Heat flow and holography

Motivating Example: $V = L^2(S^1) = L^2(S^1, \mathbb{C})$

$\theta \in (0, 2\pi)$ coordinate on S^1

$d = \frac{d}{d\theta} : \Lambda^*(M) \to \Lambda^*(M)$, its spectrum $\sigma(\frac{d}{d\theta})$ contains zero eigenvalue of infinite multiplicity & infinite dimensional space of forms not in the eigenspace of zero eigenvalue (of course, except $\dim M = 1$, $\Lambda^0(M) \cong \Lambda^1(M)$.)

We shall be interested in $\Delta = -\frac{d^2}{d\theta^2}$ acting on V;

ON-basis : $\{e^{in\theta}\}_{n\in\mathbb{Z}}$, $\Delta e^{in\theta} = n^2 e^{in\theta}$,

$\mathrm{Spec}(\Delta) = \{n^2 \mid n \in \mathbb{Z}\}$. Eigenfunction decomposition for $f \in L^2(S^1)$:

$$f = \sum_n a_n e^{in\theta} = \sum_n \langle f, e^{in\theta}\rangle e^{in\theta}, \quad a_n \in \mathbb{C},$$

with

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{S^1} f(\theta)\,\overline{g(\theta)}\,d\theta$$

Because for the sequence $\{\frac{e^{in\theta}}{n}\}_{n\in\mathbb{Z}}$ fulfills

$$\left\|\frac{e^{in\theta}}{n}\right\| \xrightarrow{n\to\infty} 0 \quad \Rightarrow \quad \left\|\Delta\left(\frac{e^{in\theta}}{n}\right)\right\| \to \infty,$$

the operator Δ is unbounded. Notice that for $\lambda \notin \sigma(D)$, $(D - \lambda\mathbb{1})^{-1}$ exists and is bounded.

A more complicated situation is $\Delta = -\frac{d^2}{dt^2}$ on $L^2(\mathbb{R})$, t the coordinate on $\mathbb{R}$. For D, a symmetric unbounded operator on a Hilbert space $\mathcal{H}$, the spectrum $\sigma(D)$ of D, $\sigma(D) \subseteq \mathbb{R}$, is defined by $\lambda \notin \sigma(D)$ iff $(D-\lambda\mathbb{1})^{-1}$ can be extended to a bounded operator on $\mathcal{H}$. Equivalently,

1/ $\lambda \notin \sigma(D)$, iff

2/ $\mathrm{Ker}(D - \lambda\mathbb{1}) = 0$, iff

3/ $\mathrm{Im}(D - \lambda\mathbb{1})$ is dense in $\mathcal{H}$, iff

4/ $(D - \lambda\mathbb{1})^{-1}$ is bounded on $\mathrm{Im}(D - \lambda\mathbb{1})$.

Set $D := \Delta$ acting on $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$.

First consider $\lambda \geq 0$, the eigenfunctions are

$$f(x) = \exp(\pm i\sqrt{\lambda}\, x)$$
$$\Delta f = \lambda f$$
$$\forall \lambda \in \mathbb{R}^+ \cup \{0\}.$$

None of functions $f(x) = \exp(\pm i \sqrt{\lambda}\, x)$ is in $L^2(\mathbb{R}, \mathbb{C}) = \mathcal{H}$

This does not mean $\sigma(\Delta) = \emptyset$:

$\varphi_N(x) \quad \mathbb{R} \to \mathbb{R} \quad : \quad \varphi_N \geq 0 \quad \forall N \in \mathbb{N}$

$\cap$

$C_0^\infty(\mathbb{R})$

$\cap$

$L^2(\mathbb{R}, \mathbb{C})$

$$\varphi_N(x) = \begin{cases} 0 & x \in (-\infty, -N] \cup [N, \infty) \\ 1 & x \in [-N+1, N-1] \end{cases}$$

For fixed $\lambda \geq 0$ $\exists\, C, C' > 0$ such that for all $N \in \mathbb{N}$:

$$\| (\Delta - \lambda \mathbb{1})(\varphi_N(x) e^{i\sqrt{\lambda} x}) \| \leq C \leq \frac{C'}{N} \| \varphi_N(x) e^{i\sqrt{\lambda} x} \|$$

(φ_N is non-constant on $[-N, -N+1] \cup [N-1, N]$ only.)

The last inequality implies $\| (\Delta - \lambda \mathbb{1})^{-1} \| = \infty$

[illegible]

For $\lambda < 0$ is $\lambda \notin \sigma(\Delta)$. Why? – [illegible]

[illegible]

First, there are no L^2-eigenfunctions for λ. Then we claim that $\operatorname{Im}(\Delta - \lambda \mathbb{1})$ is dense in $L^2(\mathbb{R}, \mathbb{C})$. If not, take $\alpha \in L^2(\mathbb{R}, \mathbb{C})$ nonzero : $\alpha \perp \operatorname{Im}(\Delta - \lambda \mathbb{1})$, and we have

$$0 = \int_{\mathbb{R}} \alpha(x) (\Delta - \lambda \mathbb{1}) f(x) \qquad \forall f \in C_0^\infty(\mathbb{R})$$

$\cap$ dense

$L^2(\mathbb{R}, \mathbb{C})$

$\Rightarrow$ α is a weak (distributional) solution to

$(\Delta - \lambda \mathbb{1}) \alpha = 0$, but according to elliptic regularity for Δ operator $\alpha \in L^2(\mathbb{R})$ $\Rightarrow$ contradiction, so $\operatorname{Im}(\Delta - \lambda \mathbb{1})$ is dense in $\mathcal{H}$ and $\lambda \notin \sigma(\Delta)$.

$\Rightarrow$ $\sigma(\Delta) = [0, \infty)$.

Fourier series decomposition:

S^1 $$f(\theta) = \sum_{n\in\mathbb{Z}} \left(\frac{1}{2\pi}\int_{S^1} f(\varphi)\, e^{-in\varphi}\, d\varphi\right) e^{in\theta}$$

$\mathbb{R}$ $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(y)\, e^{-i\xi y}\, dy\right) e^{i\xi x}\, d\xi$$

} isometries of L^2-spaces

Heat flow on S^1 and $\mathbb{R}$

The heat equation $(\partial_t + \Delta) f(t,\theta) = 0$, say for $f(t,\theta) \in C^\infty(\mathbb{R}_+, L^2(S^1))$.

If $f(t,\theta) = \sum_n a_n(t)\, e^{in\theta}$ is the Four. decomp. for $f(t,\theta)$, $\theta \in S^1$, $t \in \mathbb{R}_{\geq 0}$,

$a_n := a_n(0)$, then

$$0 = \sum_n \left[\dot{a}_n(t) + n^2 a_n(t)\right] e^{in\theta}$$

$$\Rightarrow a_n(t) = a_n e^{-n^2 t} \quad \Rightarrow \quad f(t,\theta) = \sum_n e^{-n^2 t} a_n e^{in\theta}.$$

For $t \to \infty$, $f(t,\theta) \to a_0$ (the average value of f, $a_0 = \int_{S^1} f(\theta)\, d\theta$.)

The situation on $\mathbb{R}$ is more complicated. For $f(0,x) = f(x) \in L^2(\mathbb{R},\mathbb{C})$, set

$$f(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y)\, dy. \qquad (\blacktriangle)$$

For $f(y)$ continuous, $(\partial_t + \Delta_x) f(t,x) = 0$ and $\lim_{t\to 0+} f(t,x) = f(x)$.

Why? Denote $\hat{f}(t,\xi)$ the Fourier transform in x, $(\partial_t + \Delta_x) f(t,x) = 0$ implies

$$-|\xi|^2 \hat{f}(t,\xi) = \partial_t \hat{f}(t,\xi),$$

$$\hat{f}(t,\xi) = \hat{f}(\xi)\, e^{-t|\xi|^2} = \hat{f}(\xi)\, e^{-\frac{|\xi\sqrt{2t}|^2}{2}}$$

$$= \hat{f}(\xi) \widehat{\left(\frac{1}{\sqrt{2t}} e^{-\frac{|x|^2}{4t}}\right)}(\xi)$$

$$= \widehat{\left(f * \frac{1}{\sqrt{2t}} e^{-\frac{|x|^2}{4t}}\right)}(\xi)$$

$\wedge$ ... Four. transform

$*$ ... convolution : $\mathcal{F}(- * -) = \mathcal{F}(-)\mathcal{F}(-)$.

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The inverse Fourier transform gives the formula (Δ).

The smooth function $K_{\mathbb{R}}: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$

$$(t, x, y) \mapsto K_{\mathbb{R}}(t,x,y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}}$$

is called integral kernel of heat operator. In fact, (Δ) is analogous to $f(t,\theta) = \sum_{n\in\mathbb{Z}} e^{-n^2 t} a_n e^{in\theta}$ in the sense that on S^1

$$f(t,\theta) = \sum_{n\in\mathbb{Z}}{}^{!} e^{-n^2 t} \langle f, e^{in\theta} \rangle e^{in\theta}$$

$$= \frac{1}{2\pi} \int_{S^1} \underbrace{\sum_{n\in\mathbb{Z}}{}^{!} e^{-n^2 t} e^{in\theta} \overline{e^{in\psi}}}_{K_{S^1}(t,\theta,\psi)} f(\psi)\, d\psi$$

$\psi = \hat{\theta}$ is the Fourier dual variable

$K_{S^1}(t,\theta,\psi)$ is a smooth fn on $\mathbb{R}_+ \times S^1 \times S^1$

What is the relation of $K_{\mathbb{R}}$ and K_{S^1} (in the light of the fact that $\mathbb{R}$ is locally isometric to S^1.)?

Set $\tilde{K}_{S^1}(t,\theta,\psi) := \sum_{n\in\mathbb{Z}} K_{\mathbb{R}}(t, \theta, \psi + 2\pi n)$,
which is well-defined (i.e. a function on $S^1 \times S^1$)
because $K_{\mathbb{R}}(t,x,y) = K_{\mathbb{R}}(t, x+l, y+l)$, $\forall l \in \mathbb{R}$.)

<u>Lemma</u>: $\tilde{K}_{S^1} = K_{S^1}$, i.e.,

$$K_{S^1}(t,\theta,\psi) = \sum_{n\in\mathbb{Z}}{}^{!} K_{\mathbb{R}}(t, \theta, \psi + 2\pi n).$$

<u>Pf</u>: We have $(\partial_t + \Delta_\theta) \tilde{K}_{S^1} = 0$, and

$$\lim_{t\to 0^+} \int_{S^1} \tilde{K}_{S^1}(t,\theta,\psi) f(\psi)\, d\psi = \lim_{t\to 0^+} \int_{\mathbb{R}} K_{\mathbb{R}}(t,\theta,\psi) f(\psi)\, d\psi$$

for all $f \in C^\infty(\mathbb{R})$ periodic for $\mathbb{Z}$.

The proof is finished in the light of (which can be generalized to any manifold)

<u>Lemma</u>: Let $K_1(t,\theta,\gamma)$, $K_2(t,\theta,\gamma) \in C^\infty(\mathbb{R}_+ \times S^1 \times S^1)$ satisfy

$$(\partial_t + \Delta_\theta) K_1(t,\theta,\gamma) = 0,$$
$$(\partial_t + \Delta_\theta) K_2(t,\theta,\gamma) = 0,$$

and

$$\lim_{t\to 0+} \int_{S^1} K_1(t,\theta,\gamma) f(\gamma)\, d\gamma =$$
$$= \lim_{t\to 0+} \int_{S^1} K_2(t,\theta,\gamma) f(\gamma)\, d\gamma =$$
$$= f(\theta)$$ for all $f \in L^2(S^1, \mathbb{C})$. Then

$$K_1 = K_2.$$

Convention: Δ acts in the first variable; the proof of [↖ lemma] is quite straightforward (and elementary.) One of the ingredients is the symmetry of K_1 and K_2 in θ and γ variables.

The operator taking heat distribution f to the time t heat distribution $f(t,x)$ via heat flow is denoted

$$e^{-t\Delta} : L^2(S^1, \mathbb{C}) \to L^2(S^1, \mathbb{C}),$$

with the notation for S^1 justified by multiplication by e^{-tn^2} on n^2- eigenspace of Δ on S^1 (and similarly for $\mathbb{R}$ by spectral theorem for unbounded operators.)

The trace of the heat operator on S^1 is

$$\mathrm{Tr}(e^{-t\Delta}) = \sum_{n\in\mathbb{Z}} e^{-n^2 t} = \int_{S^1} K_{S^1}(t,\theta,\theta)\, d\theta.$$

⑥ For $t \to 0^+$, the trace for heat kernel K_{S^1} is (has an asympt. behav.)

$$\sum_{n\in\mathbb{Z}}{}' e^{-n^2 t} = \int_{S^1} K_{S^1}(t,\theta,\theta)\,d\theta = \int_{S^1} \sum_{n\in\mathbb{Z}}{}' K_{\mathbb{R}}(t,\theta,\theta+2\pi n)\,d\theta$$

$$= \int_{-\pi}^{\pi} K_{\mathbb{R}}(t,x,x)\,dx + \underbrace{O(t^\infty)}$$

← all terms with $n\neq 0$

with $O(t^\infty)$ denotes terms characterized by $e^{-\frac{\alpha}{t}}$ for some $\alpha>0$ in the limit $t\to 0^+$.

Thus for $t\to 0^+$, $Tr(e^{-t\Delta})$ on S^1 behaves as $\frac{2\pi}{\sqrt{4\pi t}}$.

This can be better viewed in the light of

Def: (Vanishing to infinite order)

For two functions $f=f(t)$, $g=g(t)$, we write $f\sim g$ if $\lim_{t\to 0}\frac{f(t)-g(t)}{t^m}=0 \quad \forall m\in\mathbb{R}^+$.

Because $\sum_{n\in\mathbb{Z}}{}' e^{-n^2 t} \sim \sqrt{\pi}\, t^{-1/2}$ (as a consequence of the Poisson summation for Jacobi theta function), the same $t\to 0^+$ behaviour as $K_{\mathbb{R}}$.

For the length l circle S^1, $Tr(e^{-t\Delta}) \sim \frac{l}{\sqrt{4\pi t}}$, so because l is the only metric invariant of S^1, heat flow carries "all" info about the metric geometry of S^1.

The long time behavior $t\to\infty$ differs for compact and non-compact manifolds (e.g., S^1 and $\mathbb{R}$)

Heat equation ~~…~~

Heat kernel equation on a Riemannian manifold (M, g): (Assume M is compact.)

x, y … local coordinates on the first (resp. second) copy of M

There exists $K_M(t, x, y) \in C^\infty(\mathbb{R}_+^* \times M \times M)$ such that

$$(\partial_t + \Delta_x) K_M(t, x, y) = 0,$$

$$\lim_{t \to 0+} \int_M K_M(t, x, y) f(y) dy = f(x).$$

} well-defined Cauchy problem (the existence implies its uniqueness)

We define the heat operator $e^{-t\Delta}$ by

$$e^{-t\Delta} : f \mapsto \int_M K_M(t, x, y) f(y) \, d\mathrm{Vol}.$$

Then $e^{-t\Delta} f$ solves the heat equation with initial condition $f(x) \equiv f(0, x) \equiv \lim_{t \to 0+} f(t, x)$.

The quantities Δ, $d\mathrm{Vol}$ depend on g.

<u>Intermezzo</u>: $\forall p \in M$, $v \in T_x M$ $\exists \epsilon > 0$ and a unique geodesic $\gamma_v(t)$, $t \in (-\epsilon, \epsilon)$: $\gamma(0) = 0$, $\gamma'(0) = v$.

(geodesic: $\nabla_{\dot\gamma} \dot\gamma = 0$ for the Levi-Civita connection ∇ of g)

For $p \in M$, the exponential map $\exp_p : T_p M \to M$, $v \mapsto \exp_p(v) = \gamma_v(1)$, induces a diffeomorphism $B_\epsilon(0) \xrightarrow{\sim} U(p)$ — a neighborhood of p in M.

Riemann normal coordinates – pick an ON-basis in $T_p M$, the resulting coordinate system given by $\exp_p$. Once ~~…~~ $T_p M \to M$ is a local diff. $B_\epsilon(0) \to U(p)$

$Fr^{-1} \circ \exp_p^{-1} : \quad U(p) \to \mathbb{R}^n$

for $Fr : \mathbb{R}^n \to T_p \mathbb{R}^n$ an isomorphism given by choice of ON-frame

is a special local chart in which:
(distinguished)

- $v \in T_p M$ with components v_j, $j = 1, .., \dim M$ for the choice of Fr in the construction of normal coordinates. Let $\gamma_v(t)$ be the geodesic: $\gamma_v(0) = p$, $\dot{\gamma}_v(0) = v$. Then γ_v is in normal Riemannian coordinates given by $\gamma_v(t) = (tv^1, .., tv^n)$ (as long as this is a point in $U(p)$.)
- the coordinates of p are $(0, .., 0)$.
- In the Riemann. normal coordinates at $p \in M$, the components of Riemann. ~~tensor~~ metric g at p is $g_{ij}(p) = \delta_{ij}$.
- $\Gamma^i_{jk}(p) = 0$ (Christoffel symbols), $\partial_i g_{ij}(p) = 0$.

A special example of normal Riemannian coordinates is given by polar normal coordinates. Introduce on $T_p M$ the standard spherical
‖
radial
‖
spherical

coordinates (t, φ), $t \geq 0$ is the radial coordinate and $\varphi = (\varphi_1, .., \varphi_{m-1})$ parametrizes $(m-1)$-sphere. The composition with $\exp^{-1}$ is the corresponding chart.

The radial coordinate represents the geodesic distance to $p \in M$.

Gauss Lemma (about images of standar spheres) implies that the radial coordinate represents the geodesic distance to p.

$\langle df, dt \rangle = \frac{\partial f}{\partial t} \quad \forall f \in C^\infty(M, \mathbb{C})$, $g = \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ 0 & g_{\varphi\varphi}(t, \varphi) \end{bmatrix}$

(block diagonal form in (r, φ).)

In the spherical Riemannian ~~polar~~ normal coordinates, the Laplace operator is given by

$$(\Delta f)(q) = -(\nabla_{\partial_t}\nabla_{\partial_t} f)(q) + (\Delta_S f)(q) - \left(\frac{m-1}{t} + Det^{-1}\nabla_{\partial_t} Det\right)\nabla_{\partial_t} f(q)$$

and for $f \in C^\infty(M)$ φ-independent this simplifies to

$$(\Delta f)(q) = -\left(\frac{\partial^2}{\partial t^2} f\right) - \left(\frac{m-1}{t} + \frac{\partial_t \det(d\exp_p)}{\det(d\exp_p)}\right)\partial_t f(q).$$

Here $\det(d\exp_p)$ is the determinant of Jacobian of $\exp_p$, Δ_S is the Laplace operator on S - the sphere of constant distance r_0 from p; $S = \exp_p S'$ (S' is the sphere of radius r_0 in T_pM.)

<u>end of Intermezzo:</u> Back to heat kernel questions/problems

Let $\varphi_i \in L^2(M)$ be the eigenfunction of Δ on M, satisfying $\Delta\varphi_i = \lambda_i\varphi_i$. Assume $\{\varphi_i\}_i$ is ON-basis of $L^2(M)$.

<u>Prop:</u> Assume $K_M(t,x,y) \in C^\infty(\mathbb{R}^*_+ \times M \times M)$, the heat kernel, satisfying

$$(\partial_t + \Delta_x)K_M(t,x,y) = 0,$$

$$\lim_{t\to 0+}\int_M K_M(t,x,x)\, dVol = f(x),$$

exists for all $f \in L^2(M)$. Then we have the pointwise convergence

$$K_M(t,x,y) = \sum_i e^{-\lambda_i t}\varphi_i(x)\varphi_i(y).$$

Pf: $\{\varphi_i\}_i$ ON-basis of $L^2(M)$, fix t, x and write in $L^2(M)$:

$$K_M(t,x,-) = \sum_i f_i(t,x)\,\varphi_i(-),$$

i.e. $$f_i(t,x) = \int_M K_M(t,x,y)\,\varphi_i(y)\,dy; \quad = d\,Vol_y$$

Then $$\partial_t f_i(t,x) = \partial_t \int_M K_M(t,x,y)\,\varphi_i(y)\,dy$$

$$= -\int_M \Delta_y K_M(t,x,y)\,\varphi_i(y)\,dy$$

$$= -\int_M K_M(t,x,y)\,\Delta\varphi_i(y)\,dy$$

$$= -\lambda_i \int_M K_M(t,x,y)\,\varphi_i(y)\,dy$$

$$= -\lambda_i\, f_i(t,x)$$

provided ∂_t commutes with $\int_M$.

$\Rightarrow$ $f_i(t,x) = g_i(x)\,e^{-\lambda_i t}$ $\quad \forall i$.

For $f \in L^2(M)$ we have $f = \sum_i A_i \varphi_i$, and so

$$f(x) = \lim_{t\to 0+} \int_M K_M(t,x,y)\,f(y)\,dy$$

$$= \lim_{t\to 0+} \int_M \sum_i e^{-\lambda_i t} g_i(x)\,\varphi_i(y) \sum_j A_j \varphi_j(y)\,dy$$

$$= \lim_{t\to 0+} \sum_i e^{-\lambda t} g_i(x)\,A_i$$

$$= \sum_i g_i(x)\,A_i \quad \Rightarrow \quad g_i(x) = \varphi_i(x) \quad (\text{in } L^2(M))$$

$$\Rightarrow \quad K_M(t,x,y) = \sum_i e^{-\lambda_i t}\,\varphi_i(x)\,\varphi_i(y).$$

Convergence (pointwise) is $\forall\, t, x$ and a.e. $y \in M$.

$$\sum_{i=0}^{n_k} \varphi_i(x)\,\varphi_i(y)\,e^{-\lambda_i t} \to K_M(t,x,y)$$

Parseval equality $\langle K_M(t,x,-), K_M(t,x',-)\rangle_{L^2(M)} = \sum_i e^{-\lambda_i t}\,\varphi_i(x)\,\varphi_i(x')$

implies the continuous limit in t, x, x' $\Rightarrow$ convergence everywhere.

In fact, $\sum_i e^{-\lambda_i t} \varphi_i(x)\varphi_i(x') \to K_M(t,x,y)$ in $H_s(M\times M)$ $\forall s\in\mathbb{R}$, $t>0$

$\int_M$ commutes with ∂_t $\Leftarrow$ Sobolev space

<u>Corollary</u>: $\sum_i e^{-\lambda_i t} = Tr(e^{-t\Delta}) = \int_M K_M(t,x,x)\,dx$

Parametrix for the heat kernel equation

" approximate solution to the heat equation for x, y close $\in M$

<u>Lemma</u>: $\Delta(fg) = (\Delta f)g - 2\langle df, dg\rangle + f\Delta g$.

Pf: the statement is local; take $x\in M$ and normal Riemann coordinates around $x\in M$. $g_{ij}(x) = \delta_{ij}$, $\partial_k g_{ij}(x) = 0$ $\Rightarrow$

$$\Delta = -\frac{1}{\sqrt{\det g}}\partial_{x^i}\left(\sqrt{\det g}\, g^{ij}\partial_{x^j}\right) = -\sum_{i=1}^n \frac{\partial^2}{\partial x_j^2}$$

and

$$\langle df, dg\rangle = \left\langle \frac{\partial f}{\partial x^i}dx^i, \frac{\partial g}{\partial x^j}dx^j\right\rangle = \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}g^{ij}$$
$$= \sum_{i,j=1}^n \frac{\partial f}{\partial x^j}\frac{\partial g}{\partial x^j}.$$

Define a neigh. of the diagonal in $M\times M$, $U_\epsilon(\text{diag}) :=$

$\{(x,y)\in M\times M : y\in V_x, \underbrace{r(x,y)}_{\text{length of geod. connecting } x,y} < \epsilon\}$. Then

$$G(t,x,y) = (4\pi t)^{-n/2} e^{-\frac{r^2(x,y)}{4t}} \in C^\infty(\mathbb{R}_+^* \times U_\epsilon(\text{diag}))$$

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Heat kernel ...

<u>Lemma</u>: $\Delta(fg) = (\Delta f)g - 2\langle df, dg\rangle + f(\Delta g)$ on a Riemm. manifold (M, g), $\Delta = \Delta_g$.

Pf: The statement is local $\Rightarrow$ use normal Riem. coordinates at $x \in M$. ON-frame in T_xM $\Rightarrow$ $g_{ij}(x) = \delta_{ij}$, $\partial_k g_{ij}(x) = 0$. Thus

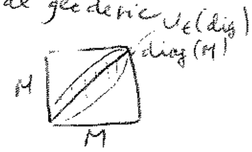
$$\Delta = -\frac{1}{\sqrt{\det g}}\partial_{x^i}\left(\sqrt{\det g}\, g^{ij}\partial_{x^j}\right) = -\sum_{i=1}^{n}\frac{\partial^2}{\partial x^{i2}},$$

and

$$\langle df, dg\rangle_{g^{-1}} = \left\langle \frac{\partial f}{\partial x^i}dx^i, \frac{\partial g}{\partial x^j}dx^j\right\rangle_{g^{-1}} = \sum_{i,j=1}^{n}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}g^{ij} = \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^i}. \quad \square$$

$x \in M$, $\exists \epsilon > 0$ $\exp_x$ $\underset{\substack{\cup\\ T_xM}}{B_\epsilon(0)} \xrightarrow[\text{diffeom.}]{\sim} \underset{\substack{\cup\\ M}}{U_\epsilon(x)}$, $y \in U_\epsilon(x)$ set $r(x,y) =$ the length of radial geodesic

$M \times M \supseteq U_\epsilon = \{(x,y) \in M \times M \mid y \in U_\epsilon(x),\ r(x,y) < \epsilon\}$.



Take $G(t,x,y) \equiv \frac{1}{(4\pi t)^{m/2}} e^{-\frac{r^2(x,y)}{4t}} \in C^\infty(\mathbb{R}^*_+ \times U_\epsilon)$,

and assume that for $t \to 0^+$ is a formal power series in t describing a deformation from the Euclidean case: fix $k \in \mathbb{N}$ and set

$$S \equiv S_k = S_k(t,x,y) = (4\pi t)^{-\frac{m}{2}} e^{-\frac{r^2(x,y)}{4t}}\left(u_0(x,y) + \dots + u_k(x,y)t^k\right)$$

for unknown $u_i \in C^\infty(U_\epsilon)$; we would like to have

$$(\partial_t + \Delta_y)S = 0.$$

A/ $\frac{\partial S}{\partial t} = G\left(\left(-\frac{m}{2t} + \frac{r^2}{4t^2}\right)(u_0 + \dots + t^k u_k) + (u_1 + 2u_2 t + \dots + k u_k t^{k-1}\right)$,

B/ $\Delta_y S = (\Delta_y G)(u_0 + \dots + u_k t^k) - 2\langle dG, d(u_0 + \dots + u_k t^k)\rangle + G\Delta_y(u_0 + \dots + u_k t^k)$,

where (G is radial fun)

C/ $\Delta_y G = -\frac{\partial^2}{\partial r^2}G - \frac{\partial G}{\partial r}\left(\frac{D'}{D} + \frac{m-1}{r}\right) = \left(\frac{m}{2t} - \frac{r^2}{4t^2}\right)G + \frac{r}{2t}\frac{D'}{D}G$

for $D = \det(d\exp_x)$,

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D/ $\langle dG, d(u_0 + \dots + u_k t^k)\rangle = \left\langle \frac{\partial G}{\partial r} dr + \frac{\partial G}{\partial \theta} d\theta, d(u_0 + \dots + u_k t^k)\right\rangle$

$= \left\langle \frac{\partial G}{\partial r} dr, \frac{\partial u_0}{\partial r} dr + \frac{\partial u_0}{\partial \theta} d\theta + \dots + t^k \frac{\partial u_k}{\partial r} dr + t^k \frac{\partial u_k}{\partial \theta} d\theta \right\rangle$

$\frac{\partial}{\partial \theta}$-derivatives do not contribute to scalar product (consequence of Gauss lemma)

$= \left\langle \frac{\partial G}{\partial r} dr, \frac{\partial u_0}{\partial r} dr + \dots + t^k \frac{\partial u_k}{\partial r} dr \right\rangle =$

$= \frac{\partial G}{\partial r}\left(\frac{\partial u_0}{\partial r} + \dots + t^k \frac{\partial u_k}{\partial r}\right) = -\frac{r}{2t}\left(\frac{\partial u_0}{\partial r} + \dots + t^k \frac{\partial u_k}{\partial r}\right) G,$

Altogether,

$$(\partial_t + \Delta_y) S = G\Big(u_1 + \dots + k t^{k-1} u_k + \frac{r}{2t}\frac{D'}{D}(u_0 + \dots + t^k u_k) + \frac{r}{t}\left(\frac{\partial u_0}{\partial r} + \dots + t^k \frac{\partial u_k}{\partial r}\right) + \Delta_y u_0 + \dots + t^k \Delta_y u_k\Big).$$

Make this expression vanish up to highest order $t^{k-\frac{m}{2}-1}$, i.e. solve $(\partial_t + \Delta_y) S = (4\pi t)^{-\frac{m}{2}} e^{-\frac{r^2(x,y)}{4t}} t^k \Delta_y u_k(x,y)$ by making the coefficients by $t^{i-\frac{m}{2}-1}$ equal zero for $i = 0, 1, \dots, k$.

This reduces to (transport) equations

$$r\partial_r u_0 + \frac{r}{2}\frac{D'}{D} u_0 = 0, \qquad \text{for } i = 0$$

$$r\partial_r u_i + \left(\frac{r}{2}\frac{D'}{D} + i\right) u_i + \Delta_y u_{i-1} = 0, \qquad \text{for } i = 1, \dots, k.$$

The first equation reduces to

$$\frac{\partial}{\partial r} \ln u_0 = -\frac{1}{2}\frac{\partial}{\partial r} \ln D \implies u_0 = D^{-1/2} = \det(d\exp_x)^{-1/2}$$

after normalisation of the solution.

$$u_0(x,y) = \frac{1}{\sqrt{\det(d\exp_x(y))}}, \quad \text{so} \quad u_0(x,x) = 1.$$

(3)

Solve first a simpler version of the second equation:

$$r \frac{\partial}{\partial r} u_i + \left(\frac{r}{2} \frac{D'}{D} + i \right) u_i = 0 ,$$

whose solution is $C r^{-i} D^{-1/2}$ ($C = C(\theta)$); assuming $u_i = k\, r^{-i} D^{-1/2}$ ($k = k(r)$), the first equation gives

$$\frac{\partial k}{\partial r} = - D^{1/2} (\Delta u_{i-1})\, r^{i-1}.$$

Let $x = x(s)$ be the unit speed geodesic from x to y, $s \in \langle 0, r \rangle$. Δu_{i-1} is a function of r along this geodesic, so the equation can be solved by r-integration:

$$u_i(x,y) = - r^{-i}(x,y) D^{-1/2}(y) \int_0^r D^{1/2}(x(s)) \Delta_y u_{i-1}(x(s), y) s^i \, ds.$$

Recursion relation for the solution of

$$(\partial_t + \Delta_y) S = (4\pi t)^{-\frac{m}{2}} e^{-\frac{r^2}{4t}} t^k \Delta u_k .$$

1/ Induction on i shows $u_i \in C^\infty(\mathcal{U}_\epsilon(x))$,

2/ $H_k := \eta S_k \in C^\infty(\mathbb{R}^*_+ \times M \times M)$ is parametrix of heat equation for $k > m/2$.
$\exists\, \eta(x,y) \in \langle 0, 1 \rangle$, $\eta = 0$ on $M \times M \setminus \mathcal{U}_\epsilon$, $\eta = 1$ on $\mathcal{U}_{\epsilon/2}$
$C^\infty(M \times M)$ (the bump function for $\mathcal{U}_\epsilon(0)$)

The parametrix $H_k(t,x,y)$ (good approx to the heat kernel for $t \sim 0$ and x, y close) & techniques of approximation allowing to promote $H_k(t,x,y)$ to the full heat kernel.

As a consequence,

<u>Prop:</u> $K_M(t,x,y)$ has the $t \to 0^+$ asymptotic expansion

$$K_M(t,x,y) \sim e^{-\frac{r(x,y)}{4t}} (4\pi t)^{-m/2} \sum_{k=0}^{\infty} u_k(x,y) t^k ,$$

so $$K_M(t,x,x) \sim (4\pi t)^{-m/2} \sum_{k=0}^{\infty} u_k(x,y) t^k .$$

(4) Asymptotic expansion for the trace of heat kernel:

Theorem: Let $\{\lambda_i\}_i$ be the spectrum of Δ on (M, g). Then

$$\sum_i e^{-\lambda_i t} \sim (4\pi t)^{-m/2} \sum_{k=0}^{\infty} a_k t^k,$$

$$a_k := \int_M u_k(x,x)\, d\,Vol_x.$$

Pf:
$$\sum_i e^{-\lambda_i t} = \int_M K_M(t,x,x)\, d\,Vol_x \sim \int_M (4\pi t)^{-m/2} \sum_k u_k(x,x) t^k \, d\,Vol_x$$
$$\sim (4\pi t)^{-m/2} \sum_k \left(\int_M u_k(x,x) d\,Vol_x \right) t^k.$$

Spectrum of Δ on M determines some geometric quantities: (M, g_M) and (N, g_N) are isospectral if the eigenvalues of their Laplacians agree (counted with multiplicity)

Corollary: (M, g_M), (N, g_N) compact isospectral Riem. manifolds. Then M, N have the same dimension and volume.

Pf: $\{\lambda_i\}_i$ ... spectrum of Δ on M, N;
if $\left.\begin{matrix} \dim M = m \\ \dim N = n \end{matrix}\right\}$

$$(4\pi t)^{-m/2} \sum_{k=0}^{\infty} \left(\int_M u_k^M(x,x) \right) t^k \sim \sum_i e^{-\lambda_i t} \sim$$
$$\sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} \left(\int_N u_k^N(x,x) \right) t^k.$$

This implies $m = n$. Thus

$$(4\pi t)^{-m/2} \left[\int_M u_0^M(x,x) - \int_N u_0^N(x,x) \right] \sim$$
$$\sim (4\pi t)^{-m/2} \sum_{k=1}^{\infty} \left(\int_M u_k^M(x,x) - \int_N u_k^N(x,x) \right) t^k.$$

This implies
$$\int_M u_0^M(x,x) = \int_N u_0^N(x,x)$$
and iterating the argument gives $\int_M u_k^M(x,x) = \int_N u_k^N(x,x)$, $\forall k \in \mathbb{N}$.

In particular, $u_0(x,x) = 1 \Rightarrow Vol(M) = Vol(N)$.

The quantity $u_1(x,x)$ gives another topological obstruction to manifolds having the same spectrum:

Proposition (M, g_M), (N, g_N) compact Riemann isospectral surfaces. Then M and N are diffeomorphic.

Pf: We have

$$\int_M u_1^M(x,x)\, d\,Vol_x = \int_N u_1^N(x,x)\, d\,Vol_x .$$

Because $u_1^M(x,x) = R(x)$ (the scalar curvature, as we shall prove the next time)

For M, N: $\left.\begin{array}{l}\dim_{\mathbb{R}} M = 2\\ \dim_{\mathbb{R}} N = 2\end{array}\right\} \Rightarrow$ scalar curvature is twice Gauss curvature

Gauss-Bonnet formula $\Rightarrow$

$$6\pi\, \chi(M) = \int_M u_1^M(x,x)\, d\,Vol_x = \int_N u_1^N(x,x)\, d\,Vol_x = 6\pi\, \chi(N).$$

Because oriented surfaces with equal Euler characteristic are diffeomorphic and the proof follows.

The next time compute $u_1^M(x,x) = R(x)$ (using normal Riemann coordinates). The next result

$$u_2(x,x) = \frac{1}{360}\left(2 R_{ijkl} R^{ijkl}(x) + 2 R_{jk} R^{jk}(x) + 5 R^2(x) - 12\, \Delta_x R(x)\right)$$

is discouraging, has no topological significance. However, is has certain geometrical significance (a close relation to Q-curvature, for example.)

Heat kernel ..

∃ infinite sequence of obstructions to two manifolds being isospectral: $\int_M u_k(x,x)\, d\,Vol_x$ $\forall k \in \mathbb{N}$.

$R_x, \nabla R_x, \nabla^2 R_x$ .. cov. der. of Riemannian curvature tensor at $x \in M$
P .. polynomial in •; P is universal provided its coefficients depend on dimension of M only
$P(R_x, \nabla R_x, .., \nabla^k R_x)$ .. always computed in a Riemannian normal coordinate chart.

Lemma: On (M, g), $\dim_{\mathbb{R}} M = m$.

$$u_1(x,x) = P_1^m(R_x),$$
$$u_i(x,x) = P_i^m(R_x, \nabla R_x, \nabla^2 R_x, .., \nabla^{2i-4} R_x), \quad i \geq 2,$$

for some universal polynomials P_i^m.

Pf: $u_0(x,x) = 1$, but for y close to x
$u_0(x,y) = \det^{-1/2} g(y)$ has a Taylor expansion with coefficients universal polynomials in $R_x, \nabla R_x, ..$, because
1/ g_{ij} has the expansion, 2/ $f^{-1/2}$ has such Taylor expansion.
The recursion formula

$$u_i(x,y) = - r^{-i}(x,y) \det(y)^{-1/2} \int_0^r \det(x(s))^{1/2} \Delta_y u_{i-1}(x(s), y)\, s^{i-1} ds,$$

$x(s)$ is a geodesic $x \to y = x(r)$.
In Riem normal coordinates at x, $r^2(y) = g_{ij} y^i y^j$:
$(y^1, .., y^m) = y$
1/ $r^{-i}(y)$ has Taylor expansion in y,
2/ if $u_{i-1}(x,y)$ satisfies ind. hypoth. $\forall x, y$ close, then $\Delta_y u_{i-1}(x,y)$
$\Rightarrow u_i(x,y)$ satisfies the hypothesis; setting $r=0$ computes $u_i(x,x)$
(= constant term in Taylor exp.).

Which (curvature) terms may occur in $u_i(x,x)$?

$\lambda \in \mathbb{R}$, $\lambda^2 g$ scaled metric, then
1/ $\Delta^{\lambda^2 g} = \lambda^{-2} \Delta^g$, so $\mathrm{Spec}(\Delta^{\lambda^2 g}) = \{\lambda^{-2}\lambda_i\}_i$, $\{\lambda_i\}_i$ spectrum for Δ^g,
2/ $\{\varphi_i\}_i$ ON basis $L^2(M, g)$, then $\{\lambda^{-\frac{m}{2}} \varphi_i\}$ is ON basis of $L^2(M, \lambda^2 g)$,
3/ R_g, ∇_g^k for g, then $\nabla^k_{\lambda^2 g} R_{\lambda^2 g} = \lambda^{-2-k} \nabla_g^k R_g$.

② $u_i^g(x,x)$ .. heat kernel coeff. for g

<u>Lemma</u>: $\lambda \in \mathbb{R}$, $u_k^{\lambda^2 g}(x,x) = \lambda^{-2k} u_k^g(x,x)$.

<u>Pf</u>: $$K_M^{\lambda^2 g}(t,x,x) = \sum_i e^{-(\lambda^{-2}\lambda_i)t} |\lambda^{-\frac{m}{2}} \varphi_i(x)|_g^2$$
$$= \lambda^{-m} \sum_i e^{-\lambda_i(\lambda^{-2}t)} |\varphi_i(x)|_g^2$$
$$= \lambda^{-m} K_M^g(\lambda^{-2}t, x, x)$$

Thus
$$\frac{1}{(4\pi t)^{\frac{m}{2}}} \sum_k u_k^{\lambda^2 g}(x,x) t^k \sim K_M^{\lambda^2 g}(t,x,x) = \lambda^{-m} K_M^g(\lambda^{-2}t, x, x)$$
$$\sim \frac{\lambda^{-m}}{(4\pi(\lambda^{-2}t))^{\frac{m}{2}}} \sum_k u_k^g(x,x)(\lambda^{-2}t)^k$$
$$= \frac{1}{(4\pi t)^{\frac{m}{2}}} \sum_k \lambda^{-2k} u_k^g(x,x) t^k,$$

and so $u_k^{\lambda^2 g} = \lambda^{-2k} u_k^g \quad \forall k \in \mathbb{N}$. □

Fix i, m, let $\mathcal{M} = \{m_j\}_j$ denote set of monomials in P_i^m, $m_j(g)$ is the monomial m_j evaluated for g on M, $\dim M = m$. Each $m_j(g)$ is of the form $(\nabla^{k_1} R)^{p_1} \cdots (\nabla^{k_q} R)^{p_q}$ for $k_1, \ldots, k_q$, $p_1, \ldots, p_q \in \mathbb{N}$.
(depending on j, not on g).

<u>Lemma</u>: $\forall$ monomial $(\nabla^{k_1} R)^{p_1} \cdots (\nabla^{k_q} R)^{p_q}$ in P_i^m, we have
$$i = \sum_{s=1}^{q} \left(1 + \frac{k_s}{2}\right) p_s.$$

<u>Pf</u>: $$u_i^{\lambda^2 g}(x,x) = \lambda^{-2i} u_i^g(x,x) = \lambda^{-2i} \sum_{m_j \in \mathcal{M}} m_j(g),$$

and by $\nabla^k_{\lambda^2 g} R_{\lambda^2 g} = \lambda^{-2-k} \nabla^k_g R_g$,
$$u_i^{\lambda^2 g}(x,x) = \sum_{m_j \in \mathcal{M}} m_j(\lambda^2 g) = \sum_{m_j \in \mathcal{M}} \lambda^{-\sum_s (2+k_s) p_s} m_j(g).$$

As a pol. in λ, the coefficients and exponents must agree $\Rightarrow$
$-2i = -\sum_s (2 + k_s) p_s$ and the proof follows.

□

(3) For $i=1$, $\forall$ mon. in P_1^m satisfies $1 = \sum_s \left(1+\frac{k_s}{2}\right)p_s$

$\Rightarrow \#s = 1$, $k_1 = 0$, $p_1 = 1$, so P_1^m is linear fcion with no constant term $\Rightarrow u_1(x,x)$ is a fcion of R at x.
(curvature tensor)

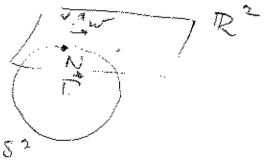
It cannot depend on R_{ijkl}, etc $\Rightarrow$ depends on $S_x := R^{ij}{}_{ij}$, the scalar curvature.
(coordinate invariance!)

$$\boxed{u_1(x,x) = C\, S_x}$$
$C = C(m)$

Lemma: $u_1(x,x) = \frac{1}{6} S_x$.

Pf: P_1^m is universal pol. $\Rightarrow$ compute for (S^m, g_0) ← the round metric

Exponential map: say, for $m=2$ $\quad S^2 \hookrightarrow \mathbb{R}^3$



$\exp: T_N S^2 \to S^2$

$(v,w) \mapsto \left(\theta = \sqrt{v^2+w^2},\ \varphi = \arctan\left(\frac{w}{v}\right)\right)$

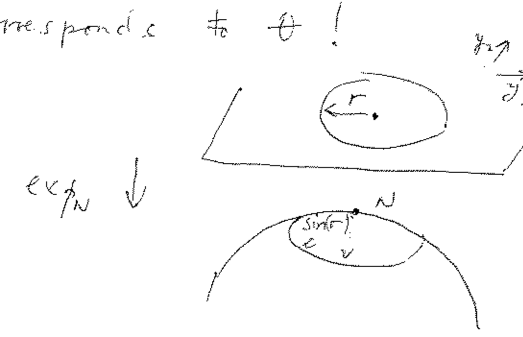
and the pull-back of g_0 by $\exp^*$:

$$\exp^*(g_0) = ds^2 = \frac{(v\,dv + w\,dw)^2}{v^2+w^2} + \sin^2\sqrt{v^2+w^2}\ \frac{(v\,dv - w\,dv)^2}{(v^2+w^2)^2}.$$

By expanding sin in a Taylor series, the metric near N is $dv^2 + dw^2 + \underset{(v,w)}{O}(v^2, w^2)$.

In our case, Riemannian normal polar coordinate are spherical (= spherical)
(S^m, g_0), $g_0 = dt^2 + \sin^2 t\, d\varphi^2$

Coordinate on $T_N S^m$, the radial coordinate (geodesic distance) corresponds to θ!



$r = \theta$!

$x = N$, y close to x on S^m

$y = \exp_x rv$, $v \in T_N S^m$, $|v| = 1$

$$\Rightarrow \det\left(T_N\left(\exp_N(rv)\right)\right) = \left(\frac{\sin r}{r}\right)^{m-1}$$

because

$$\begin{pmatrix} 1 & & 0 & \\ 0 & \frac{\sin r}{r} & 0 \ldots & \\ & 0 & \ddots & \\ & & & \frac{\sin r}{r} \end{pmatrix} = T_N\left(\exp_N(rv)\right)$$

(4)

$$\varphi(r) := u_0(x, \exp_x rv) = \left(\frac{\sin r}{r}\right)^{-\frac{(m-1)}{2}}$$

$$= 1 + \frac{(m-1)}{12} r^2 + \frac{(m-1)(5m-1)}{1440} r^4 + O(r^6), \quad \text{for } r \to 0.$$

Also,

$$\Delta_y u_0(x, \exp_x \tau r v)$$

$$= \varphi''(\tau r) + (m-1)\frac{\cos \tau r}{\sin \tau r} \varphi'(\tau r)$$

$$= \frac{m-1}{6} + O((\tau r)^2)$$

$$+ (m-1)\left(\frac{1}{\tau r}\left(1 - \frac{(\tau r)^2}{3} + O((\tau r)^4)\right)\right)\left(\frac{m-1}{6}\tau r + O((\tau r)^3)\right).$$

Taken together in

$$u_1(x, \exp_x rv) = \frac{1}{\sqrt{\det \exp_x rv}} \int_0^1 \sqrt{\det \exp_x(\tau r v)}\, \Delta_y (x, \exp_x \tau r v)\, d\tau,$$

we get $u_1(x, \exp_x r\underset{v}{\underset{\shortparallel}{y}}) = \frac{m(m-1)}{6} + O(r^2)$.

Thus $u_1(x, x) = \frac{m(m-1)}{6}$ on (S^m, g_0),

and because $S(x) = m(m-1)$ for all $x \in S^m$, the result follows.

Heat equation and holographic Laplacian and hidden conf. invariants

What we have learned so far?

① Solutions of the heat equation on (M,g) gives topological and geometrical informations about (M,g).

② $\exists$ algorithm to compute heat kernel coeff., but it is complicated.

③ In dimension 2 it holds

$$\int_M u_1(x,x)\, vol(g) = 6\pi\chi(M) \qquad \text{(Gauss-Bonnet)},$$

hence $\int_M u_1(x,x)\, vol(g)$ is a conf. inv., since r.h.s. does not depend on g.

④ The functional

$$\mathcal{E} : g \mapsto \int_M \overbrace{Scal_g}^{\sim u_1(x,x)}\, vol(g) \in \mathbb{R}$$

is called Einstein-Hilbert functional.

$\rightsquigarrow$ $\mathcal{E}$ is critical on Ricci-flat metrics g.

Now we are going to consider the heat-kernel coeff. of the heat equation

$$\partial_t + \mathcal{H}(r, g) = 0 ,$$

where

$$\mathcal{H}(r; g) : C^\infty(M) \to C^\infty(M) \quad \text{is a}$$

perturbation of the so-called Yamabe-operator

$$P_2(g) = \Delta^g - \frac{n-2}{2} J , \quad \text{i.e.} , \quad \mathcal{H}(0, g) = P_2(g) .$$

Plan:

(1) Conformal geometry

(2) Ambient metric (Poincaré-Einstein metric const.

(3) GJMS-operators and their inversion formula

(4) Holographic Laplacian $\mathcal{H}(r, g)$

(5) Heat kernel coeff of $\mathcal{H}(r, g)$

① Conformal geometry

Let (M^m, g) be a semi-Riem mfd. We say that $\hat{g} \sim_{conf} \hat{g} \Leftrightarrow \exists\ \sigma \in C^\infty(M) : \hat{g} = e^{2\sigma} g$.

This defines an equivalence relation, since

① a/ $g \sim g$ for $\sigma = 0$

b/ $g \sim \hat{g} \Rightarrow \hat{g} \sim g$

$(\hat{g} = e^{2\sigma} g \Rightarrow g = e^{-2\sigma} \hat{g})$

c/ $g \sim \hat{g} \wedge \hat{g} \sim \hat{\hat{g}} \Rightarrow g \sim \hat{\hat{g}}$

$(\hat{g} = e^{2\sigma} g \quad \hat{\hat{g}} = e^{2\tilde{\sigma}} \hat{g} = e^{2\tilde{\sigma}} e^{2\sigma} g = e^{2(\tilde{\sigma} + \sigma)} g)$

Def:

- We call $[g] =: c$ a conformal structure on M.
- We call (M, c) a conformal mfd.

What is the difference between

Riemannian geometry $\leftrightarrow$ Conformal geometry ?

(M, g) $\qquad$ (M, c)

Facts.

a/ The length of a vector $X \in T_pM$ $(p \in M)$ is

given by

$$\|X\|_g := \sqrt{|g_p(X,X)|}$$

Taking $\hat{g} = e^{2\sigma} g \in [g]$

$$\rightsquigarrow \|X\|_{\hat{g}} = \sqrt{|\hat{g}_p(X,X)|} = \sqrt{|e^{2\sigma} g_p(X,X)|}$$

$$= e^{\sigma} \|X\|_g$$

Hence a conformal str does not preserve

the length of vector.

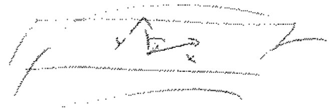
b/ Causality is preserved by c, i.e.,

If X is space / time / light - like w.r.t g

$\Leftrightarrow$ X is space / time / light - like w.r.t $\hat{g} = e^{2\sigma} g$

c/ Angles are preserved by c ~~[illegible]~~ for Riem. sign, i.e.,

$\forall\, X, Y \in T_pM$

$\measuredangle(X,Y) = \alpha$ is determined 

by

$$\cos(\alpha) = \frac{g(X,Y)}{\|X\|_g \|Y\|_g} = \frac{\hat{g}(X,Y)}{\|X\|_{\hat{g}} \|Y\|_{\hat{g}}}$$

- A scalar function / tensor / differential op $A(g)$ on (M, g) is said to be a Riemannian invariant
$:\Leftrightarrow$ $\forall$ Diffeom $\varphi : M \to M$ holds

$$\varphi^* (A(\varphi^* g)) = A(g)$$

Example:

- ~~Constant~~ Levi-Civita - connection

$$\nabla^g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$
$$X, Y \mapsto \nabla^g_X Y$$

- Riem. Curvature - tensor

$$\mathcal{R}^g : \mathfrak{X}(M)^4 \to C^\infty(M)$$

⋮

a lot and easy to find!

- A scalar function / tensor / diff. op. $A(g)$ on (M, g) is said to be a conformal "cova" variant
$:\Leftrightarrow$ $\forall$ conf diff. $\varphi : M \to M$ $\exists$ $(a,b) \in \mathbb{R}^2$ s.t.
$(\varphi^* g = e^{2\sigma} g)$

$$e^{a\sigma} \varphi^*(A(\varphi^* g)) e^{b\sigma} = A(g)$$

Example

- Weyl tensor $n \geq 4$, Cotton-tensor $n = 3$, Bach-tensor $n = 4$
- Dirac-op. / Yamabe-op

actually a lot but hard to find!

11. Conf transformation laws for "fundamental Riemannian invariants"

Lemma

Let ∇ and $\hat{\nabla}$ be the Levi Civita connection w.r.t. g and $\hat{g} = e^{2\sigma} g$. Then

$$\hat{\nabla}_X Y = \nabla_X Y + X(\sigma) Y + Y(\sigma) X - g(X,Y) \operatorname{grad}^g(\sigma)$$

$\forall$ $X, Y \in \mathfrak{X}(M)$.

⌜ Use Koszul formula.

$$2 g(\hat{\nabla}_X Y, Z) - 2g(\nabla_X Y, Z) = 2e^{-2\sigma} \hat{g}(\hat{\nabla}_X Y, Z) - 2 g(\nabla_X Y, Z)$$

$$e^{-2\sigma} X(\hat{g}(Y,Z)) + e^{-2\sigma} Y(\hat{g}(X,Z)) - e^{-2\sigma} Z(\hat{g}(X,Y))$$

$$- X(g(Y,Z)) - Y(g(X,Z)) + Z(g(X,Y))$$

$$X(\hat{g}(Y,Z)) = X(e^{2\sigma} g(Y,Z))$$

$$= X(e^{2\sigma}) \, g(Y,Z) + e^{2\sigma} X(g(Y,Z))$$

$$= 2e^{2\sigma} X(\sigma) \, g(Y,Z) + e^{2\sigma} X(g(Y,Z))$$

$$Y(\hat{g}(X,Z)) = 2 e^{2\sigma} Y(\sigma) \, g(X,Z) + e^{2\sigma} Y(g(X,Z))$$

$$Z(\hat{g}(X,Y)) = 2 e^{2\sigma} Z(\sigma) \, g(X,Y) + e^{2\sigma} Z(g(X,Y))$$

$$\Rightarrow 2 g(\hat{\nabla}_X Y, Z) - 2 g(\nabla_X Y, Z)$$

$$= 2 g(X(\sigma) Y, Z) + 2 g(Y(\sigma) X, Z) - 2 Z(\sigma) g(X,Y)$$

The result follows from $Z(\sigma) = g(\operatorname{grad}^g \sigma, Z)$ ⌟

On (M,g) consider the differential

$$d : \Omega^p(M) \to \Omega^{p+1}(M) \, ,$$

given by the formula

$$d\omega(X_0 \ldots X_p) = \sum_{i=0}^{p} (-1)^i \left(\nabla_{X_i} \omega\right)(X_0 \ldots \hat{X}_i \ldots X_p) \, .$$

The Hodge-op

$$* : \Omega^p(M) \to \Omega^{n-p}(M)$$

(has a formula, but we don't need)

and the co-diff.

$$\delta : \Omega^p(M) \to \Omega^{p-1}(M) \, , \quad \delta := (-1)^{n(p+1)+1} * d * \, .$$

<u>Lemma:</u> d and δ in terms of Levi-Civita

$$d\omega(X_0 \ldots X_p) = \sum_{i=0}^{p} (-1)^i \left(\nabla_{X_i} \omega\right)(X_0 \ldots \hat{X}_i \ldots X_p)$$

$$\delta\omega(X_1 \ldots X_{p-1}) = - \sum_{i=1}^{n} \left(\nabla_{s_i} \omega\right)(s_i, X_1 \ldots X_{p-1}) \, ,$$

where $\{s_i\}$ is a local g-ONB.

[standard text books ⊥

Conf. trafos for d, $*$ and δ

Lemma

For $\hat{g} = e^{2\sigma} g$ it holds

① $e^{a\sigma} \hat{d} \circ e^{-a\sigma} \omega = d\omega - a\, d\sigma \wedge \omega$

③ $e^{(a+2)\sigma} \hat{\delta} \circ e^{-a\sigma} \omega = \delta\omega - (n-2p-a)\, i_{\operatorname{grad}\sigma} \omega$

② $\hat{*}\omega = e^{(n-2p)} * \omega$, $\forall\, \omega \in \Omega^p(M)$

Especially we have for $\Delta := \delta d$ that

$$e^{(a+2)\sigma} \hat{\Delta} e^{-a\sigma} f = \Delta f - a\, \delta(d\sigma \wedge f) - (n-2-a)\, i_{\operatorname{grad}} df$$
$$+ a(n-2-a)\, |d\sigma|^2 f \qquad (*)$$

⌜ ① $\hat{d} = d$ + product rule

② $\hat{*}\omega = e^{(n-2p)}\omega$ via local formula for $*$ in terms of ONB.

③ using ① and ② for $\delta = (-1)^{n(p+1)+1} * \circ d \circ *$ ⌟

A direct consequence ⇒

Lemma

Let (M, g) be a 2-dim mfd.

$$\Rightarrow \quad e^{2\sigma} \hat{\Delta} \omega = \Delta \omega \qquad \forall \omega \in \mathcal{D}^0(M)$$

"conf. covariance".

From now on we will be interested only in

$C^\infty(M) = \mathcal{D}^0(M)$.

Question:

For general (M, g) does there exist correction term (curvature) such that Δ can be corrected to become conf. covariant?

Prop:

On (M^n, g), $n \geq 3$ the operator

$$P_2(g) := \Delta - \frac{n-2}{2} J \quad : C^\infty(M) \to C^\infty(M)$$

is conf. cov, i.e.,

$$e^{\frac{n+2}{2}\sigma} P_2(\hat{g}) e^{-\frac{n-2}{2}\sigma} = P_2(g)$$

⌈ Based on $e^{2\sigma} \hat{J} = J + \Delta\sigma - \frac{n-2}{2} (d\sigma)^2$ and $(*)$ ⌋

Rmk.

Let $(M,g) = (\mathbb{R}^n, \langle\cdot,\cdot\rangle)$

$\rightsquigarrow J = 0 \rightsquigarrow P_2(g) = \Delta$.

Furthermore: The op $P_{2N}(\langle\cdot,\cdot\rangle) := \Delta^N : C^\infty(\mathbb{R}^n) \circlearrowleft$

is conf. cov., i.e.,

$$e^{\frac{n+2N}{2}\sigma} \circ P_{2N}(\hat{g}) \circ e^{-\frac{n-2N}{2}\sigma} = P_{2N}(\langle\cdot,\cdot\rangle).$$

Next-time:

- Ambient / Poincaré-Einstein-const.
- GJMS-op. on (M,g)
- Inversion-formula
- Holographic Laplacian and its heat-kernel coeff.

Ambient metric construction

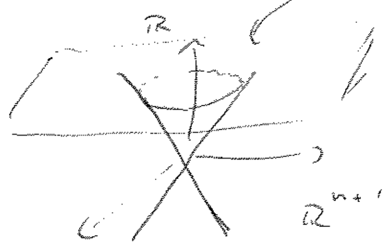
Idea: Let $(M^{p,q}, c)$ be a conf. mfd.

Can we construct a pseudo Riem. mfd $(\widetilde{G}^{p+1,q+1}, \widetilde{g})$ in a "unique" way s.t. Riem invariants on $(\widetilde{G}, \widetilde{g})$ give conf. invariants on (M, c)?

The model:

Consider the ~~[illegible]~~ sphere S^n with conf. class c induced by the round metric.

$\Rightarrow (\widetilde{G}^{1,n+1}, \widetilde{g}) = (\mathbb{R}^{1,n+1}, \langle\cdot,\cdot\rangle)$



light cone C^+

$i: C^+ \hookrightarrow \mathbb{R}^{n+1}$,

$g_0 = i^* \langle\cdot,\cdot\rangle$ is degenerate in one direction, but it ~~becomes~~ induces a metric on $\mathbb{P}C^+$ (projective light cone), moreover it induces a conf. class on $\mathbb{P}C^+$.

Hence, since $\mathbb{P}C^+ \cong S^n$ we recover the conf. class on S^n.

Now consider the general case:

Let (M, c) be a conf. mfd and define the metric bundle

$$Q := \dot{\bigcup_{x \in M}} \{ g_x : T_xM \times T_xM \to \mathbb{R} \mid g \in c \}$$

$$\subseteq S^2(T^*M) .$$

It carries a $\mathbb{R}^+$-action

$$\delta : Q \times \mathbb{R}^+ \to Q$$

$$(g_x, s) \longmapsto s^2 g_x =: \delta_s(g_x)$$

and a projection

$$\pi : Q \to M$$

$$g_x \mapsto x .$$

$\Rightarrow$ $(Q, \pi, M, \mathbb{R}^+)$ is a $\mathbb{R}^+$-principal bdl,

and every section $\sigma \in \Gamma(Q)$ gives

an representation of the conf. str., i.e.,

$$\sigma : M \to Q \quad , \quad \sigma(x) = g_x .$$

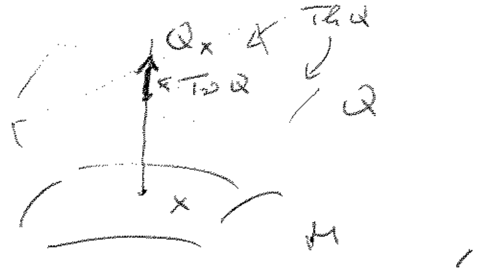
On Q we define a symmetric bilinear form

$$g_0 : TQ \times TQ \to C^\infty(Q) \quad \text{via}$$

$$(X, Y) \mapsto g_0(d\pi(X), d\pi(Y)) .$$

Note that g_0 is degenerate in direction

$T_v Q := \ker(d\pi)$ (vertical tangent space)



and is homogeneous with respect to δ, i.e.,

$$\delta_s^* g_0 = s^2 g_0 .$$

Def: (Fefferman/Graham)

An ambient metric space for $(M^{p,q}, c)$ is a pseudo-Riem. mfd. $(\tilde{G}^{p+1,q+1}, \tilde{g})$ s.t.

1) $\exists$ embedding $i : Q \hookrightarrow \tilde{G}$ with

$$i^* \tilde{g} = g_0$$

2) $\exists$ $\mathbb{R}^+$-action $\tilde{\delta} : \tilde{G} \times \mathbb{R}^+ \to \tilde{G}$ with

$$\tilde{\delta}_s^* \tilde{g} = s^2 \tilde{g} \quad \text{and} \quad i \circ \delta_s = \tilde{\delta}_s \circ i .$$

<u>Theorem</u> (Fefferman/Graham '85)

Let (M^n, c) be a n-dim. conf. mfd and set $\tilde{G} := \mathcal{Q} \times [-1,1]_\rho$.

a) n odd. Up to an $\mathbb{R}^+$-equivariant diffeom. fixing $\mathcal{Q}$, there is a unique formal power series solution $\tilde{g}$ to 1) and 2) which satisfies $Ric(\tilde{g}) = 0$.

b) n even. Up to an $\mathbb{R}^+$-equivariant diffeom. fixing $\mathcal{Q}$, there is a formal power series solution $\tilde{g}$ uniquely determined up to $O(\rho^{\frac{n}{2}})$ to 1) and 2) which satisfies

$$Ric(\tilde{g})|_{\mathcal{Q}} = O(\rho^{\frac{n-4}{2}}) \quad \text{and} \quad Ric(\tilde{g})|_{\mu} = O(\rho^{\frac{n-2}{2}}).$$

⌈ Idea of the proof:

Fix a metric $h \in c$ (but the const. will be independent of h), and assume <u>n odd</u>.

① $\exists$ coordinates (t, x, ρ) (geodesic coordinates) on $\tilde{G} = \mathbb{R}^+ \times M \times (-1,1)$

s.t. $$\hat{g}_{(t,x,\rho)} = t^2 \sum_{i,j=1}^{n} g_{ij}(x,\rho)\, dx_i dx_j + 2t\, dt\, d\rho + e(x,\rho)\, dt^2 + t \sum_{k=1}^{n} d_k(x,\rho)\, dx_k\, dt$$

with $\quad g_{ij}(x,0) = h_{ij}(x)\,,\quad e(x,0) = 0 = d_a(x,0)\,.$

② $\quad Ric(\tilde{g})_{00} = O(\rho^\infty)$ and $Ric(\tilde{g})_{0a} = O(\rho^\infty)$

$$\Longleftrightarrow$$

$$e(x,\rho) - 2\rho = O(\rho^\infty) \quad \text{and} \quad d_a(x,\rho) = 0\,.$$

③ ~~By induction~~ Inductively prove

a) $\quad Ric(\tilde{g})_{ij} = O(\rho) = Ric(\tilde{g})_{im}$

$$\Longleftrightarrow$$

$g_{ij}(x,\rho) = h^{(1)}_{ij}(x,\rho) + O(\rho^2)$ with

$h^{(1)}_{ij}(x,\rho) = h_{ij}(x) + 2\rho\, P_{ij}$

$\uparrow$ Schouten-tensor w.r.t. h.

$\frac{1}{n-2}\left(Ric^{(h)} - J h\right)_{ij}$

b) $\quad Ric(\tilde{g})_{ij} = O(\rho^s) = Ric(\tilde{g})_{im}$ and

$$Ric(\tilde{g})_{mm} = O(\rho^{s-1})$$

$$\Longleftrightarrow$$

$g_{ij}(x,\rho) = h^{(s)}_{ij}(x,\rho) + O(\rho^{s+1})$ with

$h^{(s)}_{ij}(x,\rho) = h^{(s-1)}_{ij}(x,\rho) + \tau^{(s)}_{ij}(x)\,\rho^s$ and

$\tau^{(s)}_{ij}(x) = \frac{2}{s(n-2s)}\left(H^{(s-1)}_{ij} - \frac{tr_h\left(H^{(s-1)}\right)}{s(n-s)}\, h_{ij}\right)$ for

$$Ric\left(h^{(s-1)}\right)_{ij} = H^{(s-1)}_{ij}(x)\,\rho^{s-1}\,.$$

Summary:

If n is odd there exist a complete power series (determined by $h \in c$)

$$g_{ij}(x,\rho) = h_{ij}(x) + 2\rho P_{ij} + \dots \qquad \text{s.t.}$$

$$\tilde{g}_{(t,x,\rho)} = \begin{pmatrix} 2\rho & 0 & \cdots & 0 & t \\ 0 & & & & 0 \\ \vdots & & t^2 g_{ij}(x,\rho) & & \vdots \\ 0 & & & & 0 \\ t & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

If n is even we can only uniquely determine $g_{ij}(x,\rho)$ by

$$g_{ij}(x,\rho) = h_{ij}(x) + 2\rho P_{ij} + \dots + \rho^{\frac{n}{2}-1} S_{ij} + O(\rho^{\frac{n}{2}}).$$

Poincaré - Einstein metrics

Poincaré - Einstein metric spaces carry on their boundary a conf. mfd.

The model:

- Consider the $(n+1)$ - dimensional unit ball

$$B^{n+1} := \{ x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle \leq 1 \}$$ with

hyperbolic metric

$$g_{hyp} = \frac{4}{(1-|x|^2)^2} \langle \cdot , \cdot \rangle \qquad (*)$$

$\rightsquigarrow (B^{n+1}, g_{hyp})$ has $(S^n, [g_{S^n}])$ as conf. infinity.

Note that $(*)$ can be rewritten as

$$r(x) := \frac{1-|x|}{1+|x|} \quad \leftarrow \text{defining function for } S^n, \text{ i.e. } S^n = \{r=0\}.$$

$$g_{hyp} = r^{-2} \left(dr^2 + \frac{1}{4}(1-r^2)^2 g_{S^n} \right)$$

Now we consider the general case.

Let (M, c) be an n-dim. conf. mfd.

<u>Def.</u>

Let X^{n+1} be a mfd with boundary M.

A metric g_+ on $\mathrm{int}(X)$ is called a

Poincaré-Einstein metric iff

1) $\exists$ a defining function

$r: X \to \mathbb{R}^{\geq 0}$ with $\{r=0\} = M$

and $dr|_M \neq 0$

2) $\bar{g} := r^2 g_+$ is smooth on X

3) $\bar{g}|_{TM \times TM} \in c$

<u>Rmk</u>

The metric $\bar{g}|_{TM \times TM}$ depends on r

and thus only determines a conf

structure on M.

Theorem (Fefferman/Graham '85)

Let (M^n, c) be a conf. mfd and set $X := [0,\varepsilon) \times M$.

a) n odd. Up to a diffeom. fixing M, there is a formal power series solution of a Poincaré-Einstein metric g_+ with $Ric(g_+) = -n g_+$

b) n even. Up to a diffeom. fixing M, there is a formal power series solution of a Poincaré-Einstein metric g_+, uniquely determined up to order $O(r^{n-2})$, s.t. $Ric(g_+) + n g_+ = O(r^{n-2})$.

Idea of the proof: Assume n odd.

- Make an ansatz $X := [0,\varepsilon)_r \times M_x$, $h \in c$

$$g_+ = r^{-2}(dr^2 + h_r)$$ for a 1-parameter family of metric h_r on M.

- a choice of $h \in c$ determine the defining function (uniquely) by

$$|dr|_{\bar{g}} = 1 \quad \text{on } X$$

- $h_r = \sum_{i=0}^{\infty} h^{(i)}(x)\, r^i$ has to solve

$$\mathrm{Ric}(g_+) = -n\, g_+ \, ,$$

hence h_r is uniquely determined by h.

$\lrcorner$

<u>Summary</u>

Let (M, c) be a conf. mfd. with $h \in c$.

$\Rightarrow \exists$ Poincaré-Einstein metric g_+ on $X = [0,\varepsilon) \times M$

s.t. $\quad g_+ = r^{-2}(dr^2 + h_r) \quad$ an

$$h_r = h + r^2 h_{(2)} + \dots \qquad (n \text{ odd})$$

$$h_r = h + r^2 h_{(2)} + \dots + r^{(n-2)} h_{(n-2)} + O(r^n)$$

(~~another~~ $h_{(2)}, \dots, h_{(n-2)}, \dots$ are determined by h.)

Holographic Laplacian and its heat kernel coefficients

$(M^n, [h])$ ~~ex~~ n-dim conf. mfd.

⇝ Poincaré-Einstein metric space (X, g_+)

[$h \in [h]$ ⇝ $X \cong (0,1) \times M$, $g_+ = r^{-2}(dr^2 + h_r)$
(r, x)

for $h_r = h + r^2 h^{(2)} + r^4 h^{(4)} + \dots$]

1) holographic volume coefficients (n even)

$$v(r) := \frac{\sqrt{\det(h_r)}}{\sqrt{\det(h)}} = 1 + r^2 v_2 + r^4 v_4 + \dots + r^n v_n + \dots$$

$d\mathrm{vol}(h_r) = v(r)\, d\mathrm{vol}(h)$,

v_{2k} ~ holographic coefficients

v_n ~ holographic anomaly

Partition functions of CFT on M are related to $\mathrm{Vol}(X)$

Prop: (Graham '99)

M^n compact, n even.

$\Rightarrow \int_M v_n \,\mathrm{vol}(h)$ is a conf invariant

$\left(\left(\int_M v_{2k}(h)\, \mathrm{vol}(h) \right)^{\bullet}[\varphi] = (n-2k) \int_M v_{2k}(h)\, \mathrm{vol}(h) \right)$

Ex. $n = 2$ $\quad v_2 = -\frac{1}{2} J$

$\rightsquigarrow \int_{M^2} v_2 \,\mathrm{vol}(h) = -2\pi \chi(M^2)$

$n = 4$ $\quad v_4 = \frac{1}{8}(J^2 - |P|^2)$

$\rightsquigarrow \int_{M^4} v_4 \,\mathrm{vol}(h) = 2(2\pi)^2 \chi(M^4) - \frac{1}{4} \int |W|^2 \mathrm{vol}(h)$

2) Q-curvature

Consider the GJMS-operator

$$P_{2N}(h) : C^\infty(M) \to C^\infty(M) \quad \text{and}$$

define

$$\left(\frac{n}{2}-N\right)(-1)^N Q_{2N}(h) := P_{2N}(h)(1)$$

(1 ↑ constant function.)

⇝ $Q_{2N}(h)$ is a Riemannian invariant of order $2N$ (w.r.t. h).

<u>Ex</u>

$Q_2(h) = J$

$Q_4(h) = \frac{n}{2} J^2 - 2|P|^2 - \Delta J$

For even n we call $Q_n(h)$ the critical Q-curvature.

<u>Prop:</u>

$$e^{n\sigma} Q_n(e^{2\sigma} h) = Q_n(h) + P_n(h)(\sigma)$$

and $\left(\int_M Q_{2N}(h)\, vol(h)\right)^\bullet [\sigma] = (n-2N) \int_M Q_{2N}(h)\, dvol(h)$)

<u>Corollary</u> M^n closed mfd.

$\int_{M^n} Q_n(h)\, vol(h)$ is conf. invariant.

total Q-curvature.

$Q_2 = -2 v_2$

$Q_4 = -P_2(Q_2) - Q_2^2 + 16 v_4$

⋮

<u>Theorem</u> (Graham/Juhl '07)

n even. Q_n (critical Q-curvature) naturally decompose $\mathbb{R}[v_n, P(v_{2k})]$ (← diff. op.).

3) Heat kernel coeff for $P_2(g)$ - conf. Laplacian

(M,g) comp. mfd

let $K_M \in \mathcal{C}^\infty(\mathbb{R}_+ \times M \times M)$ be the heat kernel

of $P_2(g)$.

$t \to 0^+$ asymptotic ex.

$$K_M(t, x, x) \overset{t \to 0^+}{\sim} (4\pi t)^{-\frac{n}{2}} \sum_{k=0}^{\infty} a_{2k}(x,x)\, t^k$$

↗ heat kernel coeff.

($a_{2k} = m_k$ from previous lectures)

Ex: $6\, a_2 = -(n-4)\, J$

$$360\, a_4 = -(n-6)\left[\, 2(n-2)\, |P|^2 - (5n-16)\, J^2 - 6\Delta J \,\right] + 2\, |W|^2$$

(Parker/Rosenberg '87 Branson/Ørsted '86)

Prop:

$n = 4 \Rightarrow a_2$ is a conf. inv.

$n = 6 \Rightarrow a_4$ is a conf. inv.

Prop. (Parker/Rosenberg '87). (M^n, g) closed n-dim

$\int_M a_{2n} \, vol(g)$ is a conf. inv.

~~$\int_{M^4} a_4 \, vol(g)$~~

$\left(\int_M \dot{a}_{2k}(g) \, vol(g) \right)^{\cdot}[\sigma] = (n-2k) \int_M a_{2k}(g) \, vol(g) \right)$

Summary:

Let (M^n, g) be closed n-dim Riem. mfd, n even

$\leadsto$ $\{ v_2, v_4, \ldots, v_n \}$

$\{ Q_2, Q_4, \ldots, Q_n \}$ Riem. invariants w.r.t. g

$\{ a_2, a_4, \ldots, a_n \}$ total integral is a conf. inv.

and

Relation between

$\{ a_{2k}, Q_{2n} \}$ $\to$ generalization of Polyakov's formula

$3 a_2 = (n-4) v_2$ $\longrightarrow$ $\{ a_{2k}, v_{2k} \}$ $\to$ gauge gravity dualities

$180 \, a_4 = (n-6) \left(8(n-2) v_4 + 6(n-4) v_2^2 - 6 \Delta v_2 \right) + |W|^2$

4) holographic Laplacian

a) Inversion formula:

~~Th.~~ ∃ sequence of second order ~~self~~ formally self-adjoint op. $\{M_2, M_4, \dots\}$ s.t.

$$P_{2N} \in \mathbb{R}[M_2, \dots, M_{2N}]$$

$$M_{2N} \in \mathbb{R}[P_2, \dots, P_{2N}]$$

Ex. · $M_2 = P_2$

· $M_4 = P_4 - P_2^2$ vs. $P_4 = M_2^2 + M_4$

(· $M_6 = P_6 - 2(P_2 P_4 + P_4 P_2) + 3 P_2^3$

vs. $P_6 = M_2^3 + 2(M_2 M_4 + M_4 M_2) + M_6$)

Now define

$$\mathcal{H}(r; h) := \sum_{N \geq 1} M_{2N} \frac{1}{(N-1)!^2} \left(\frac{r^2}{4}\right)^{N-1}$$

holographic Laplacian

⇝ $\mathcal{H}(0; h) = M_2 = P_2$ Yamabe op.

Theorem (holographic deformation of the Yamabe operator)

$$\mathcal{H}(r;h) = -\delta^h(h_r^{-1} d) - r^2 \left(\Delta_{g_+}(\log w) - |d \log w|^2_{g_+} \right)$$

for g_+ - Poincaré-Einstein metric

$w(r) := \sqrt{v(r)}$

Remark:

This formula was recovered by Fefferman/Graham 2012

Ex: see later for (M,g) Einstein.

5) Heat kernel coefficients of $\mathcal{H}(r,h)$

Consider the heat equation

$$\partial_t + \mathcal{H}(r;h) = 0 \qquad \text{for small } r \in [0,\varepsilon].$$

Theorem (Juhl 2014)

Let $K(x,y,t;r)$ be the kernel of $e^{t\mathcal{H}(r)}$,

i.e. $e^{t\mathcal{H}(r)}(u)(x) = \int_M K(x,y,t,r)\, u(y)\, dvol(h)$,

$$\Rightarrow \quad K(x,x,t;r) \overset{t\to 0^+}{\sim} (4\pi t)^{-\frac{n}{2}} \sum_{j\geq 0} t^j a_{2j}(x;r)$$

for $a_{2j}(x;r) \in C^\infty(M \times (-\varepsilon,\varepsilon))$.

Rmk:

The proof of that Theorem also yields a recursive structure determining the heat kernel coefficients (see previous lectures).

Corollary

Consider

$$a_{2j}(x;r) := \sum_{k \geq 0} a_{2j,2k}(x)\, r^{2k} .$$

Then $a_{2j,2k} \in C^\infty(M)$ are scalar Riem. invariants, and $a_{2j}(x;0) = a_{2j}(\cdot)$

← heat kernel coeff for $\Delta_2(g)$.

Examples

- $a_0(\cdot\,; r) = v(r) = 1 + r^2 v_2 + \dots$

- $$a_2(\cdot, r) = \Big[\frac{scal(g_r)}{6} + \frac{1}{2} \Delta^{g_r} \log(v) - \frac{1}{4} |d \log v|^2 + \mathcal{U}(r) \Big]\, v(r)$$

for $\mathcal{U}(r) = -v(r)^{-1} \big[\partial_r^2 - (n-1) r^{-1} \partial_r - \delta(g_r^{-1} d) \big]\, v(r)$

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~~Furthermore, it holds~~

7) Applications

a) Recover the heat kernel coeff. of $P_2(g)$

$$a_{2j}(0;g) = a_{2j}$$

b) (M,g) Einstein

$$a_{2j}(r,g) = \left(1 - \frac{2r^2}{4}\right)^{-2} a_{2j}(g)$$

c) ∃ conf. variation formula for $a_{2j}(r,g)$ on Einstein manifolds

$\int_{M^n} a_{2j}($

$$\left(\int_{M^n} q_{(2j,2k)}(g)\, dvol(g)\right)^{\bullet}[\varphi]$$

$$= (n-2j-2k)\int_{M^n} \varphi\, q_{(2j,2k)}(g)\, vol(g)$$

$\leadsto$ $\int_{M^n} q_{(2j,n-2j)}(x)\, dvol(g)$ is a conf. invariant.

(Holds maybe in general)

6) ~~Applications~~ Properties of $\mathcal{H}(r,g)$

- (M,g) Einstein

$\leadsto \quad P_{2N}(g) = \prod_{e=1}^{N} (\Delta + c_e J)$ for some constant $c_e \in \mathbb{R}$.

~~As a~~

Furthermore

$$g_r^{-1} = (1 - r^2/2J)^{-2}$$

$$w(r) = \sqrt{\det(1 - \tfrac{r^2}{2J})}$$

$$\leadsto \mathcal{H}(r;g) = \left(1 - \frac{r^2}{4}\right)^{-2} \left(\Delta - \tfrac{n}{2}\left(\tfrac{n}{2}+1\right)\right)$$

(M,g) round sphere

$$\mathcal{H}(r;g) = \left(1 - \frac{\lambda r^2}{4}\right)^{-2} P_2(g) \quad \text{for } \lambda = \frac{\tau}{n(n-1)}$$

constant.

- Conformal variation of $\mathcal{H}(r;g)$:

(conf. weights of $P_2(g)$)

$$\frac{d}{dt}\Big|_{t=0} \left(e^{(\frac{n}{2}+1)t\sigma} \mathcal{H}(r, e^{2t\sigma}) e^{-(\frac{n}{2}-1)t\sigma} \right)$$

$$= -\frac{1}{2} \partial_r \left(\varphi \cdot \mathcal{H}(r;g) + \mathcal{H}(r;g)(\varphi \cdot) \right)$$

$$- \left[\mathcal{H}(r,g), \left[\mathcal{K}(r,g), \sigma \right] \right]$$

for $\mathcal{K}(r;g) = \frac{1}{2} \int_0^r s\, \mathcal{H}(s,g)\, ds$