^n x_i^2 = 1$ and $x_i > 0$. Also ψ_i is smooth because ψ_i can be extended to a smooth map $\Psi_i: W_i \to \mathbb{R}^{n-1}$. Define $\varphi_i: D^{n-1} \to W_i \cap S^{n-1}$

$$(x_1, \dots, x_{n-1}) \mapsto \varphi_i(x_1, \dots, x_{n-1}) =$$
$$= \left(x_1, \dots, x_{i-1}, \sqrt{1 - \sum_{j=1}^{n-1} x_j^2}, x_i, \dots, x_{n-1}\right)$$

(9) Then $\varphi_i = \psi_i^{-1}$, φ_i is smooth (no need to extend from D^{n-1}, because we stay in the domain away from the non-smooth point 0 of the square root.) The second case $y_i < 0$ can be treated analogously for

$$\tilde{\varphi}_i : (x_1, \dots, x_{n-1}) \mapsto \left(x_1, \dots, x_{i-1}, -\sqrt{1 - \sum_{j=1}^{n-1} x_j^2}, x_i, \dots, x_{n-1}\right).$$

The definition of tangent space for $x \in M \subseteq \mathbb{R}^k$, $x \in U \subseteq M$, $(TU)_x := \mathbb{R}^k$;

for $f : U \to \mathbb{R}^l$, $(df_x)(h)$ for $0 \neq h \in \mathbb{R}^k$ is equal to ordinary directional derivative Df: $(Df)(x)(h) = (df_x)(h)$.

$D = d$ is the linearization of f at x in the direction h:

$$f(x + th) = f(x) + Df(x)(th) + \rho(th),$$

$$\frac{1}{\|h\|} \lim_{t \to 0} \frac{\rho(th)}{t} = 0$$

($Df(x)$ is the affine approximation to f near x:

$$y \mapsto f(x) + Df(x)(y - x) \;)$$

There are many rules: transitivity, identity map, inverse function theorem, etc.

For example: $f : \mathbb{R}^k \supseteq M \to N \subseteq \mathbb{R}^l$ a smooth map between smooth manif.
$x \mapsto f(x) = y$

$x \in M$: $df_x : (TM)_x \to (TN)_{f(x)}$ is defined by extending f locally at x to a smooth map $F : W \to \mathbb{R}^l$, and $(df_x)(h) := dF_x(h)$, $h \in \mathbb{R}^k$.

It fulfills:
- df_x does not depend on the choice of $F : W \to \mathbb{R}^l$,
- $(dF_x)(TM_x) \subseteq TN_{f(x)}$.

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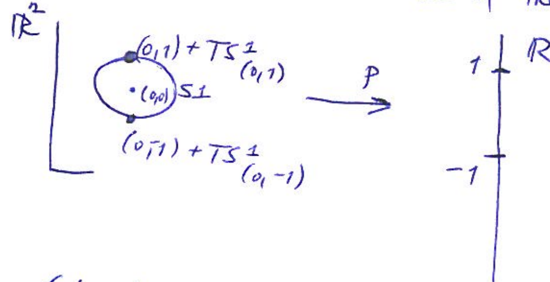
Regular/critical values and Sard's theorem

M, N - smooth manifolds, $f: M \to N$ smooth map ($\dim M = m$, $\dim N = n$)

<u>Def</u>: A point $x \in M$ is called a regular point of f if $(df)_x : T_xM \to T_{f(x)}N$ is surjective. Let $C = C(f) \subseteq M$ denote the set of points at which f is not regular ($C \equiv$ critical). Then $f(C) \subseteq N$ is the set of critical values of f, and $N \setminus f(C)$ is the set of regular values of f.

<u>Ex</u>: 1/ If $y \in N$ but $y \notin \mathrm{Im}(f)$ (so that $f^{-1}(y) = \emptyset$), y is by abuse of notation a regular value of f.

2/ Let $p|_{S^1} : S^1 \to \mathbb{R}$ be the restriction of the projection $p : \mathbb{R}^2 \to \mathbb{R}$ to S^1 ($S^1 \subset \mathbb{R}^2$).
$(x_1, x_2) \mapsto x_2$



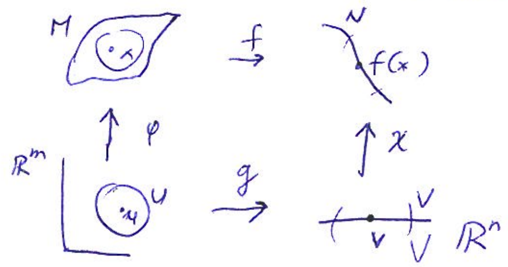
$x \in \mathbb{R}^2$; $\ker(dp_x) \simeq \mathbb{R} \times \{0\}$, but $TS^1_x \simeq \mathbb{R} \times \{0\}$ precisely for $x = (0,-1)$ and $x = (0,1)$. So $C(f) = \{-1, 1\}$, $(0, \pm 1)$ are critical pts of p (dp_x is onto for $x \neq (0, \pm 1)$, and is zero map at $(0, \pm 1)$).

3/ If $m < n$, $\forall x \in M$ is critical (the set of crit. values is $f(M)$.)

<u>Theorem</u>: M, N, f, m, n as above, x is a regular point of f. Then there $\exists$ parametrizations $\varphi : U \to M$ at x and $\chi : V \to N$ at y such that $(\chi^{-1} \circ f \circ \varphi)(x_1, \dots, x_m) = (x_1, \dots, x_n)$.

<u>Pf</u>: For U, V chosen, let us consider the commutative diagram ($g := \chi^{-1} \circ f \circ \varphi$)

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \varphi\uparrow & & \uparrow\chi \\ U & \xrightarrow{g} & V \end{array}$$



w.l.o.g.: $f(\varphi(U)) \subseteq \chi(V)$ (otherwise shrink U)

Since $dg_u : \underset{T_u\mathbb{R}^m}{\overset{\shortparallel}{\mathbb{R}^m}} \to \underset{T_{g(u)}\mathbb{R}^n}{\overset{\shortparallel}{\mathbb{R}^n}}$ is surjective (requires $m \geq n$, $\varphi(u) = x$),

there $\exists$ invertible matrices A, B: $A(dg_u)B = (I_n, 0_{n \times (m-n)})$

identity matrix

(dg_u can be brought into the form $(I_n, 0_{n\times(m-n)})$ by suitable row & column operations.) Replace φ by $\varphi\circ B$, χ by $\chi\circ A^{-1}$ (matrices represent linear maps), g is identified with the new map but we keep the old notation for it. Let $G: U \to \mathbb{R}^m$ be defined by $G(x) := (g(x), x_{n+1}, \dots, x_m)$ for $x = (x_1, \dots, x_m)$. Then $dG_u = Id_{\mathbb{R}^m} = I_m$ is invertible, so G is locally invertible at $u \in U \subseteq \mathbb{R}^m$ by the inverse mapping theorem. Let $G^{-1}: U' \to U$ be a local inverse (we might shrink U in order to have G invertible). Then there is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ U' \arrow[u, "\varphi\circ G^{-1}"] \arrow[r] \arrow[d, "\subseteq"'] & V \arrow[u, "\chi"'] \arrow[d, "\subseteq"] \\ \mathbb{R}^m \arrow[r] & \mathbb{R}^n \end{tikzcd}$$

and $(\chi^{-1}\circ f\circ\varphi\circ G^{-1})(x_1, \dots, x_m) = (x_1, \dots, x_n)$, because $(x_1, \dots, x_m) = G(y) = (g(y), x_{n+1}, \dots, x_m)$ for some $y \in U$, implies
$$(\chi^{-1}\circ f\circ\varphi\circ G^{-1})(G(y)) = g(y) = (x_1, \dots, x_n),$$
so replacing φ by $\varphi\circ G^{-1}$ gives the conclusion. ▨

<u>Remark</u>: $\exists$ maps s.t. f near a regular pt. is given by projection. Local description of smooth maps = singularity theory.

<u>Dűsledek</u>: $f: M \to N$ smooth, $y \in N$ regular value. Then $f^{-1}(y) \subseteq M \subseteq \mathbb{R}^k$ is a smooth manifold of dim $= k = m-n$.

Pf: $\forall x \in f^{-1}(y)$ is a regular pt of f. By the final parametr. φ in the last Theorem with $\varphi(u) = x$ and $u = (u_1, \dots, u_m)$, it follows that
$$\varphi\big|_{(\{(u_1,\dots,u_n)\}\times\mathbb{R}^{m-n})\cap U}$$ is a parametrization of $f^{-1}(y)$ at x with the inverse $\varphi^{-1}\big|_{(f^{-1}(y)\cap\varphi(U))}$. ▨

$M' \subseteq M$, $T_xM' \subseteq T_xM$ $\forall x \in M'$. The normal space to $M' \subseteq M$ at $x \in M'$ is $n(M', M)_x := \{v \in T_xM \mid \langle v, T_xM'\rangle = 0\}$, where $\langle x, y\rangle = \sum_{i=1}^{k} x_i y_i$ is the inner product ($M \subseteq \mathbb{R}^\ell$ for $\ell \gg 0$), $\dim n(M', M)_x = \dim M - \dim M'$.

(12) Corollary: f a smooth map, y its regular value;
$T_x(f^{-1}(y)) = \operatorname{Ker}(df_x)$, and

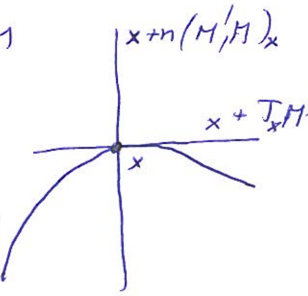
$$df_x\big|_{n(f^{-1}(y),M)_x} : n(f^{-1}(y),M)_x \to T_yN$$

is an isomorphism of vector spaces for all $x \in f^{-1}(y)$.

Pf: Consider the commutative diagram

$$\begin{tikzcd} f^{-1}(y) \arrow[r, "\subseteq"] \arrow[d, "f|_{f^{-1}(y)}"'] & M \arrow[d, "f"] \\ \{y\} \arrow[r, "\subseteq"] & N \end{tikzcd}$$

$\Rightarrow df_x(T_x(f^{-1}(y)))=0$ because $T(\{y\}_y)=\{0\}$



But $\dim T(f^{-1}(y)_x) = m-n = \dim(f^{-1}(y))$.
and $\dim(\operatorname{Ker}(df_x)) = m-n$ } dimension formula for linear spaces

Since $T(f^{-1}(y))_x \subseteq \operatorname{Ker}(df_x)$, these spaces are equal and df_x induces an isomorphism of vector spaces. ▨

Examples: 1/ $f : \mathbb{R}^m \to \mathbb{R}$

$x \mapsto f(x) = x_1^2 + \dots + x_m^2$), then $df_x = (\operatorname{grad} f)(x) = 2(x_1, \dots, x_m)$

$\Rightarrow$ for $r \neq 0$ is $(x_1, \dots, x_m)$ a regular value of f and for $r>0$
$f^{-1}(r) = \{x \in \mathbb{R}^m \mid \|x\|^2 = r\}$ is a smooth manifold in $\mathbb{R}^m$
of dim $= m-1$; for $r<0$ is $f^{-1}(r) = \emptyset$, and $f^{-1}(0) = \{0\}$
(diff. dimension than in Corollary 11.

2/ $M(n)$ ... vector space of $n\times n$ real matrices, $\cong \mathbb{R}^{n^2}$ and so $M(n)$ is a smooth man,
$\operatorname{Sym}(n) := \{B \in M(n) \mid B^T = B\} \subseteq M(n)$ a linear subspace of symmetric $n\times n$ matrices
(dim $= \frac{n(n+1)}{2}$)

$O(n) := \{A \in M(n) \mid AA^T = I\} \subseteq M(n)$
the subset of OG-matrices, $I = I_n$ the identity matrix

Lemma: $O(n) \subseteq \mathbb{R}^{n^2}$ is a smooth manifold of dimension $\frac{n(n-1)}{2}$.

Pf: $f : M(n) \to \operatorname{Sym}(n)$

$A \mapsto AA^T$ a smooth (polynomial) map,

$f^{-1}(I) = O(n)$; is I a regular value?

let $A \in f^{-1}(I)$, $B \in T_A M(n) \simeq M(n)$; then

$$(df)_A(B) = \lim_{t\to 0} \frac{f(A+tB)-f(A)}{t} = \lim_{t\to 0} \frac{(A+tB)(A+tB)^T - AA^T}{t}$$

$$= \lim_{t\to 0} (BA^T + AB^T + tBB^T) = BA^T + AB^T;$$

$Sym(n) \subseteq M(n)$ is a vector subspace, so $T_C(Sym(n)) = Sym(n)$ $\forall C \in Sym(n)$;

Let $C \in Sym(n)$ and $B := \frac{1}{2}CA$, then

$df_A(B) = \frac{1}{2}CAA^T + A(\frac{1}{2}A^TC^T) = \frac{1}{2}C + \frac{1}{2}C^T = C \Rightarrow df_A$ is surjective

$\Rightarrow \dim(O(n)) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$ and $O(n)$ is compact.

Because $\det : O(n) \to \{-1, 1\}$ is continuous (it is polynomial), $O(n)$ is not connected; it can be shown that $SO(n) := \det^{-1}(1)$ is connected (its dimension is again $\frac{n(n-1)}{2}$.)

<u>Th</u>: $f: M \to N$ smooth map, M compact and $\dim M = \dim N$, $y \in N$ a regular value of f. Then $f^{-1}(y)$ is a finite set, and $\exists$ an open neigh V of y s.t. $\# f^{-1}(y) = \# f^{-1}(y')$ $\forall y' \in V$ ($\#$ is the cardinality.)

<u>Pf</u>: $f^{-1}(y)$ is a closed subset of (the compact space) $M \Rightarrow f^{-1}(y)$ is compact; by the previous lecture $\forall x \in f^{-1}(y)$ $\exists$ U_x neigh. of x s.t. $f|_{U_x}: U_x \to f(U_x)$ is a diff (an identity in a suitable param.), $U_x \cap f^{-1}(y) = \{x\}$, and $\{U_x \cap f^{-1}(y)\}_{x \in f^{-1}(y)}$ is an covering of (the compact space) $f^{-1}(y)$; a finite subcovering contains finitely many points. So $f^{-1}(y) = \{x_1, \dots, x_r\}$, $r \in \mathbb{N}$, $U_{x_i} =: U_i$, $f(U_i) =: V_i$.

The set $V := (V_1 \cap \dots \cap V_r) \setminus f(M \setminus (U_1 \cup \dots \cup U_r))$ is an open subset of M ($U_1 \cup \dots \cup U_r$ is open, $M \setminus (U_1 \cup \dots \cup U_r)$ is compact $\Rightarrow f(M \setminus \dots)$ is compact $\Rightarrow f(M \setminus \dots)$ is closed.)

Now for $y' \in V \Rightarrow y' \in V_i$ $\forall i = 1, \dots, r \Rightarrow \exists$ unique $x_i' \in U_i$ $\forall$ $i = 1, \dots, r$ with $f(x_i') = y' \Rightarrow \# f^{-1}(y') \geq r$. Suppose $\# f^{-1}(y') > r$, then $\exists$ $x \in M \setminus (U_1 \cup \dots \cup U_r)$ with $f(x) = y' \in V$, a contradiction ($U_1, \dots, U_r$ is a covering of $f^{-1}(y)$.) ∎

<u>Rem</u>: $\underset{\substack{\| \\ C}}{C_f(M)}$ ... the critical points of f $\Rightarrow$

$$\begin{aligned} N \setminus f(C) &\longrightarrow \mathbb{N} \cup \{0\} \\ y &\longmapsto \# f^{-1}(y) \end{aligned}$$

is a locally constant function. (it is constant on $\forall$ connected component of $N \setminus f(C)$.)

We now prove $N \setminus f(C)$ is open in N.

<u>Th</u>: M, N, f as above; then

$$\begin{aligned} M &\longrightarrow \mathbb{N} \cup \{0\} \\ x &\longmapsto \operatorname{rank}(df_x) \end{aligned}$$

has the property:

$\forall x \in M$ $\exists$ a neigh. U s.t. $\operatorname{rank}(df_a) \geq \operatorname{rank}(df_x)$ $\forall a \in U$. (The rank can not drop locally.)

Pf: sufficient to prove this for $f: V \to \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ open. Then df_x is a lin. map defined by Jac. matrix; when $\operatorname{rank}(df_x) = d$, then $\exists$ $d \times d$ minor in $\left\{ \left(\frac{\partial f_i}{\partial x_j}\right)(x) \right\}_{\substack{1 \le i \le d \\ 1 \le j \le n}}$ of rank d (non-zero determinant.) f is smooth $\Rightarrow$ the matrix entries are continuous on x: if $\|x - x'\| < \delta \Rightarrow \forall\, 1 \le r, s \le d$: $\left\| \frac{\partial f_{i_r}}{\partial x_{j_s}}(x) - \frac{\partial f_{i_r}}{\partial x_{j_s}}(x') \right\| < \epsilon$.

Now consider the map
$$\lambda : V \longrightarrow \mathbb{R}$$
$$a \longmapsto \det \left(\frac{\partial f_{i_r}}{\partial x_{j_s}}(a) \right)_{\substack{1 \le r \le d \\ 1 \le s \le d}},$$
which is continuous ($\det : M(d) \simeq \mathbb{R}^{d^2} \to \mathbb{R}$ is continuous, and the composition of cont. maps is cont.)

Since $\mathbb{R} \setminus \{0\}$ is open, $U := \lambda^{-1}(\mathbb{R} \setminus \{0\})$ is open in $M(d)$ and $\forall\, a \in U$ the Jac. matrix has a submatrix of rank d ($\Rightarrow$ is of the rank $\geq d$.) The proof is complete. ∎

Because C is the complement of $\{x \in M \mid \operatorname{rank}(df_x) = n\}$, we have

Corollary: $M, N, f, C \subseteq M$ as above. Then C is closed, and if M is compact then $f(C) \subseteq N$ is compact (hence closed.)

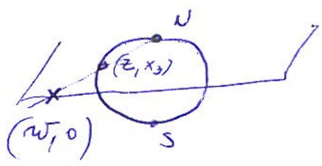
Theorem (Fundamental theorem of algebra): Let $P(z) = \sum_{i=0}^{n} a_{n-i} z^i$ be a complex polynomial, $n \geq 1$ & $a_0 \neq 0$. Then P has a root. $a_i \in \mathbb{C}$

Pf: $S^2 \subseteq \mathbb{R}^3$ (Riemann sphere), $N = (0,0,1)$, $S = (0,0,-1)$. Coordinate systems:

$h_+ : S^2 \setminus N \to \mathbb{R}^2 \simeq \mathbb{C}$, $h_+(z, x_3) = \dfrac{z}{1 - x_3}$

$h_- : S^2 \setminus S \to \mathbb{R}^2 \simeq \mathbb{C}$, $h_-(z, x_3) = \dfrac{z}{1 + x_3}$,

$z = x_1 + i x_2$, $\mathbb{R}^2 \xrightarrow{\sim} \mathbb{C}$, $\mathbb{R}^3 \xrightarrow{\sim} \mathbb{C} \times \mathbb{R}$



$$h_+^{-1}(z) = \left(\frac{2z}{|z|^2 + 1},\ 1 - \frac{2}{|z|^2 + 1} \right),$$
$$h_-^{-1}(z) = \left(\frac{2z}{|z|^2 + 1},\ \frac{2}{|z|^2 + 1} - 1 \right).$$

We notice $h_+ h_-^{-1} = h_- h_+^{-1}(z) = \dfrac{1}{\bar{z}}$.

(16) We first prove: <u>Claim</u>: P a given polynomial, $f: S^2 \to S^2$ defined by $f(x) := h_+^{-1} P h_+(x)$ for $x \neq N$, and $f(N) = N$. Then f is smooth.

Pf: Clear for $x \neq N$, so let $x = N$. Then $S^2 \setminus S$ is an open neigh. of N, coordinate system given by h_-. So f is smooth at N if and only if $Q(z) := h_- f h_-^{-1}(z)$ is smooth at 0. Now for $z \neq 0$:

$$Q(z) = h_- f h_-^{-1}(z) = h_- h_+^{-1} P h_+ h_-^{-1}(z) = \frac{1}{\overline{P(\frac{1}{\bar z})}} =$$

$$= \frac{1}{\overline{a_0 \bar z^{-n} + \dots + a_n}} = \frac{z^n}{\bar a_0 + \dots + \bar a_n z^n} \quad \Rightarrow Q \text{ is smooth}$$

at $z = 0$ because $a_0 \neq 0$. ▨

Now a critical point of f is N or $h_+^{-1}(z)$ for z such that $P'(z) = \sum a_{n-j} j z^{j-1} = 0$, because for $x \neq N$ $df_x = (dh_+^{-1})_{Ph_+(x)} \circ dP_{h_+(x)} \circ (dh_+)_x$.

Now $f(C) \subseteq S^2$ is compact and consists of finitely many points (btw. the number of zeroes of P' is finite.) Since $S^2 \setminus f(C)$ is connected, the map $S^2 \setminus f(C) \to \mathbb{N} \cup \{0\}$ is constant.

$$y \mapsto \# f^{-1}(y)$$

But $f^{-1}(y) \neq \emptyset$ for some $y \in S^2 \setminus f(C)$ (otherwise $\forall$ values taken by the fion f were critical which contradicts to their finite number.)

Thus $\# f^{-1}(y) > 0$ $\forall$ regular values (because it is true for some regular value and $S^2 \setminus f(C)$ is connected.)

Because $\# f^{-1}(y) > 0$ $\forall$ critical values $\Rightarrow$ f is onto, and so there $\exists$ $x \neq N$ such that $f(x) = S$ (we know $f(N) = N$.) In other words, $\exists$ $z \in \mathbb{C}$ such that $P(z) = 0$. ▨

How often/rare are the regular values? ⇐ Sard's theorem

Several concepts: $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n) \in \mathbb{R}^n$, $a_i < b_i$ $\forall i = 1, \dots, n$

Then $W(a, b) := \prod_{i=1}^{n} \langle a_i, b_i \rangle \subseteq \mathbb{R}^n$ n-dim rectangle

17 The lengths of sides are $d_i = b_i - a_i$, $1 \le i \le n$, and its volume is $Vol(W(a,b)) = \prod_{i=1}^{n} (b_i - a_i)$.

<u>Def</u>: $A \subseteq \mathbb{R}^n$ is a set of measure zero if $\forall \epsilon > 0$ the set A can be covered by count. many rectangles, i.e. $\exists \{W_i\}_{i \in \mathbb{N}}$ with $\sum_{i=1}^{\infty} Vol(W_i) < \epsilon$.

<u>Notation</u>: $c \in \mathbb{R}^k$, $\mathbb{R}_c^{n-k} := \{c\} \times \mathbb{R}^{n-k} \subset \mathbb{R}^n$; the rectangles W can be replaced by open rectangles / cubes

For Sard's theorem, $k=1$ is sufficient:

<u>Lemma</u>: 1/ Countable union of sets of measure zero (or, subsets of sets of measure zero), are sets of measure zero.

2/ $\mathbb{R}^{n-1} \times \{0\} \subseteq \mathbb{R}^n$ is a set of measure zero; ~~and~~ an open subset of $\mathbb{R}^n$ is not a set of measure zero.

3/ If $U \subseteq \mathbb{R}^n$ is open and $A \subseteq U$ is a set of measure zero, and if $f: U \to \mathbb{R}^n$ is smooth, then $f(A) \subseteq \mathbb{R}^n$ is a set of measure zero.

4/ $A \subseteq \mathbb{R}^n$ closed s.t. $A \cap \mathbb{R}_c^{n-k}$ is a set of measure zero for all $c \in \mathbb{R}^k$. Then A is a set of measure zero. (Measure zero set Fubini th.)

<u>Pf</u>: 1/ $N_1, \dots, N_i, \dots$, a sequence of sets of measure 0. For i fixed, let W_i^j, $j = 1, 2, \dots$, be a covering of N_i with $\sum_{j=1}^{\infty} Vol(W_i^j) < \frac{\epsilon}{2^i}$.

Then $\{W_i^j \mid i = 1, 2, \dots, j = 1, 2, \dots\}$ is a covering of $N_1 \cup N_2 \cup \dots$, and $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} Vol(W_i^j) < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon \sum_{i=1}^{\infty} 2^{-i} = \epsilon$.

The assertion about subsets is obvious.

2/ We begin with proving the claim for $\forall$ compact subsets $K \subseteq \mathbb{R}^{n-1}$. Then K is closed & bounded $\Rightarrow K \subseteq S$ for a suff. large rectangle in $\mathbb{R}^{n-1}$. Then $\forall \delta > 0$: $S' := S \times \langle -\frac{\delta}{2}, \frac{\delta}{2} \rangle$ is a rectangle in $\mathbb{R}^n$, $K \subseteq \mathbb{R}^{n-1}$. Let $\delta < \frac{\epsilon}{Vol(S)}$, then $vol(S') = vol(S) \times \delta < \epsilon$. The proof for $\mathbb{R}^{n-1}$: $\mathbb{R}^{n-1} = \bigcup_{i=1}^{\infty} C_i$ with C_i compact, and apply 1/.

If $U \subseteq \mathbb{R}^n$ is open, then U contains an open rectangle of volume $\delta > 0$. Then the volume of any covering of U by rectangles will be $\geq \delta$, so U is not a set of measure zero.

As for 3/, for example, we use: f smooth on K, K compact $\Rightarrow \exists C \in \mathbb{R}: \forall x, y \in K$ $|f(x) - f(y)| \leq C|x-y|$. ☒

<u>Exercise</u>: A rectangle $W(a,b)$, $a = (a_1, .., a_n)$ and $b = (b_1, .., b_n)$ in $\mathbb{R}^n$, whose vertices have rational coefficients ($a_i, b_j \in \mathbb{Q}$) and whose volume is strictly less than some $\epsilon > 0$, can be covered by finitely many cubes whose total volume is still less than ϵ.

<u>Def</u>: $N \subseteq \mathbb{R}^\ell$ be a smooth manifold of dimension n. Then $A \subseteq N$ is called a set of measure 0 if for some covering of N by coordinate systems $\varphi_j: W_j \to \mathbb{R}^n$, $W_j \subseteq N$ open, the sets $\varphi_j(W_j \cap N)$ are sets of measure zero in $\mathbb{R}^n$ for all j.

By 3/ of the last Lemma, the property "set of measure zero" is independent of $\{\varphi_j\}_j$.

<u>Theorem</u>: $A \subseteq N$ a set of measure zero in a smooth manifold of dimension n. Then $N \setminus \{A\}$ is dense in N, i.e. $cl_N(N \setminus A) = \overline{N \setminus A} = N$.

<u>Pf</u>: Suppose the claim is not true, i.e. $\exists\, y \in N$ with $y \notin \overline{N \setminus A}$. So there is an open neigh. V of $y \in N$ in $N \setminus (\overline{N \setminus A}) \subseteq A$ and a corresponding coordinate system (φ, W) at y, $y \in W$, mapping $W \cap V$ onto an open subset of $\mathbb{R}^n$. But $\varphi(W \cap V \cap A) = \varphi(W \cap V)$ is open in $\mathbb{R}^n$ ($W \cap V$ is open, φ is a homeom.) and has measure zero (by Lemma ↑, 1/ & 3/). This contradicts Lemma ↑, 2/. ☒

<u>Theorem</u> (Sard's) M, N smooth man, $f: M \to N$ smooth maps. Then the set of critical values of f is a set of measure zero. In particular, the regular values are dense.

Notice: for $\dim M < \dim N$ this proves $f(M) \subseteq N$ is a set of measure zero.

Pf: Let $C_f \subseteq M$ be the set of critical pts., consider a countable covering of M by parametrizations $\varphi_i : U_i \to M$, $i \in \mathbb{N}$. Then $f(C) = \bigcup_{i\in\mathbb{N}} (f\circ\varphi_i)(C_i)$ with $C_i \subset U_i$ the set of critical points of $f\circ\varphi_i$. Moreover, $x \in M$ is a critical point for $\chi\circ f$ and $\forall$ coordinate system χ for N at $f(x)$. Sard's theorem is a consequence of its Euclidean version:

Lemma (Sard's theorem for Euclidean space.)

Let $U \subseteq \mathbb{R}^m$ be open and $f: U \to \mathbb{R}^n$ be smooth. Then $f(C) \subseteq \mathbb{R}^n$ is a set of measure zero, where C is the set of critical points of f.

Pf: By induction on m; $m=0$ is obvious. It follows from the Corollary (Gancalf, C is closed) and continuity of derivatives that we have a sequence of closed sets: $C \supset C_1 \supset C_2 \dots \supset C_i \supset C_{i+1} \supset \dots$,

with $C_i := \{ x \in U \mid \forall$ partial der. of f of order $\leq i$ vanishing at $x \}$

e.g.: $C_1 := \{ x \in U \mid df_x = 0 \}$. The proof is based on three lemmas:

Lemma 1: $f(C \setminus C_1)$ is a set of measure zero.

Pf: $\forall x \in C \setminus C_1$ we construct an open set $V = V_x$ such that $f(V_x \cap (C \setminus C_1))$ has measure zero. Because of our earlier result (a subset $X \subseteq \mathbb{R}^k$ is second countable), $C \setminus C_1$ is covered by countably many sets V_i, $i \in \mathbb{N}$, $V_i \subset V_x$ for some $x \in C \setminus C_1$, since $f(V_i \cap (C \setminus C_1))$ has measure zero the claim follows. It remains to construct V_x:

For $x \in C \setminus C_1$, there is a non-vanishing partial derivative: $\frac{\partial f}{\partial x_1}(x) \neq 0$. Define $h: U \to \mathbb{R}^m$ by

$h(x) = (f_1(x), x_2, \dots, x_m)$, $(dh)_x = (\mathrm{grad}(f_1)(x^T), e_2, \dots, e_m)^T$,

so h is a diffeomorphism (of an open neigh. V of x onto $V' \subseteq \mathbb{R}^m$)

The map $g := f \circ h^{-1} : V' \to \mathbb{R}^n$ has the same regular values as $f|_V$, and $g(t, x_2, \dots, x_m) = (t, y_2, \dots, y_n)$ for suitable

(20) [Pf. of this claim: let $(t, x_2, \dots, x_m) \in V'$ be arbitrary, we know $(t, x_2, \dots, x_m) = h(x) = (f_1(x), x_2, \dots, x_m)$ for some $x = (x_1, \dots, x_m) \in V$; therefore, $g(t, x_2, \dots, x_m) = f(h^{-1}(h(x))) = f(x) =$
$= (f_1(x), y_2, \dots, y_n)$ and $t = f_1(x)$.]

Now $\forall\, t$:

$$g^t := g\big|_{(\{t\} \times \mathbb{R}^{m-1}) \cap V'} : (t \times \mathbb{R}^{m-1}) \cap V' \to t \times \mathbb{R}^{n-1},$$

and we know the image of the linear map $\left(\frac{\partial g_i}{\partial x_j}\right) = \begin{pmatrix} 1 & 0 \ \dots & 0 \\ * & \left(\frac{\partial (g^t)_i}{\partial x_j}\right) \end{pmatrix}$

is spanned by columns; $(t, x_2, \dots, x_m)$ is critical for g^t iff it is critical for g and not in C_1. By induction hypothesis, the set of critical values of g^t in $t \times \mathbb{R}^{n-1}$ has measure zero $\underset{\text{Fubini measure zero theorem}}{\Longrightarrow}$ the claim is true for g. ▨

<u>Remark</u>: The set of critical values of g is not necessarily closed.
Namely, for $C \subseteq V$ closed $\exists\, \tilde{C} \subseteq \mathbb{R}^m$ closed s.t. $\tilde{C} \cap V = C$, e.g. $\tilde{C} = cl_{\mathbb{R}^m} C$. Now $V = \bigcup_{j \in \mathbb{N}} K_j$ for $K_j \subseteq V$ compact subsets, and so $C = \bigcup_{j \in \mathbb{N}} \tilde{C} \cap K_j = \bigcup_{j \in \mathbb{N}} C \cap K_j$;
because $\tilde{C} \cap K_j$ is closed, $C \cap K_j$ is closed and hence compact $\Rightarrow$ C is countable union of compact sets $\Rightarrow$ $g(C)$ is a countable union of compact sets.

This Remark is then used in the following lemma, a variation on the previous proof.

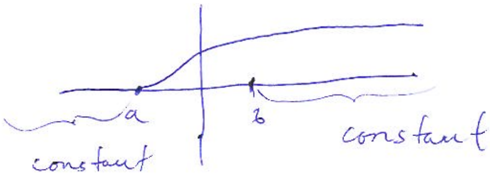
<u>Lemma</u>: For $k \geq 1$ the set $f(C_k \setminus C_{k+1})$ has measure zero.

<u>Lemma</u>: For $k > \frac{m}{n} - 1$, $f(C_k)$ is a set of measure zero.

01

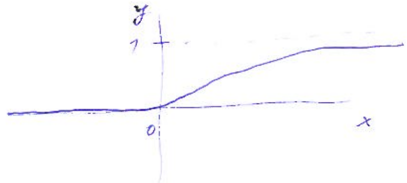
Exercise / homework (later related to the notion of smooth homotopy)

$a \in \mathbb{R}$
$b \in \mathbb{R}$



?

Define $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$

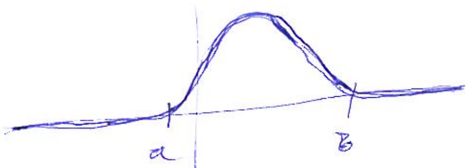


claim: f is a smooth function!

(smooth in $x<0$, left limit $x \to 0^-$;
— " — $x>0$, ? right limit $x \to 0^+$
of $\forall$ derivatives ?)

Construct ~~the~~ function $E_a^b : \mathbb{R} \to I = \langle 0, 1 \rangle$ such that it is smooth, identically zero for $x \leq a$ and identically one for $x \geq b$, $a, b \in \mathbb{R}$ $a < b$.

Start with the bump f-ion $\beta : \mathbb{R} \to I$ $\quad \beta(x) = \begin{cases} e^{-\frac{1}{x-a}} e^{\frac{-1}{x-b}} & a<x<b \\ 0 & \text{otherwise} \end{cases}$



and normalize it.

$$E_a^b(x) = \frac{\int_a^x \beta(t)\,dt}{\int_a^b \beta(t)\,dt}$$

(1') Notice (Some vocabulary):

Def: A map $f: M \to N$ without critical points is called a submersion.

Ex: The inclusion $i: U \hookrightarrow M$ of an open subset $U \subseteq M$ in M is a submersion.

Ex: Let $m > n$. The projection $p: \mathbb{R}^m \to \mathbb{R}^n$, defined by $p(x^1, \dots, x^m) = (x^1, \dots, x^n)$ is a submersion.

We already know: $f: M \to N$ smooth, $x \in M$ not a critical point. Then $\exists$ $U \ni x$ an open neigh. s.t. $f|_U : U \to N$ is a submersion.

Def: An immersion is a smooth map $f: M \to N$ s.t. $\forall x \in M$ the differential $\underset{f_*}{\underset{\shortparallel}{(df)_x}} : T_x M \to T_{f(x)} N$ is injective.

Ex: see the discussion on page (2')

Because immersions exhibit quite strange behaviour, we restrict to a special case known as embedding:

Def: An immersion $f: M \to N$ is called an embedding if the map $f: M \to f(M)$ is a homeomorphism onto its image.

We call a subset $M \subseteq N$ an embedded submanifold if the inclusion $M \hookrightarrow N$ is an embedding. If $M \subseteq N$ is an embedded submanifold, we consider $T_x M$ as a subset of $T_x N$, $x \in M$.

Ex: $i: U \hookrightarrow M$ inclusion of an open subset in a smooth manifold is an embedding.

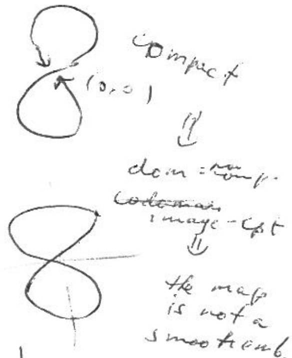
Ex: Let $m < n$. The map $i: \mathbb{R}^m \to \mathbb{R}^n$, $i(x^1, \dots, x^m) = (x^1, \dots, x^m, \underbrace{0, \dots, 0}_{n-m})$ is an embedding.

Ex: $M \subseteq \mathbb{R}^N$ be a smooth manifold. Then the inclusion $M \hookrightarrow \mathbb{R}^N$ is an embedding.

Analogously to the case of submersion: $f: M \to N$ smooth, $x \in M$ s.t. $(df)_x : T_x M \to T_{f(x)} N$ is injective. Then $\exists$ $U \ni x$ open s.t. $f|_U : U \to N$ is an immersion.

2'

(immersion is locally embedding): Examples: non-cpt $0 \longleftrightarrow 2\pi$



1/ $I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$

the map is 2π-period.

$(0, 2\pi) \ni t \longmapsto (x = \cos(\pi/2 + t), y = \sin(2t))$

is an immersion, fails to be an embedding (the crossing pt)
(the diff. is injective on the domain)
$\Rightarrow S^1$ is not homeom to its image (in the induced topology)

Notice: the image can be realised by the map $t \mapsto (\cos t, \sin 2t)$, which is not injective, however.

2/ The map $q \circ i$, where $i : t \mapsto (t, \pi t)$ $\mathbb{R} \to \mathbb{R}^2$

$q \circ i : \mathbb{R} \to \mathbb{T}^2$ (2-tori) $q : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 \simeq \mathbb{T}^2$ the quotient map

The map $q \circ i$ is injective, $(q \circ i)_* = d(q \circ i)$ is injective on $\mathbb{R}$, but in the induced topology on $Im(q \circ i)$ for $\mathbb{T}^2/\mathbb{Z}^2$, the domain of $q \circ i$ $(\mathbb{R})$ is not homeom. with $Im(q \circ i)$ $(\mathbb{R})$. This map is an immersion, but not embedding.

3/ The map $\mathbb{R} \to \mathbb{R}^2$

$t \mapsto (t^2, t^3)$ is injective, the diff. is injective on $\mathbb{R}^*$ (not for $t = 0$!) $\Rightarrow$ this map is not an immersion (hence not an embedding)

<u>Question</u>: let $f : \mathbb{R}P^2 \longrightarrow \mathbb{R}^3$

$[x, y, z] \longmapsto (xy, xz, yz).$

Show f is a (well-defined) smooth fcion.
Is f bijection? Is f immersion?

Let $g : \mathbb{R}P^2 \longrightarrow \mathbb{R}^4$

$[x, y, z] \longmapsto (xy, xz, yz, x^4).$

Is g a smooth fcion? Is it immersion, is it embedding?