(1)

A representation of a group G is a homomorphism $G \to GL(V)$, where $V = V/\mathbb{C}$ is a vector space often equiped by structure of a Hilbert space. The notions of irreducible, unitary, direct sum and equivalence, applied to representations of a fixed group G (finite, discrete, topological, compact, smooth, ...), will be reviewed later.

There are two general problems to consider, for any given G:

1/ Classify the set $\Pi(G)$ of equivalence classes of irreducible unitary representations of G.

2/ If $R : G \to GL(V)$ is some natural unitary representation of G, decompose R explicitly into irreducible representations. This amounts to find a G-equivariant isomorphism $V \xrightarrow{\sim} W$, where W is a space buildt explicitly out of irreducible representations as a direct sum

$$W \simeq \bigoplus_{\pi \in \Pi(G)} n_\pi V_\pi \quad , \quad n_\pi \in \mathbb{N} \cup \{\infty\},$$

or perhaps in some more general fashion.

Harmonic analysis aims to resolve 1/, 2/ by analytical tools; it is based on the following vocabulary between geometrical-topological and spectral quantities:

	Geometric/topological objects	Spectral = analytical objects
Linear algebra	sum of diagonal entries of a square matrix	sum of eigenvalues of the matrix
Finite groups	Conjugacy classes	Irreducible characters
Number theory	Logarithms of powers of prime numbers	Zeroes of zeta functions, i.e., $\zeta(s)$
Automorphic forms	Rational conjugacy classes	Automorphic representations
Differential geometry	Length of closed geodesics	Eigenvalues of the Laplace operator
Algebraic geometry	Algebraic cycles	Motives
Topology	Singular homology	De-Rham cohomology
Physics	Particles in classical mechanics	Waves in quantum mechanics

Here are two well-known examples from the basic course of analysis:

Example 1: (Fourier analysis and compact - discrete duality):
~~(Compact)~~

$$G = \mathbb{R}/\mathbb{Z} \simeq S^1, \quad V = L^2(\mathbb{R}/\mathbb{Z}),$$

$$(R(y)f)(x) = f(x+y), \quad x, y \in G, \ f \in V$$

Fourier analysis on $\mathbb{R}/\mathbb{Z}$, the set $\Pi(\mathbb{R}/\mathbb{Z})$ is parametrized by $\mathbb{Z}$:

$$\pi \in \Pi(\mathbb{R}/\mathbb{Z}) \iff V_\pi \simeq \mathbb{C}$$

$$\pi(y)\, v = e^{-2\pi i n y} v, \quad v \in V_\pi, \ n \in \mathbb{Z}.$$

The space

$$\hat{V} = L^2(\mathbb{Z}) = \{ c = \{c_n\}_{n \in \mathbb{Z}} : \sum_{n \in \mathbb{Z}} |c_n|^2 < \infty \}$$

supports the representation

$$(\hat{R}(y)c)_n = e^{2\pi i n y} c_n, \quad c = \{c_n\}_{n \in \mathbb{Z}}$$

which is a direct sum of all irreducible representations of G, each occuring with multiplicity one.

The Fourier coefficients $\{\hat{f}_n\}_{n \in \mathbb{Z}}$ of f,

$$f \to \hat{f}_n := \int_{\mathbb{R}/\mathbb{Z}} f(x) e^{-2\pi i n x}\, dx,$$

gives the isomorphism from V to $\hat{V}$ such that $R \simeq \hat{R}$. It satisfies the Plancherel formula

$$\int_{\mathbb{R}/\mathbb{Z}} |f(x)|^2\, dx = \sum_{n \in \mathbb{Z}} |\hat{f}_n|^2.$$

④ Example 4: $G = \mathbb{R}$ (additive group), $V_R = L^2(\mathbb{R})$,

$$(R(y)f)(x) = f(x+y), \quad y \in G, \ f \in V_R.$$

In this case is $\Pi(G)$ parametrized by $\mathbb{R}$:

$$\pi \in \Pi(G) \Leftrightarrow V_\pi \simeq \mathbb{C}$$

$$\pi(y)v = e^{-i\lambda y}v, \quad v \in V_\pi, \ \lambda \in \mathbb{R}.$$

We define $\hat{V} = L^2(\mathbb{R})$, and

$$(\hat{R}(y)\varphi)(\lambda) = e^{i\lambda y}\varphi(\lambda), \quad \varphi \in \hat{V}, \ \lambda \in \mathbb{R}.$$

Then $\hat{V}$ is a "continuous direct sum", or direct integral of irreducible representations.
The Fourier transform

$$f \to \hat{f}(\lambda) = \int_{\mathbb{R}} f(x) e^{-i\lambda x}\,dx, \quad f \in C_c^\infty(\mathbb{R}),$$

where $C_c^\infty(\mathbb{R})$ are smooth compactly supported functions (dense in $L^2(\mathbb{R})$), extends to a topological isomorphism $V \to \hat{V}$ that satisfies Plancherel formula

$$\int_{\mathbb{R}} |f(x)|\,dx = \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{f}(\lambda)|^2\,d\lambda.$$

Harmonic analysis on finite groups

Notation:

X ... finite set (top. space with the discrete topology), $\mathcal{F}(X) := \{f : X \to \mathbb{C}\}$ ... (fin.-dim) vector space of $\mathbb{C}$-valued functions on X

$\dim_{\mathbb{C}} \mathcal{F}(X) = |X|$

$x \in X$, $\delta_x \in \mathcal{F}(X)$ Dirac function supported at $x \in X$:

$$\delta_x(y) = \begin{cases} 1 & y = x \\ 0 & y \neq 0 \end{cases}$$

$\{\delta_x \mid x \in X\}$ ... natural basis for $\mathcal{F}(X)$,

$f \in \mathcal{F}(X) \Rightarrow f = \sum_{x \in X} f(x)\, \delta_x$.

$\mathcal{F}(X)$ carries scalar product $\langle , \rangle : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{C}$

$$f_1, f_2 \mapsto \langle f_1, f_2 \rangle := \sum_{x \in X} f_1(x) \overline{f_2(x)}$$

and set $\|f\|^2 := \langle f, f \rangle$.

The basis $\{\delta_x \mid x \in X\}$ is ON with respect $\langle , \rangle$.

For $Y \subseteq X$, 1_Y ... characteristic function of Y

$$1_Y(x) = \begin{cases} 1 & x \in Y \\ 0 & x \notin Y \end{cases}$$

($Y = X$ write $1 = 1_X$.)

$X = Y_1 \dot{\cup} Y_2$ disjoint union ($Y_1 \cap Y_2 = \emptyset$), iterated for any finite collection of subspaces.)

⑥

$A : \mathcal{F}(X) \to \mathcal{F}(X)$ linear operator;

set $a(x,y) := (A\delta_y)(x)$, $x, y \in X$ arbitrary

$$(Af)(x) = \sum_{y\in X}^{!} a(x,y) f(y), \quad \forall f \in \mathcal{F}(X)$$

so the operator A is represented by matrix $a = \{a(x,y)\}_{x,y\in X}$

For $A_1, A_2 : \mathcal{F}(X) \to \mathcal{F}(X)$ linear operators

$A_1 \leftrightarrow \{a_1(x,y)\}_{x,y\in X}$

$A_2 \leftrightarrow \{a_2(x,y)\}_{x,y\in X}$ $\Rightarrow$ $A_1 \circ A_2$ is represented by product of matrices

$$a(x,y) = \sum_{z\in X} a_1(x,z)\, a_2(z,y)$$

Identity operator is represented by identity matrix $I = \{\delta_x(y)\}_{x,y\in X}$.

Harmonic analysis on finite cyclic groups

$C_n = \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$ cyclic group of order n

$\shortparallel$

$\mathbb{Z}/n\mathbb{Z}$ $\quad \bar{x} = x + n\mathbb{Z}$

$\bar{x} + \bar{y} = \overline{x+y}$, notation $\bar{x} = x$ (and $-$ denotes complex conjugation)

$f \in \mathcal{F}(C_n) \simeq \{ f : \mathbb{Z} \to \mathbb{C} \mid f(x+n) = f(x) \ \forall x \in \mathbb{Z} \}$

The functions χ_x, $x \in C_n$, defined by

$$\chi_x(y) := \omega^{xy}, \quad x, y \in C_n, \quad \omega = e^{\frac{2\pi i}{n}}$$

are characters (homom. $C_n \to \mathbb{C}^*$), such that $\chi_x(y) = \chi_y(x)$

$\chi_y(-x) = \overline{\chi_y(x)}$, $\chi_0 = 1$.

In fact, any $\varphi : C_n \to S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ satisfying

$$\left.\begin{array}{l} \varphi(x+y) = \varphi(x)\varphi(y) \\ \varphi(0) = 1 \end{array}\right\} \Rightarrow \varphi = \chi_z \text{ for some } z \in C_n.$$

Lemma 1: (ortogonality relations) Set $\delta_0(x) = \begin{cases} 1 & x \underset{n}{=} 0 \\ 0 & \text{otherwise} \end{cases}$

Then $\sum_{y=0}^{n-1} \chi_{x_1}(y)\overline{\chi_{x_2}(y)} = n\,\delta_0(x_1 - x_2)$.

Pf: $\chi_{x_1}(y)\overline{\chi_{x_2}(y)} = \omega^{y(x_1-x_2)} = \left(\omega^{(x_1-x_2)}\right)^y$. For $x_1 \underset{\text{mod } n}{\neq} x_2$

$z = \omega^{(x_1-x_2)}$ satisfies $z^n - 1 = 0$ but not $z - 1 = 0$,

and from $z^n - 1 = (z-1)(1 + z + \dots + z^{n-1}) = (z-1)\sum_{y=0}^{n-1} z^y$

$\Rightarrow \sum_{y=0}^{n-1} \chi_{x_1}(y)\overline{\chi_{x_2}(y)} = \sum_{y=0}^{n-1} z^y = \dfrac{z^n - 1}{z-1} = 0$.

For $x_1 \underset{\text{mod } n}{=} x_2$ then $\omega^{(x_1-x_2)} = 1 \Rightarrow$ sum $= n$. ☒

By $\chi_x(y) = \chi_y(x) \Rightarrow \sum_{x \in C_n} \chi_x(y_1)\overline{\chi_x(y_2)} = n\,\delta_0(y_1 - y_2)$.

$\dim_{\mathbb{C}} \mathcal{F}(C_n) = n \Rightarrow \{\chi_x \mid x \in C_n\}$ is an OG-basis of $\mathcal{F}(C_n)$

Def 2: (Fourier transform) The Fourier transform for $f \in \mathcal{F}(C_n)$ is $\hat{f} \in \mathcal{F}(C_n)$:

$$\hat{f}(x) := \langle f, \chi_x \rangle = \sum_{y \in C_n} f(y)\overline{\chi_x(y)}.$$

χ_x's are OG-basis for $\mathcal{F}(C_n)$:

Theorem 3: (Fourier inversion formula) $|C_n| = n$

$$f = \frac{1}{n}\sum_{x \in C_n} \hat{f}(x)\chi_x \qquad \forall f \in \mathcal{F}(C_n).$$

Pf: $\frac{1}{n}\sum_{x \in C_n} \hat{f}(x)\chi_x(y_1) = \frac{1}{n}\sum_{x \in C_n}\sum_{y \in C_n} f(y)\overline{\chi_x(y)}\chi_x(y_1)$

$= \frac{1}{n}\sum_{y \in C_n} f(y) \sum_{x \in C_n} \overline{\chi_x(y)}\chi_x(y_1) = \frac{1}{n}\sum_{y \in C_n} f(y)\,\delta_0(y - y_1)_n$

$= f(y_1)$

☒

Th 4: (Plancherel formula) For $f \in \mathcal{F}(C_n)$ we have $\|f\|\sqrt{n} = \|\hat{f}\|$.

Pf:

$$\|\hat{f}\|^2 = \langle \hat{f}, \hat{f} \rangle = \sum_{x \in C_n} \hat{f}(x)\overline{\hat{f}(x)} =$$

$$= \sum_{x \in C_n} \left(\sum_{y_1 \in C_n} f(y_1) \overline{\chi_x(y_1)} \right) \left(\sum_{y_2 \in C_n} \overline{f(y_2)} \chi_x(y_2) \right)$$

$$= \sum_{\substack{y_1 \in C_n \\ y_2 \in C_n}} f(y_1)\overline{f(y_2)} \sum_{x \in C_n} \overline{\chi_x(y_1)} \chi_x(y_2)$$

$$= n \sum_{y \in C_n} f(y)\overline{f(y)} = n \|f\|^2$$

∎

$f_1, f_2 \in \mathcal{F}(C_n)$, the convolution of f_1, f_2 is

$$(f_1 * f_2)(y) := \sum_{x \in C_n} f_1(y - x) f_2(x), \quad y \in C_n.$$

A/k ... algebra (vectorspace $/k$, ring structure)

In our case : $A/k = \mathcal{F}(C_n)$ with convolution

Lemma 5: $f_1, f_2, f_3 \in \mathcal{F}(C_n)$

1/ $f_1 * f_2 = f_2 * f_1$ (commutativity)

2/ $(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$ (associativity)

3/ $(f_1 + f_2) * f_3 = f_1 * f_3 + f_2 * f_3$ (distributivity)

4/ $\delta_0 * f = f * \delta_0 = f$.

5/ $\widehat{f_1 * f_2} = \hat{f_1} \cdot \hat{f_2}$

Pf: 5/

$$\widehat{f_1 * f_2}(y) = \sum_{x \in C_n} (f_1 * f_2)(x) \overline{\chi_y(x)}$$

$$= \sum_{x \in C_n} \sum_{t \in C_n} f_1(x-t) f_2(t) \overline{\chi_y(x-t)}\, \overline{\chi_y(t)}$$

$$= \hat{f_1}(y)\, \hat{f_2}(y).$$

(9) For $f=\delta_x$, we have $\hat{\delta}_x = \overline{\chi_x}$ $\forall x\in C_n$.

The translation operator

$$T_x : \mathcal{F}(C_n) \to \mathcal{F}(C_n)$$
$$f \longmapsto (T_x f)(y) = f(y-x) \quad \forall x,y\in C_n.$$

We have $\widehat{(T_x f)}(y) = \overline{\chi_y(x)}\,\hat{f}(y)$, as follows from

$$(T_x f) = f * \delta_x$$

Let $R : \mathcal{F}(C_n) \to \mathcal{F}(C_n)$ be a linear operator, associated with matrix $\{r(x,y)\}_{x,y\in C_n}$, i.e.

$$(Rf)(x) = \sum_{y\in C_n} r(x,y) f(y).$$

We say R is C_n-invariant if commutes in $\mathrm{End}(\mathcal{F}(C_n))$ with T_x $\forall x\in C_n$, i.e.

$$R T_x = T_x R \quad \forall x\in C_n.$$

We say R is a convolution operator if there exists $h\in\mathcal{F}(C_n)$ such that $Rf = f*h$ $\forall f\in\mathcal{F}(C_n)$. The function h is called the convolution kernel of R.

<u>Lemma 6</u> The linear operator R associated with the matrix $\{r(x,y)\}_{x,y\in C_n}$ is C_n-invariant iff

$$r(x-z,\ y-z) = r(x,y), \quad \forall x,y,z\in C_n.$$

<u>Pf:</u> R is C_n-invariant iff $(T_z(Rf))(v) = (R(T_z f))(v)$ iff

$$\sum_{u\in C_n} r(v-z,u) f(u) = \sum_{u\in C_n} r(v,u) f(u-z)$$

equivalent to $(u\to u-z)$

$$\sum_{u\in C_n} r(v-z,\ u-z) f(u-z) = \sum_{u\in C_n} r(v,u) f(u-z).$$

☒

(10) Theorem 7: $R : \mathcal{F}(C_n) \to \mathcal{F}(C_n)$ a linear operator. TFEAE:

1/ R is C_n-invariant,

2/ R is convolution operator,

3/ Every χ_x is an eigenvector of R.

Pf: 1/ $\Rightarrow$ 2/

By previous lemma, $r(x,y) = r(x-y, 0)$, and so for $h(x) := r(x,0)$ we have

$$(Rf)(x) = \sum_{y \in C_n} h(x-y) f(y) = (h * f)(x),$$

so that R is convolution operator.

The converse 2/ $\Rightarrow$ 1/ is easy, because $\forall$ convolution operator is C_n-invariant.

2/ $\Rightarrow$ 3/ $\exists h \in \mathcal{F}(C_n)$: $R = *h$, $Rf = f * h$ $\forall f \in \mathcal{F}(C_n)$.

Then

$$(R\chi_x)(y) = \sum_{t \in C_n} \chi_x(y-t)\, h(t) = \chi_x(y) \sum_{t \in C_n} \overline{\chi_x(t)}\, h(t) =$$

$$= \hat{h}(x)\, \chi_x(y) \Rightarrow \forall \chi_x \text{ is an eigenvector}$$

with eigenvalue $\hat{h}(x)$.

3/ $\Rightarrow$ 2/ By Fourier inversion formula $\left(f = \frac{1}{n} \sum_{x \in C_n} \hat{f}(x)\, \chi_x\right)$ and linearity of R we have

$$(Rf)(z) = \frac{1}{n} \sum_{x \in C_n} \hat{f}(x)\, \lambda(x)\, \chi_x(z)$$

$$= \sum_{y \in C_n} f(y)\, \frac{1}{n} \sum_{x \in C_n} \lambda(x)\, \chi_x(z-y)$$

$$= (h * f)(z), \quad h(y) = \frac{1}{n} \sum_{x \in C_n} \lambda(x)\, \chi_x(y).$$

$\blacksquare$

Důsledek 8: R ... conv operator with kernel $h \in \mathcal{F}(C_n)$. Then convolution

$R\chi_x = \hat{h}(x)\chi_x$. The spectrum of R is given by $\sigma(R) = \{\hat{h}(x) \mid x \in C_n\}$.

A matrix of the form
$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & & & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix}$$
$a_0, a_1, \dots, a_{n-1} \in \mathbb{C}$

is circulant. $\mathcal{C}_n \equiv$ the set of all $n \times n$ circulant matrices.

Dovětek:
If $A, B \in \mathcal{C}_n$, then $AB = BA$ and $AB \in \mathcal{C}_n$. Since $aA + bB \in \mathcal{C}_n$ $\forall a, b \in \mathbb{C}$, $\mathcal{C}_n$ is commutative algebra with unit (identity matrix)

For $\mathcal{F}(C_n)$ take the basis $B = \{\delta_0, \delta_1, \dots, \delta_{n-1}\}$. A lin. op. $T: \mathcal{F}(C_n) \to \mathcal{F}(C_n)$ is convolution op $Tf = f * h$ iff in B can be written as
$$\begin{pmatrix} f(0) \\ f(1) \\ \vdots \\ f(n-1) \end{pmatrix} \longmapsto \begin{pmatrix} h(0) & h(n-1) & \cdots & h(1) \\ h(1) & h(0) & \cdots & h(2) \\ \vdots & \vdots & & \vdots \\ h(n-1) & h(n-2) & \cdots & h(0) \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \\ \vdots \\ f(n-1) \end{pmatrix}$$

i.e., iff the matrix of T is circulant. In particular, $\mathcal{C}_n \simeq \mathcal{F}(C_n)$ as algebras.

Denote
$$F = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & & 1 \\ 1 & \omega^{-1} & \omega^{-2} & & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & & \omega^{-2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & & \omega^{-(n-1)^2} \end{pmatrix}$$

F is symmetric $\Rightarrow$ $F^* = \overline{F}$, $\langle\, , \rangle$ is adjoint of F

ON-relations for characters $\Rightarrow$ F is unitary

(12) A matrix A $(n \times n)$ belongs to $\mathcal{C}_n$ iff FAF^* is diagonal (note that columns of F are given by values of characters on C_n.
The matrix FAF^* is called the matrix form of the Fourier transform on C_n (discrete Fourier transform.)

Harmonic analysis on finite group / set

X ... a finite set, G ... a finite group

<u>Def</u>: A (left) action of G on X: $G \times X \to X$

$(g, x) \mapsto g\,x$

1/ $(gh).x = g.(h.x)$ $\forall g, h \in G$, $x \in X$,

2/ $e.x = x$ $\forall e \in G$

transitive action: $\forall x_1, x_2 \in X$ $\exists g$: $g x_1 = x_2$

fixed point: $x \in X$ is fixed by $g \in G$ if $g.x = x$

$Stab_G(x) = \{g \in G \;\; g.\cancel{x} = \cancel{x}\}$ ... stabiliser of $x \in X$

$Orb_G(x) = \{g\,\cancel{x} \; : g \in G\}$ ... G-orbit of G

$x_1 \in Orb_G(x) \Leftrightarrow Orb_G(x) = Orb_G(x_1)$; $Orb_G(x) \subseteq X$

"$\sim$": equivalence relation $x_1 \sim x_2 \Leftrightarrow x_1 = g\,x_2$ for some $g \in G$

transitive action $\Leftrightarrow$ $Orb_G(x) = X$ $\forall x \in X$

$X = \dot{\bigcup}_{x \in \Gamma} Orb_G(x)$, Γ ... a set of representatives of the orbits

<u>Ex</u>: G, $K \leq G$, $X = G/K = \{gK \; g \in K\}$

G acts from the left ~~to~~ on X by $g'.(gK) = (\cancel{g}'g).K$, $g', g \in G$.

This action is transitive. The action of $K \leq G$ is not transitive, e.g., K stabilizes eK: $Orb_K(eK) = eK$; K-orbits correspond to double cosets $K \backslash G / K$.

S_X ... the group of bijections $X \to X$, $\sigma \in S_X$ is a permutation of X. For $X = \{1, \dots, n\}$, $S_X \equiv S_n$.

<u>Lemma</u>. Assume G acts on X.

1/ $\forall g \in G$ induces $\sigma_g \in S_X$,

2/ The map $G \to S_X$, $g \mapsto \sigma_g$ is a group homomorphism.

3/ There is bijection between $\forall$ actions $G \times X \to X$ and the set of homomorphisms $G \to S_X$.

(14)

$G \times X \to X$, $\mathrm{Ker}(\sigma_g) = H \leq G$

$= \{ g \in G : gx = x \ \forall x \in X \}$

kernel of G acting on X

If $H = e$, the action is faithful ($G \leq S_X$)

Two G-spaces X, X' are isomorphic if $\exists$ a bijection $\varphi : X \to X'$ fulfilling $\varphi(g \cdot x) = g \cdot \varphi(x)$ $\forall x \in X, g \in G$ (G-equivariant)

<u>Lemma</u> X a G-space (~~*~~G acts trans. on X), $x \in X$, $K = \mathrm{Stab}_G(x)$. Then the map

$$\Psi : G/K \to X$$
$$gK \mapsto g \cdot x$$

is a G-equivariant bijection ($G/K \underset{G\text{-spaces}}{\simeq} X$)

<u>Pf</u>: 1/ Ψ is well-defined + injective:

$g_1 K = g_2 K \Leftrightarrow g_1^{-1} g_2 \in K \Leftrightarrow (g_1^{-1} g_2) \cdot x = x \Leftrightarrow$

$\Leftrightarrow g_1^{-1} \cdot (g_2 \cdot x) = x \Leftrightarrow g_2 \cdot x = g_1 \cdot x$.

2/ Ψ is surjective (clear)

3/ G-equivariance Ψ:

$g_1 \cdot \Psi(g_2 K) = g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x = \Psi(g_1 g_2 K)$

$= \Psi(g_1 \cdot g_2 K)$ ▨

<u>Lemma</u>: Let G act on X. Then

$$|G| = |\mathrm{Stab}_G(x)| \cdot |\mathrm{Orb}_G(x)| \quad \forall x \in X$$

<u>Pf</u>: The previous lemma applied to G acting on $\mathrm{Orb}_G(x)$, $x \in X$ arbitrary: $|\mathrm{Orb}_G(x)| = |G/\mathrm{Stab}_G(x)| = |G| / |\mathrm{Stab}_G(x)|$. ▨

For $\varphi : X \to X'$ an isomorphism of G-spaces we have $\forall x \in X$:

$$\mathrm{Stab}_G(x) \simeq \mathrm{Stab}_G(\varphi(x))$$

(15)

<u>Lemma:</u> X a G-space. Then $\underset{\text{in } G}{\mathrm{Stab}_G(g\cdot x)} = \underset{\text{in } G}{g\,\mathrm{Stab}_G(x)\,g^{-1}}$

<u>Pf:</u> $h \in \mathrm{Stab}_G(g\cdot x) \Leftrightarrow h\cdot(g\cdot x) = g\cdot \cancel{x} \Leftrightarrow g^{-1}\cdot[h\cdot(g\cdot x)] = (\cancel{g}^{-1}hg)x = x$
$\Leftrightarrow g^{-1}hg \in \mathrm{Stab}_G(x)$. ▨

<u>Lemma:</u> $H, K \leq G$ subgroups. Then G/H, G/K are isomorphic G-spaces iff H, K are conjugate in G (i.e., $\exists g \in G$ $K = g^{-1}Hg$.)

<u>Pf:</u> $\Leftarrow$ $\exists g \in G$: $K = g^{-1}Hg$, $X = G/H$, $x = g^{-1}H$, then
$K = \mathrm{Stab}_G(x) \underset{\text{Lemma for } \Psi}{\Rightarrow} G/K \underset{G\text{-space}}{\xrightarrow{\sim}} X = G/H$.

$\Rightarrow$ $\varphi: G/H \to G/K$ is G-equiv. isom, let $g \in G$ such that $\varphi(H) = gK$. Then by Lemma above $\Rightarrow$ $\mathrm{Stab}_G(H) = \mathrm{Stab}_G(gK)$. Since $\mathrm{Stab}_G(gK) = gKg^{-1}$ and $\mathrm{Stab}_G(H) = H \Rightarrow H = gKg^{-1}$. ▨

<u>Ex 1:</u> (trivial action)
G, X : $g\cdot x = x$ $\forall g \in G, x \in X$ trivial action.
$\forall x \in X$: $\mathrm{Orb}_G(x) = \{x\}$,
$\mathrm{Stab}_G(x) = G$.

<u>Ex 2:</u> G, $X = G$, G acts on X by left action.
A/ $G \times G \to G$
$\underbrace{(g\cdot h) := gh \qquad \text{B/ } G\times G \to G, \ (g\cdot h) := hg^{-1}}_{\text{transitive, faithful action}}$

<u>Ex 3:</u> $G \times G \to G$
$g\cdot h := ghg^{-1}$, the orbits = conjugacy classes
conj. action
stabilizer of $x \in G$ - $Z(x)$ centralizer of x in G
A fixed point = central element (commutes $\forall g \in G$)
G abelian $\Rightarrow$ $\forall$ element is central

Ex: Let Ω denote the set of all subgroups $H \leq G$. Define the action of G on Ω: $g \cdot H := gHg^{-1} = \{ghg^{-1} \mid h \in H\}$. (Conjugation action on subgroups.) $H \leq G$ is fixed by conjugation action if it is normal subgroup. The stabilizer of H is called the normalizer of H in G.

Ex: (Diagonal action) G acts on Ω. Define the action of G on $\Omega \times \Omega$ by $g \cdot (\omega_1, \omega_2) = (g \cdot \omega_1, g \cdot \omega_2)$, and if the action of G on Ω is transitive, the description of G-orbits on $\Omega \times \Omega$ is known (maybe will be discussed as an exercise). It is diagonal action of $G \times G$ on $\Omega \times \Omega$, i.e. $\tilde{G} = \{(g, g) : g \in G\}$ acting on $\Omega \times \Omega$.

Ex: G group, Ω_1, Ω_2 sets. If G acts on Ω_1, it acts on

$$\Omega_2^{\Omega_1} := \{f : \Omega_1 \to \Omega_2\} \text{ by}$$

$$(g \cdot f)(\omega_1) = f(g^{-1} \cdot \omega_1), \quad \omega_1 \in \Omega_1, g \in G, f \in \Omega_2^{\Omega_1}.$$

When $\Omega_1 = G$, $\Omega_2 = \mathbb{C}$, $\Omega_2^{\Omega_1} \equiv L(G)$ is the left (right) regular representation (see Cayley action)

When the action of G on Ω_1 is transitive, $\Omega_1 = G/K$ (K is the stabilizer of $[g] \in G/K$) and $\Omega_2 = \mathbb{C}$, $L(X)$ is called permutation represent. on $L(X)$.

Representations, irreducibility, unitarity, ...

$V/_{\mathbb{C}}$ ... fin. dim vect. space, $GL(V)$ ... group of invert. transformations ($A : V \to V$ invert. map)

Def: A repr. (ρ, V) of G on V is

$$G \times V \to V$$
$$(g, v) \mapsto \rho(g)v$$

ρ is homomorphism: $\rho(g) \in GL(V)$

1/ $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$,

2/ $\rho(e) = \mathbb{I}_V$.

When a basis is chosen in V, then $GL(V) \cong GL(n, \mathbb{C})$ for $n = \dim V$.

(ρ, V) a represent. of G; $W \leq V$ a G-invariant subspace, i.e. $\rho(g)w \in W$ $\forall g \in G$, $w \in W$ - $\rho(G)W \leq W$. Then

$\rho_W : g \in G \mapsto \rho(g)|_W \in GL(W)$ on W is subrepr. of G

$(\rho_V, V) \geq (\rho_W, W)$. We use the notation $\rho_W \leq \rho$ (clearly $\rho \leq \rho$)

(ρ, V) is irreducible if G-invariant subspaces are $W = \{0\}$ or $W = V$ (the improper subspaces only.) E.g., all 1-dim representations are irreducible.

Equivalence of repr. : $(\rho, V) \sim (\sigma, W)$ for two G-repr.

$J : V \to W$ linear bijection such that $\sigma(g) J = J \rho(g)$ $\forall g \in G$.

Let V be endowed with a scalar product $\langle \, , \, \rangle$.

$(V, \langle \, , \, \rangle_V)$, $(W, \langle \, , \, \rangle_W)$ $\quad T : V \to W$

$T^* : W \to V$ adjoint of T

$$\langle w, Tv \rangle_W = \langle T^* w, v \rangle_V \quad \forall v \in V, w \in W.$$

A linear op. $U : V \to W$ is unitary if $U^* U = Id = U U^*$, i.e.

$$\langle Uv, Uv' \rangle_W = \langle v, v' \rangle_V \quad \forall v, v' \in V.$$

Its spectrum is characterized by $\sigma(U) \subseteq \{z \in \mathbb{C} : |z| = 1\}$.

A repr. is unitary if it preserves the scalar product, i.e., $\langle \rho(g) v, \rho(g) v' \rangle = \langle v, v' \rangle$ $\forall g \in G$, $v, v' \in V$. Equivalently, (ρ, V) is unitary if $\rho(g) \leq U(V)$ is a subgroup of unitary group.

Given (ρ, V), it is possible to endow it with an inner product for which the action is unitary:

$$(v, w) = \sum_{g \in G} \langle \rho(g) v, \rho(g) w \rangle, \quad v, w \in V.$$

<u>Lemma</u>: The representation $(\rho, V_{(\, , \,)})$ is unitary and equivalent to $(\rho, V_{\langle \, , \, \rangle})$. In particular, every representation is equivalent to a unitary repr.

(18)

Pf: $(\, , \,)$ is clearly an inner product on V. We have

$$(\rho(h)v, \rho(h)w) = \sum_{g\in G} \langle \rho(g)\rho(h)v, \rho(g)\rho(h)w\rangle$$
$$= \sum_{g\in G} \langle \rho(gh)v, \rho(gh)w\rangle$$
$$= \sum_{k\in G} \langle \rho(k)v, \rho(k)w\rangle$$
$$= (v, w) \qquad \Rightarrow G\text{-invariant}$$

The equivalence $(\rho, V_{(\,,\,)})$ and $(\rho, V_{\langle\,,\,\rangle})$ is trivial (given by Id_V). ▨

Equivalence class of representations $\Rightarrow$ unitary representations.
Unitarity assumption: $\rho(g^{-1}) = \rho(g)^{-1} = \rho(g)^* \quad \forall g\in G$.
Unitary equivalence = equivalence + $\exists\, U: V \to W$ unitary operator
$\sigma(g)U = U\rho(g)$

<u>lemma</u>: Suppose $(\rho, V), (\sigma, W)$ are unitary representations of finite group G. If they are equivalent $\Rightarrow$ they are also unitarily equivalent.

Pf: Assume $\exists\ T: V \to W$

$$\rho(g) = T^{-1}\sigma(g)T \qquad \forall g\in G$$

$\Rightarrow$ for $\rho(g)^* = \rho(g^{-1})$

$$\rho(g) = T^*\sigma(g)(T^*)^{-1}$$

$\Rightarrow$

$$\rho(g)^{-1}(TT^*)\rho(g) = T^*T \qquad \forall g\in G$$

$\Rightarrow$ Taking $T = U|T|$ the polar decomposition of T, where U is unitary and $|T|$ is positive ~~[illegible]~~ linear operator $T = \sqrt{T^*T}$ (pos. square root)

$$\rho(g)^{-1}\,|T|\,\rho(g) = |T|$$

$\Rightarrow$ $|T|$ is invertible, so

$$U\rho(g)U^{-1} = T|T|^{-1}\rho(g)|T|T^{-1} = T\rho(g)T^{-1} = \sigma(g).$$

▨

(19)

(ρ_i, W_i) represent.,

$\rho = \rho_1 \oplus \cdots \oplus \rho_k$ is repr. on $W_1 \oplus \cdots \oplus W_k$
direct sum representation

$$\rho(g)v = \rho_1(g)w_1 + \cdots + \rho_k(g)w_k$$
$$\forall\ v = w_1 + \cdots + w_k \in V,\ w_i \in W_i,\ g \in G.$$

$W \subseteq V$ a G-invariant subspace of (ρ, V),
$W^\perp := \{v \in V \mid \langle v, w\rangle = 0\ \forall w \in W\}$ the orthogonal complement of W, is also G-invariant: for $v \in W^\perp$, $w \in W$

$$\langle \rho(g)v, w\rangle = \langle v, \rho(g^{-1})w\rangle = \langle v, \underbrace{w'}_{\in W}\rangle = 0 \quad \forall w \in W.$$

$\rho_1 = \rho_W$, $\rho_2 = \rho_{W^\perp}$: $\rho = \rho_W \oplus \rho_{W^\perp}$
This implies, by induction:
Lemma: $\forall$ representation of G is a direct sum of a finite number of irreducible repr.

Def: G a finite group, $\hat{G} :=$ a complete set of irreducible pairwise (the dual of G) inequivalent (unitary) represent. of G.

Ex: $L(G) := \{f : G \to \mathbb{C}\}$ the space of $\mathbb{C}$, the scalar product is
$$\langle f_1, f_2\rangle = \sum_{g \in G} f_1(g)\overline{f_2(g)}$$
and the G-action
$$(\rho(g)f)(h) = f(g^{-1}h).$$

Ex: $C_n = \langle 1, a, a^2, \ldots, a^{n-1}\}$ cyclic group of order n, set
$\omega = e^{\frac{2\pi i}{n}}$, $(\rho_k, \mathbb{C})$ is defined by $\rho_k(a^h) = \omega^{kh}$
$k = 0, 1, \ldots, n-1$
is 1-dim ($\Rightarrow$ irreducible) repr. of C_n.

Intertwining maps and Schur's lemma

$(\rho, V), (\sigma, W)$ repr. of G, $L : V \to W$ a linear map
L intertwines ρ, σ : $L\rho(g) = \sigma(g)L \quad \forall g \in G.$

Lemma (Schur) irred vs $\exists$ of intertw. map:
$(\rho, V), (\sigma, W)$ irred. repr. of G. If L intertwines ρ, σ, then either $L = 0$ or L is an isomorphism.

Pf: $V_0 = \{v \in V \mid L(v) = 0\}$, $W_0 = \{L(v) \mid v \in V\} \leq W$

kernel range of L

L is intertwinor $\Rightarrow$ V_0, W_0 are inv. subspaces.

Irreducibility $\Rightarrow$ 1/ $V_0 = V$, $W_0 = \{0\}$, or

(and non-triviality 2/ $V_0 = \{0\}$, $W_0 = W$.
of V, W)

$\Rightarrow$ 1/ L is trivial,

2/ L is an isomorphism. ▨

Cor: (ρ, V) irrep. of G. $L: V \to V$ intertwining ρ, $L\rho(g) = \rho(g)L$ $\forall g \in G$. Then $L \in \mathbb{C} \cdot Id_V$.

Pf: $\lambda \in \sigma(L)$. Then $L - \lambda Id_V$ also intertwines ρ with itself, by previous lemma: $L - \lambda Id_V$ is either invertible or zero. By definition of eigenvalue, it is not invertible $\Rightarrow$ it is zero $\Rightarrow$ $L \in \mathbb{C} \cdot Id_V$. ▨

Ex: $\rho \in \hat{G}$, $g_0 \in Z(G) = \{$ ~~...~~ $g \in G : gh = hg \ \forall h \in G\}$

$\boxed{\text{Matrix coefficients and their orthogonality relations}}$ ...

(ρ, V) - unitary repr. of G; $v, w \in V$ $u_{v,w}(g): G \to \mathbb{C}$

$u_{v,w}(g) = \langle \rho(g) w, v \rangle$, $g \in G$.

$u_{v,w}$ - matrix coefficient of (ρ, V)

$\{v_1, \ldots, v_n\}$ - ON-basis for V, $\rho(g)$ - $(n \times n)$-matrix $(u_{v_i, v_j}(g))_{i,j}$ $\forall g \in G$

Any $u: G \to \mathbb{C}$ can be realized as a coefficient of a unitary repr.:
$V = L(G)$ with scalar product $\langle \, , \, \rangle = \frac{1}{|G|} \sum_{g \in G} \ldots$
$u(g) = \langle \rho(g) \delta_e, \bar{u} \rangle$, ρ is left regular repr. (δ_e is the Dirac funct. supported on $e \in G$.)

Coefficients of irred. repr. constitute an ON-basis of $L(G)$!

Lemma: $(\rho, V), (\sigma, W)$ are irred. non-equivalent repr. of G. Then all coefficients of ρ are orthogonal to all coefficients of σ.

Pf: $v_1, v_2 \in V$, $w_1, w_2 \in W$. We show

$$u(g) = \langle \rho(g) v_1, v_2 \rangle_V, \quad u'(g) = \langle \sigma(g) w_1, w_2 \rangle_W$$

are OG in $\mathcal{L}(G)$. Consider

$$L : V \to W, \quad Lv = \langle v, v_2 \rangle_V \, w_2 \quad \forall v \in V.$$

Then $\tilde{L} : V \to W$

$$\tilde{L} = \sum_{g \in G} \sigma(g^{-1}) L \rho(g)$$

satisfies $\tilde{L}\rho(g) = \sigma(g)\tilde{L}$ $\forall g \in G$ (i.e., $\tilde{L}$ is an intertwiner)

Schur Lemma $\Rightarrow$ $\tilde{L}$ is either invertible or $\tilde{L} = 0$. As $\rho \not\sim \sigma$ (non-equivalent repr.) $\Rightarrow$ $\tilde{L} = 0$ and

$$\underset{\forall w_1 \in W}{0 = \langle \tilde{L} v_1, w_1 \rangle_W} = \sum_{g \in G} \langle L\rho(g) v_1, \sigma(g) w_1 \rangle_W$$

$$= \sum_{g \in G} \langle \rho(g) v_1, v_2 \rangle_V \langle w_2, \sigma(g) w_1 \rangle_W = \sum_{g \in G} u(g) \overline{u'(g)}.$$

$\square$

Corollary: G finite group. Then there exist only finit many irred. (unitary) repr. pairwise non-equivalent, $|\hat{G}| < \infty$.

Pf: $\mathcal{L}(G)$ is finite dim., contains fin. many distinct pairwise OG-fcions, the statement follows from previous Lemma. $\square$

~~Lemma: $\langle u_{ij}, u_{kh} \rangle$~~

(ρ, V) irred. repr., $\{v_1, \dots, v_d\}$ OG-basis of V, $\dim V = d$.

Lemma: The matrix coefficients $u_{ij}(g) = \langle \rho(g) v_j, v_i \rangle$ are pairwise OG. In particular,

$$\underset{V}{\langle u_{ij}, u_{kh} \rangle_{\mathcal{L}(G)}} = \frac{|G|}{d} \delta_{ik} \delta_{jh}.$$

Pf: $i, k = 1, \dots, d$

(22) Pf: Fix $i, k \in \{1, \dots, d\}$ and consider $L_{ik} : V \to V$
$$L_{ik}(v) := \langle v, v_i \rangle_V \, v_k.$$
Observe $\mathrm{Tr}(L_{ik}) = \delta_{ik}$. Define $\widetilde{L}_{ik} = \frac{1}{|G|} \sum_{g \in G}' \rho(g^{-1}) L_{ik} \rho(g)$ and observe $\widetilde{L}_{ik} \rho(g) = \rho(g) \widetilde{L}_{ik}$. As ρ is irreducible, $\widetilde{L}_{ik} = \alpha \, Id_V$ for some $\alpha \in \mathbb{C}$. Because $d\alpha = \mathrm{Tr}(\widetilde{L}_{ik}) = \mathrm{Tr}(L_{ik}) = \delta_{ik}$ (take the trace of) , $\alpha = \frac{\delta_{ik}}{d}$. Then $\widetilde{L}_{ik} = \frac{1}{d} \delta_{ik} Id_V$, $\langle \widetilde{L}_{ik} v_j, v_\ell \rangle_V = \frac{1}{d} \delta_{j\ell} \delta_{i\ell}$. So

$$\begin{aligned}
\langle \widetilde{L}_{ik} v_j, v_\ell \rangle_V &= \frac{1}{|G|} \sum_{g \in G}' \langle L_{ik} \rho(g) v_j, \rho(g) v_\ell \rangle_V \\
&= \frac{1}{|G|} \sum_{g \in G} \langle \rho(g) v_j, v_i \rangle_V \langle v_k, \rho(g) v_\ell \rangle_V \\
&= \frac{1}{|G|} \langle u_{ij}, u_{k\ell} \rangle_{L(G)}.
\end{aligned}$$
∎

Matrix coefficients of unitary representations have further properties:

Lemma: $\forall g, g_1, g_2 \in G$, $1 \le i, j, k \le d$:

1/ $u_{i,j}(g_1 g_2) = \sum_{h=1}^{d} u_{i,h}(g_1) u_{h,j}(g_2)$,

2/ $u_{i,j}(g^{-1}) = \overline{u_{j,i}(g)}$,

3/ $\sum_{j=1}^{d} u_{j,i}(g) \overline{u_{j,k}(g)} = \delta_{i,k}$;

Pf: 2/ follows from $\rho(g)^* = \rho(g^{-1})$, $\langle x, y \rangle = \overline{\langle y, x \rangle}$ $\forall g \in G$, $x, y \in V$.

1/, 3/ analogous. ∎

$\boxed{\text{Characters} \dots}$

(ρ, V) irred. repr., $\chi_\rho = \mathrm{Tr}\,\rho$ a function on G
$$g \mapsto \mathrm{Tr}\rho(g) \in \mathbb{C}$$
$\mathrm{Tr} : V \to \mathbb{C}$ a unique lin. functional characterized by

1/ $\mathrm{Tr}(TS) = \mathrm{Tr}(ST)$ $\forall S, T \in \mathrm{End}(V)$,

2/ $\mathrm{Tr}(Id_V) = \dim V$.

$Tr(\rho(g)) = \sum_{j=1}^{d} u_{jj}(g)$, independent of ON-basis $\{v_1, .., v_d\}$ of V and of a representative of the equivalence class of representations; if $\dim V = 1$, $\rho = \chi_\rho$

Proposition: (ρ, V) a repr. of G, χ_ρ its character, $\dim V = d$. Then for all $s, t \in G$:

1/ $\chi_\rho(Id_G) = d$,

2/ $\chi_\rho(s^{-1}) = \overline{\chi_\rho(s)}$,

3/ $\chi_\rho(t^{-1}st) = \chi_\rho(s)$.

Pf: 1/, 3/ clear. Unitarity of $(\rho, V) \Rightarrow$ 2/

$\chi_\rho(s^{-1}) = Tr(\rho(s^{-1})) = Tr(\rho(s)^*) = \overline{\chi_\rho(s)}$,

since $\rho(s^{-1}) = \rho(s)^*$, $Tr(A^*) = \overline{Tr(A)}$. ∎

Characters

$\forall$ irred. repr. (or, its equivalence class) $\rightsquigarrow \chi_\rho$; $\chi_\rho \in \mathcal{F}(G)$

$g \mapsto \chi_\rho(g) := \mathrm{Tr}(\rho(g))$

V... fin dim complex vectorspace, $\mathrm{End}(V)$... algebra of linear maps $T: V \to V$, $\mathrm{Tr}: \mathrm{End}(V) \to \mathbb{C}$ is a linear form characterized by

- $\mathrm{Tr}(TS) = \mathrm{Tr}(ST)$, $S, T \in \mathrm{End}(V)$,
- $\mathrm{Tr}(1_V) = \dim V$.

Ex (Basic properties of trace) V... fin. dim. vect space, $\langle\, , \rangle : V \times V \to \mathbb{C}$ scalar product, $\{v_1, \dots, v_d\}$ an ON-basis of V.

Then $\mathrm{Tr}(T) = \sum_{i=1}^{d} \langle T v_i, v_i \rangle$, $T \in \mathrm{End}(V)$, and

$$\mathrm{Tr}(TS) = \sum_{i,j=1}^{d} \langle T v_j, v_i \rangle \langle S v_i, v_j \rangle, \quad S, T \in \mathrm{End}(V).$$

(ρ, V), $\chi_\rho = \chi_V$

Lemma: (ρ, V) a representation of G, χ_ρ its character. Denote by $d = d_\rho = \dim V$ its degree. Then for all $s, t \in G$:

1/ $\chi_\rho(e) = d$,

2/ $\chi_\rho(s^{-1}) = \overline{\chi_\rho(s)}$,

3/ $\chi_\rho(t^{-1} s t) = \chi_\rho(s)$.

Pf: 2/ ρ is unitary $\Rightarrow$ $\rho(s^{-1}) = \rho(s)^*$, and so

$$\chi_\rho(s^{-1}) = \mathrm{Tr}\big(\rho(s^{-1})\big) = \mathrm{Tr}(\rho(s)^*) = \overline{\chi_\rho(s)},$$

since $\mathrm{Tr}(A^*) = \overline{\mathrm{Tr}(A)}$.

1/, 3/ are clear. ∎

The consequence of orthogonality of matrix functions of irreducible representations:

Lemma: Let ρ, σ be irreducible representations of a group G.

1/ If ρ, σ are inequivalent, then $\langle \chi_\rho, \chi_\sigma \rangle = 0$,

2/ $\langle \chi_\rho, \chi_\rho \rangle = |G|$.

Prop.: Let ρ, σ be two repr. of a group G. Suppose ρ decomposes into irreducibles as $\rho = \rho_1 \oplus \dots \oplus \rho_k$, $k \in \mathbb{N}$, and σ is irreducible. Then for $m_\sigma = |\{j : \rho_j \sim \sigma\}|$, one has

$$m_\sigma = \frac{1}{|G|} \langle \chi_\rho, \chi_\sigma \rangle.$$

(In particular, m_σ does not depend on the chosen decomposition of ρ.)

(2)

Pf: $\rho \simeq \rho_1 \oplus \rho_2 \oplus \dots \oplus \rho_k$ of irred. subrepr., then $\chi_\rho = \sum_{j=1}^{k} \chi_j$.
This implies the claim. ▨

m_σ is called the multiplicity of σ as a subrepresentation of ρ.

Důsl 1: Let ρ be a repr. of G. Then $\rho = \bigoplus_{\sigma \in \widehat{G}} m_\sigma \sigma$, where
$m_\sigma \sigma = \underbrace{\sigma \oplus \dots \oplus \sigma}_{m_\sigma \times}$, $\chi_\rho = \sum_{\sigma \in \widehat{G}} m_\sigma \chi_\sigma$.

Důsl 2: ρ, σ repr. of G. Suppose $\rho \simeq \bigoplus_{i \in I} m_i \rho_i$, $\sigma = \bigoplus_{j \in J} n_j \rho_j$, where ρ_i, ρ_j are irreducible. Then denoting by $I \cap J$ the subset of common irreducible repr., we have
$$\frac{1}{|G|} \langle \chi_\rho, \chi_\sigma \rangle = \sum_{i \in I \cap J} m_i n_i .$$

Důsl 3: Reprezentace ρ grupy G je ireducibilní $\Leftrightarrow$ $\langle \chi_\rho, \chi_\rho \rangle = |G|$.

Důsl 4: ~~D~~ Two representations ρ, σ are equivalent iff $\chi_\rho = \chi_\sigma$.

Theorem (Peter-Weyl): G .. finite group, $(\rho, \mathcal{F}(G))$ - left regular representation.

1/ $\forall$ irred. repr. (ρ, V), $\rho \in \widehat{G}$, appears in the decomposition $(\rho, L(G))$ with multiplicity equal to its dimension d_ρ.

2/ $u^\rho_{i,j}$ matrix coefficients of $\rho \in \widehat{G}$ w. r. to an orthonormal basis, then the functions
$$\left\{ \sqrt{\frac{d_\rho}{|G|}}\, u^\rho_{i,j} \;\middle|\; i,j = 1, \dots, d_\rho,\ \rho \in \widehat{G} \right\}$$
is an ON-basis of $\mathcal{F}(G)$.

3/ $|G| = \sum_{\rho \in \widehat{G}} d_\rho^2$, $\mathcal{F}(G) = \bigoplus_{\rho \in \widehat{G}} d_\rho V$.

Pf: $(\rho, \mathcal{F}(G)) \simeq \bigoplus_{\rho \in \widehat{G}} m_\rho \rho$, $m_\rho \in \mathbb{N}$.

For the (complete) ON-system $\{\delta_g\}_{g \in G}$ in $\mathcal{F}(G)$, we obtain $\chi_{(\rho, \mathcal{F}(G))}(e) = |G|$ and $\chi_{(\rho, \mathcal{F}(G))}(g) = 0$ if $g \neq e$; this follows from the fact that if $g, h \in G$, $\rho(h) \delta_g = \delta_{hg}$.

On the other hand, if $\rho \in \widehat{G} \underset{\text{Prop.}}{\Longrightarrow} m_\rho = \frac{1}{|G|} \langle \chi_\rho, \chi_{(\rho, \mathcal{F}_L(G))} \rangle = \chi_\rho(e)$ (because $\chi_{(\rho, \mathcal{F}(G))}(g) = 0$ for $g \neq e$.). This implies $m_\rho = d_\rho$,
$|G| = \dim \mathcal{F}(G) = \sum_{\rho \in \widehat{G}} m_\rho d_\rho = \sum_{\rho \in \widehat{G}} d_\rho^2$; this also follows from
$$\chi_{(\rho, \mathcal{F}(G))}(e) = |G| = \sum_{\rho \in \widehat{G}} m_\rho \chi_\rho(e).$$
We know that $\sqrt{\frac{d_\rho}{|G|}}\, u^\rho_{ij}$, $\rho \in \widehat{G}$, $1 \le i,j \le d_\rho$ is ON-system for $\mathcal{F}(G)$.
This system is complete, as $\sum_{\rho \in \widehat{G}} d_\rho^2 = |G|$ implies
$$|G| = \left| \left\{ \sqrt{\tfrac{d_\rho}{|G|}}\, u^\rho_{ij} \;\middle|\; \rho \in \widehat{G},\ 1 \le i,j \le d_\rho \right\} \right| = \dim \mathcal{F}(G).$$ ▨

(3)

Example: $D_n = \langle a, b \mid a^2 = b^n = 1, \ aba = b^{-1} \rangle$
be the dihedral group of degree n (the group of isometries of a regular polygon with n-vertices.)
Let n be even. There are 4 1-dim. representations (identified with their characters.) For $h = 0, 1$
$k = 0, 1, \dots, n-1$

$$\chi_1(a^h b^k) = 1,$$
$$\chi_2(a^h b^k) = (-)^h,$$
$$\chi_3(a^h b^k) = (-)^k,$$
$$\chi_4(a^h b^k) = (-)^{h+k}.$$

Set $\omega = e^{2\pi i/n}$, for $t = 0, \dots, n$ define the representation

$$\rho_t(b^k) = \begin{pmatrix} \omega^{tk} & 0 \\ 0 & \omega^{-tk} \end{pmatrix}, \quad \rho_t(ab^k) = \begin{pmatrix} 0 & \omega^{-tk} \\ \omega^{tk} & 0 \end{pmatrix}.$$

Show:
0/ ρ_t is a representation,
1/ $\rho_t \sim \rho_{n-t}$,

2/ $\chi_{\rho_0} = \chi_1 + \chi_2$, $\chi_{\rho_{\frac{n}{2}}} = \chi_3 + \chi_4$,

3/ ρ_t for $t = 1, \dots, \frac{n}{2} - 1$ are pairwise non-equivalent repr. by

1/ inspecting invariant subspaces and intertwining operators, or

2/ computing characters and their inner products.

4/ $\chi_1, \chi_2, \chi_3, \chi_4, \chi_{\rho_t}$, $1 \le t \le \frac{n}{2}$
constitute a complete list of ired. repr.

(4) **Def:** Convolution and Fourier transform

$P, Q \in \mathcal{F}(G)$, $P, Q : G \to \mathbb{C}$. The convolution of P, Q is defined by
$$\underset{\mathcal{F}(G)}{\underset{\cap}{(P*Q)}}(g) = \sum_{h\in G} P(gh^{-1})\,Q(h) = \sum_{\substack{h,k\in G\\ kh=g}} P(k)\,Q(h).$$

Rem: $P*Q \neq Q*P$, convolution is commutative iff G is abelian.
$a, b \in G$: $\delta_a * \delta_b = \delta_{ab}$, $\delta_b * \delta_a = \delta_{ba}$ (δ is the Dirac function.)
(commut. of group law is necessary cond.)
G abelian $\Rightarrow P*Q = Q*P$

Lemma: The space $\mathcal{F}(G)$ endowed with convolution product is an algebra $/\mathbb{C}$ satisfying:
1/ $\mathcal{F}(G)$ is a vectorspace $/\mathbb{C}$,
2/ $*$ is distributive (both left, right):
$$(P+Q)*R = P*R + Q*R,\quad R*(P+Q) = R*P + R*Q,$$
3/ $\mathcal{F}(G)$ has δ_e as a unit with respect to the convolution product on $\mathcal{F}(G)$: $P*\delta_e = P = \delta_e * P \;\forall P\in\mathcal{F}(G)$,
4/ The convolution is associative: $(P*Q)*R = P*(Q*R)$.

Pf: 1/, 2/, 3/ obvious.
4/ $P, Q, R \in \mathcal{F}(G)$: $\big(P*(Q*R)\big)(g) = \sum_{h\in G} P(gh^{-1})(Q*R)(h)$
$$= \sum_{h,t\in G} P(gh^{-1})\,Q(ht^{-1})\,R(t) = \sum_{\substack{h=mt\\ t,m\in G}} P(gt^{-1}m^{-1})\,Q(m)\,R(t)$$
$$= \sum_{t\in G} (P*Q)(gt^{-1})\,R(t) = \big[(P*Q)*R\big](g). \quad \blacksquare$$

Rem: $(\mathcal{F}(G), *)$ is called the group algebra. The center (central subalgebra) is given by $P\in\mathcal{F}(G)$: $P*Q = Q*P \;\forall Q\in\mathcal{F}(G)$.

Lemma: $P\in\mathcal{F}(G)$ is central iff $P(a^{-1}ta) = P(t)\;\forall a, t\in G$, i.e., it is constant on each conjugacy class.

Pf: P is in the center iff

$$\sum_{h\in G} Q(gh^{-1})P(h) = \sum_{h\in G} P(gh^{-1})Q(h) \quad \forall Q \in \mathcal{F}(G), \forall g\in G.$$

For $Q=\delta_a$, $g=ta \Rightarrow P(a^{-1}ta)=P(t)$ $\forall a,t\in G$. The converse is trivial. ▨

Def: Let $P\in\mathcal{F}(G)$, (ρ, V) a representation of G. The Fourier transform of P with respect to (ρ, V) is a linear operator $\hat{P}(\rho): V\to V$ defined by $\hat{P}(\rho) := \sum_{g\in G} P(g)\rho(g)$.

Lemma: $\forall P, Q\in\mathcal{F}(G)$, $\forall (\rho, V)$ a representation of G, we have

$$\widehat{P*Q}(\rho) = \hat{P}(\rho)\hat{Q}(\rho).$$

Pf:

$$\widehat{(P*Q)}(\rho) = \sum_{g\in G}\Big(\sum_{h\in G} P(gh^{-1})Q(h)\Big)\rho(g)$$

$$= \sum_{g,h\in G} P(gh^{-1})Q(h)\rho(gh^{-1})\rho(h)$$

$$= \sum_{h\in G}\Big(\sum_{g\in G} P(gh^{-1})\rho(gh^{-1})\Big)Q(h)\rho(h) = \hat{P}(\rho)\hat{Q}(\rho)$$

▨

Lemma: If P is a central function, then its Fourier transform with respect to irreducible repr. (ρ, V) of G is given by

$$\hat{P}(\rho) = \lambda\, Id_V, \quad \lambda = \frac{1}{d_\rho}\sum_{g\in G} P(g)\chi_\rho(g) = \frac{1}{d_\rho}\langle P, \overline{\chi_\rho}\rangle.$$

Pf: $\forall g\in G$

$$\rho(g)\hat{P}(\rho)\rho^{-1}(g) = \sum_{h\in G} P(h)\rho(g)\rho(h)\rho(g^{-1}) = \sum P(h)\rho(ghg^{-1})$$

$$= \sum_{h\in G} P(ghg^{-1})\rho(ghg^{-1}) = \hat{P}(\rho),$$

so $\hat{P}(\rho)$ intertwines $\rho \Rightarrow \hat{P}(\rho) = \lambda\, Id_V$ (follows from irreducib. of (ρ, V).)

Taking traces $Tr(\hat{P}(\rho)) = \sum_{h\in G} P(h)\chi_\rho(h) = \lambda d_\rho$. ▨

Theorem (Fourier inversion formula.)

For $P\in\mathcal{F}(G)$ the formula

$$P(g) = \frac{1}{|G|}\sum_{\rho\in\hat{G}} d_\rho\, Tr(\rho(g^{-1})\hat{P}(\rho)) \quad \forall g\in G.$$

(6) In particular if $P_1, P_2 \in \mathcal{F}(G)$ satisfy $\hat{P}_1(\rho) = \hat{P}_2(\rho)$ $\forall \rho \in \hat{G}$, then we have $P_1 = P_2$.

Pf: We know: $\sqrt{\frac{d_\rho}{|G|}}\, u^\rho_{i,j}$, computed with resp. to ON-basis $\{v^\rho_1, \dots, v^\rho_{d_\rho}\}$ $\forall \rho \in \hat{G}$, constitute ON basis of $\mathcal{F}(G)$. The same is true for the complex conjugates, therefor $\forall P \in \mathcal{F}(g)$

$$P(g) = \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_\rho \sum_{i,j=1}^{d_\rho} \langle P, \overline{u^\rho_{i,j}} \rangle \overline{u^\rho_{i,j}(g)},$$

$1 \le i,j \le d_\rho$. Because $\hat{P}(\rho) = \sum_{g \in G} P(g)\rho(g)$, $u^\rho_{ij}(g) = \langle \rho(g) v^\rho_j, v^\rho_i \rangle$, we have

$$\langle P, \overline{u^\rho_{i,j}} \rangle = \sum_{g \in G}{}' P(g) u^\rho_{i,j}(g) = \sum_{g \in G} P(g) \langle \rho(g) v^\rho_j, v^\rho_i \rangle = \langle \hat{P}(\rho) v^\rho_j, v^\rho_i \rangle$$

and
$$\sum_{i,j=1}^{d_\rho}{}' \langle P, \overline{u^\rho_{i,j}} \rangle \overline{u^\rho_{i,j}(g)} = \sum_{i,j=1}^{d_\rho} \langle \hat{P}(\rho) v^\rho_j, v^\rho_i \rangle \langle v^\rho_i, \rho(g) v^\rho_j \rangle$$
$$= \sum_{i,j=1}^{d_\rho} \langle \hat{P}(\rho) v^\rho_j, v^\rho_i \rangle \langle \rho(g^{-1}) v^\rho_i, v^\rho_j \rangle = Tr\left(\rho(g^{-1}) \hat{P}(\rho)\right)$$

$S, T \in End(V)$, v_i basis of V
$$Tr(ST) = \sum_{i,j=1}^{\dim V} \langle S v_i, v_j \rangle \langle T v_j, v_i \rangle.$$

$\square$

Let us define $C(G) := \bigoplus_{\rho \in \hat{G}} \{ \hat{P}(\rho) \mid P \in \mathcal{F}(G) \}$.

Corollary: The Fourier transfom $P \in \mathcal{F}(G) \mapsto \hat{P} \in C(G)$ and the map $Q \in C(G) \mapsto \check{Q} \in \mathcal{F}(G)$ for $\check{Q}(g) = \frac{1}{|G|} \sum_{\rho \in \hat{G}}{}' d_\rho \, Tr\left(\rho(g^{-1}) Q(\rho)\right)$, are bijective inverses one each other. We have

$$C(G) = \bigoplus_{\rho \in \hat{G}} End(V_\rho).$$

Theorem: The characters $\{\chi_\rho, \rho \in \hat{G}\}$ constitute ON-basis for the subspace of central functions. In particular, $|\hat{G}|$ equals the number of conjugacy classes in G.

Pf: Characters of irred. repr. are central functions, pairwise OG for non-equiv. repr. If P is a central function OG to all charact. of irred. repr., $\hat{P}(\rho) = 0$ $\forall \rho \in \hat{G}$ $\underset{\text{Fourier inversion}}{\Rightarrow}$ $P = 0$.

(7) The claim follows. ∎

<u>Theorem</u> (Plancherel formula): $P, Q \in \mathcal{F}(G)$. Then

$$\langle P, Q\rangle = \frac{1}{|G|}\sum_{\rho\in\hat{G}} d_\rho \operatorname{Tr}\left(\hat{P}(\rho)\,\hat{Q}(\rho)^*\right).$$

<u>Pf</u>: The proof is analogous to Fourier inversion formula:

$$\langle P, Q\rangle = \sum_{\rho\in\hat{G}} \frac{d_\rho}{|G|} \sum_{i,j=1}^{d_\rho} \langle P, \overline{u^\rho_{i,j}}\rangle \langle \overline{u^\rho_{i,j}}, Q\rangle,$$

and the formula from the previous lemma implies

$$\langle P, Q\rangle = \frac{1}{|G|}\sum_{\rho\in\hat{G}} d_\rho \sum_{i,j=1}^{d_\rho} \langle \hat{P}(\rho) v_j^\rho, v_i^\rho\rangle \langle v_i^\rho, \hat{Q}(\rho) v_j^\rho\rangle$$

$$= \frac{1}{|G|}\sum_{\rho\in\hat{G}} d_\rho \sum_{i,j=1}^{d_\rho} \langle \hat{P}(\rho) v_j^\rho, v_i^\rho\rangle \langle v_i^\rho, \hat{Q}(\rho) v_j^\rho\rangle$$

$$= \frac{1}{|G|}\sum_{\rho\in\hat{G}} d_\rho \operatorname{Tr}\left(\hat{P}(\rho)\hat{Q}(\rho)^*\right). \quad ∎$$

Description of convolution of matrix coefficients of representations:

<u>Lemma</u>: Let $\rho, \sigma \in \hat{G}$, u^ρ_{ij}, $u^\sigma_{h,k}$, $1 \le i,j \le d_\rho$, $1 \le h,k \le d_\sigma$ matrix coefficients (for ON bases of V^ρ, V^σ.) Then $u^\rho_{ij} * u^\sigma_{h,k} = \frac{|G|}{d_\rho}\delta_{j,h}\delta_{\rho,\sigma} u^\rho_{i,k}$

<u>Pf</u>: OG-relations for matrix coefficients imply

$$\left[u^\rho_{ij} * u^\sigma_{h,k}\right](g) = \sum_{s\in G} u^\rho_{ij}(gs)\, u^\sigma_{h,k}(s^{-1}) =$$

$$= \sum_{\ell=1}^{d_\rho} u^\rho_{i,\ell}(g) \sum_{s\in G} u^\rho_{\ell,j}\, \overline{u^\sigma_{k,h}(s)}$$

$$= \sum_{\ell=1}^{d_\rho} u^\rho_{i,\ell}(g)\, \delta_{\ell,k}\delta_{j,h}\delta_{\rho,\sigma} \frac{|G|}{d_\rho}$$

$$= \frac{|G|}{d_\rho}\delta_{j,h}\delta_{\rho,\sigma} u^\rho_{i,k}(g). \quad ∎$$

Exercises

Restricted / Induced modules

Def. $H \leq G$ finite groups, (ρ, V) a finite dim represent. of G; Then the restriction of ρ to H is denoted $Res_H^G(\rho)$. The restriction of character of G, χ_ρ, to H is a character denoted $Res_H^G(\chi)$

Ex: S_3, and its 2-dim repr. ρ (see the character table for S_3 and its character.)

Take $A_3 \leq S_3$, the alternating subgroup ($A_3 \simeq \mathbb{Z}/_3$.) Determine $Res_{A_3}^{S_3}(\rho)$.

Def: $H \leq G$, $\rho: H \to GL(W)$. The induced represent. $Ind_H^G(\rho)$ is

1/ $\{b_1, .., b_r\}$ left H-cosets in G; $g \in G$ $g = b_j h$ for some $b_j \in \{b_1, .., b_r\}$ and $h \in H$ (b_j, h are unique)

2/ $\mathbb{C}[G/H]$ is $\mathbb{C}$-vector space and basis $\{b_1, .., b_r\}$,

3/ Let $V = \mathbb{C}[G/H] \otimes W$ as a vector space,

4/ $g \in G$ acts on $b_i \otimes w \in V$: $\exists!$ $b_i \in \{b_1, .., b_r\}$, $h \in H$: $g b_i = b_j h$,

$$g \cdot (b_i \otimes w) = b_j \otimes hw.$$

One can also write $g = b_j h b_i^{-1}$,

$$(b_j h b_i^{-1}) \cdot (b_i \otimes w) = (b_j h b_i^{-1} b_i) \otimes w = b_j h \otimes w = b_j \otimes hw$$

5/ Extend this to G-action on V by linearity.

Lemma: $Ind_H^G(\rho)$ is independent on the choice of representatives $\{b_1, .., b_r\}$. Moreover,

$$\chi_{Ind_H^G(\rho)}(g) = \frac{1}{|H|} \sum_{\substack{k \in G \\ k^{-1}gk \in H}} \chi_\rho(k^{-1} g k)$$

The proof is easy.

Regarding $\mathrm{Ind}_H^G(\rho)(g)$ as a $r \times r$ matrix: $\forall g \in G$ acts on that by $r\times r$-matrix (in the basis of $\{b_1, \dots, b_r\}$.)

$$\chi_{\mathrm{Ind}_H^G(\rho)}(g) = \mathrm{tr}\left(g : \mathbb{C}[G/H] \otimes_{\mathbb{C}} W \to \mathbb{C}[G/H] \otimes_{\mathbb{C}} W\right)$$
$$= \sum_{\substack{i \in [r]:\\ g b_i = b_i h \\ \text{for some } \exists h \in H}} \chi_\rho(h) = \dots$$

$$= \frac{1}{|H|} \sum_{\substack{k \in G:\\ k^{-1} g k \in H}} \chi_\rho(k^{-1} g k)$$ (if $\nexists$ any $k \in G$ $\Rightarrow \sum$ is interpreted to be zero)

Frobenius reciprocity (adjunction of Res, Ind):

__Theorem:__ $H \le G$, ~~χ a character of H~~ (ρ_H, V) a representation of H, (ρ_G, W) a represent. of G, Then

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G \cancel{\rho_H} V, W) = \mathrm{Hom}_H(V, \mathrm{Res}_H^G W).$$

__Ex:__ $\mathbb{C}[G/H]$ for $G = S_3$, $H = e$ (trivial group)

$\mathrm{Ind}_H^G(1)$ = functions on G $= \mathbb{C}[G]$

Frob. recipr. (H has just triv. repr.) $\Rightarrow$ $\chi_{\mathrm{Ind}_e^G(1)} = \chi_{\mathbb{C}[G]} =$

$= \chi_1 + \chi_2 + 2\chi_3$
(triv.) (sign) (2-dim) of $G = S_3$

	1	(12)	(123)
χ_{triv}	1	1	1
χ_{sign}	1	-1	1
$\chi_{2\text{-}dim}$	2	0	-1

$\Rightarrow \chi_{reg} = (6, 0, 0)$ (as it should be)

Harmonic analysis of Lie groups

classical topic of spherical harmonic; we shall treat one explicit induced representation for the Lie group $SO(d,\mathbb{R})$ and its subgroup $SO(d-1,\mathbb{R})$.

Spherical harmonics and orthogonal bases

$\mathbb{R}^d$, $x=(x_1,\dots,x_d)\in\mathbb{R}^d$, $\langle x,y\rangle=\sum_{i=1}^d x_iy_i$, $\|x\|=\sqrt{\langle x,x\rangle}$, $\mathbb{N}_0=\{1,2,\dots\}$

For $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{N}_0^d$, a monomial $x^\alpha=x_1^{\alpha_1}\dots x_d^{\alpha_d}$, its degree is $|\alpha|=\sum_{j=1}^d\alpha_j$, a homogeneous polynomial P, $\deg P=n$, is $P(x)=\sum_{|\alpha|=n}c_\alpha x^\alpha$ for $c_\alpha\in\mathbb{R},\mathbb{C}$. A polynomial of degree at most n is $P(x)=\sum_{|\alpha|\le n}c_\alpha x^\alpha$.

P_n^d ... vector space of hom n polyn on $\mathbb{R}^d$,

Π_n^d ... — " — pol. degree $\le n$ — " — .

Counting the dimensions of $\{\alpha\in\mathbb{N}_0^d \mid |\alpha|=n\}$, $\{\alpha\in\mathbb{N}_0^d\mid|\alpha|\le n\}$, get

$$\dim P_n^d=\binom{n+d-1}{n}=\binom{n+d-1}{d-1},\quad \dim\Pi_n^d=\binom{n+d}{n}.$$

We denote $\partial_i=\frac{\partial}{\partial x_i}$, $i=1,\dots,d$ and Laplace operator $\Delta=\partial_1^2+\dots+\partial_d^2$.

__Def__: For $n\in\mathbb{N}_0$, let $\mathcal{H}_n^d$ be the vector space of harmonic pol. of degree n on $\mathbb{R}^d$, $\mathcal{H}_n^d:=\{P\in P_n^d \mid \Delta P=0\}$.

Spherical harmonics are restrictions of elements in $\mathcal{H}_n^d$ to $S^{d-1}\subseteq\mathbb{R}^d$. If $Y\in\mathcal{H}_n^d$, then $Y(x)=\|x\|^nY(x')$, where $x=\|x\|x'$, $x'\in S^{d-1}$. We will not distinct between $\mathcal{H}_d^n$ and its restriction to the sphere S^{d-1}, perhaps will write $\mathcal{H}_n^d|_{S^{d-1}}$, $P_n^d(S^{d-1}):=P_n^d|_{S^{d-1}}$, $\Pi_n^d(S^{d-1}):=\Pi_n^d|_{S^{d-1}}$.

So we shall call $\mathcal{H}_n^d$ the space of spherical harmonics.

For two smooth functions $f,g\in\mathcal{F}(S^{d-1})$, we define the scalar product

$$\langle f,g\rangle_{S^{d-1}}:=\frac{1}{\omega_d}\int_{S^{d-1}}f(x)g(x)\,d\sigma(x),$$

$d\sigma$ is area measure of S^{d-1},

$$\omega_d=\int_{S^{d-1}}d\sigma=\frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

(2)

Th: If $Y_n \in \mathcal{H}_n^d$, $Y_m \in \mathcal{H}_m^d$, and $m \neq n$. Then $\langle Y_n, Y_m \rangle_{S^{d-1}} = 0$.

Pf: $\frac{\partial}{\partial r}$ .. normal derivative (vector field) to S^{d-1} in $\mathbb{R}^d$.

Since Y_n is homogeneous, $Y_n(x) = r^n Y_n(x')$, where $x = rx'$, $x' \in S^{d-1}$.

Then $\frac{\partial}{\partial r} Y_n(x') = n Y_n(x')$, $x' \in S^{d-1}$, $n \geq 0$. By Green's theorem,

$$(n-m) \int_{S^{d-1}} Y_n Y_m \, d\sigma = \int_{S^{d-1}} \left(Y_m \frac{\partial Y_n}{\partial r} - Y_n \frac{\partial Y_m}{\partial r} \right) d\sigma$$

$$= \int_{\mathbb{B}^d} \left(Y_m \Delta Y_n - Y_n \Delta Y_m \right) dx = 0$$

because $\Delta Y_n = 0$, $\Delta Y_m = 0$. The proof follows. ∎

Th: For $n \in \mathbb{N}_0$, there is a decomposition of $\mathcal{P}_n^d$,

$$\mathcal{P}_n^d = \bigoplus_{0 \leq j \leq n/2} \|x\|^{2j} \mathcal{H}_{n-2j}^d .$$

i.e, $\forall P \in \mathcal{P}_n^d$, there $\exists !$ decomposition

$$P(x) = \sum_{0 \leq j \leq n/2} \|x\|^{2j} P_{n-2j}(x), \quad P_{n-2j}(x) \in \mathcal{H}_{n-2j}^d .$$

Pf: By induction. We have $\mathcal{P}_0^d = \mathcal{H}_0^d$, $\mathcal{P}_1^d = \mathcal{H}_1^d$. Since $\Delta \mathcal{P}_n^d \subseteq \mathcal{P}_{n-2}^d$, $\dim \mathcal{H}_n^d \geq \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d$. Suppose the statement holds for $m = 0, 1, \dots, n-1$ $\overset{?}{\Rightarrow}$ does it hold for n?

Polynomial ring is a domain, i.e. $\|x\|^2 \mathcal{P}_{n-2}^d$ is a subspace of $\mathcal{P}_n^d$ isomorphic to $\mathcal{P}_{n-2}^d$. By induction hypothesis,

$$\|x\|^2 \mathcal{P}_{n-2}^d = \bigoplus_{0 \leq j \leq n/2 - 1} \|x\|^{2j+2} \mathcal{H}_{n-2-2j}^d .$$

By previous theorem, $\mathcal{H}_n^d$ is orthogonal to $\|x\|^2 \mathcal{P}_{n-2}^d$, so that $\dim \mathcal{H}_n^d + \dim \mathcal{P}_{n-2}^d \leq \dim \mathcal{P}_n^d$. Consequently, $\mathcal{P}_n^d = \mathcal{H}_n^d \oplus \|x\|^2 \mathcal{P}_{n-2}^d$.

Corollary: For $n \in \mathbb{N}_0$,

$$\dim \mathcal{H}_n^d = \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d = \binom{n+d-1}{n} - \binom{n+d-3}{n-2},$$

where $\dim \mathcal{P}_{n-2}^d = 0$ for $n = 0, 1$.

(3) <u>Corollary</u>: For $n \in \mathbb{N}_0$,

1/ $\Pi_n(S^{d-1}) = \mathcal{P}_n(S^{d-1}) \oplus \mathcal{P}_{n-1}(S^{d-1})$,

2/ $\dim \Pi_n(S^{d-1}) = \dim \mathcal{P}^d_n + \dim \mathcal{P}^d_{n-1} = \binom{n+d-1}{n} + \binom{n+d-2}{n-1}$.

<u>Pf</u>: By previous Theorem, $\|x\| = 1$ for the restriction to S^{d-1} and $\Pi_n(S^{d-1})$ is a sum of $\mathcal{H}^d_k$ for $0 \le k \le n$, and the first claim follows. Moreover,

$$\dim \Pi_n(S^{d-1}) = \sum_{k=0}^{n} \dim \mathcal{H}^d_k = \sum_{k=0}^{n} \left(\dim \mathcal{P}^d_k - \dim \mathcal{P}^d_{k-2} \right),$$

which simplifies to the equation in 2/; the proof is complete. ☒

<u>Prop</u>: If $\mathcal{P}$ is a homogeneous polyn of degree n and $\mathcal{P}$ is OG to all pol of degree less than n with respect to $\langle \, , \rangle_{S^{d-1}}$, then $\mathcal{P} \in \mathcal{H}^d_n$.

<u>Pf</u>: $\mathcal{P} \in \mathcal{P}^d_n$, $\mathcal{P}$ can be expressed as $\mathcal{P}(x) = \sum_{0 \le j \le \frac{n}{2}} \|x\|^{2j} \mathcal{P}_{n-2j}(x)$, $\mathcal{P}_{n-2j} \in \mathcal{H}^d_{n-2j}$. The OG-nality then shows that $\mathcal{P} = \mathcal{P}_n \in \mathcal{H}^d_n$. ☒

Let $O(d)$ be the orthogonal group (the group of $d \times d$ orthogonal matrices), $SO(d) = \{ g \in O(d) \mid \det g = 1 \}$ special orthogonal group. Any rotation in $\mathbb{R}^d$ is determined by an element in $SO(d)$.

<u>Theorem</u>: The space $\mathcal{H}^d_n$ is invariant for the action $f \mapsto f(Q-)$, $f(x) \mapsto f(Qx)$, $\forall Q \in O(d)$.

Moreover, if $\{Y_\alpha\}_{\alpha \in I}$ is an ON-basis of $\mathcal{H}^d_n$, then so is $\{Y_\alpha(Q-)\}_{\alpha \in I}$, $\# I < \infty$.

<u>Pf</u>: Since Δ is invariant under $O(d)$ (i.e., $\Delta f(Q-) = Q(\Delta f)$ - $\forall f$ smooth functions) - this will be treated later, $\Delta = \nabla \cdot \nabla$ and the change of variables does the work - if $Y \in \mathcal{H}^d_n$ and $Q \in O(d)$, $Y(Qx) \in \mathcal{H}^d_n$:

$$\Delta Y(Qx) = \Delta Q Y(x) = Q(\Delta Y)(x) = 0 \Rightarrow QY \text{ is harmonic polynomial.}$$

(4)

ON-mality of $\{Y_\alpha(Q-)\}_{\alpha\in I}$ for $\mathcal{H}_n^d$:

$$\frac{1}{\omega_d}\int_{S^{d-1}} Y_\alpha(Qx)\,Y_\beta(Qx)\,d\sigma(x) = \frac{1}{\omega_d}\int_{S^{d-1}} Y_\alpha(x)\,Y_\beta(x)\,d\sigma(x) = \delta_{\alpha,\beta},$$

which follows by change of variables and the fact that $d\sigma$ is $O(d)$-invariant

Besides $\langle f,g\rangle_{S^{d-1}}$, another useful inner product can be defined on $\mathcal{P}_n^d$. For $\alpha\in\mathbb{N}_0^d$, let $\partial^\alpha := \partial_1^{\alpha_1}\dots\partial_d^{\alpha_d}$. Let $(a)_n := a(a+1)\dots(a+n-1)$ be the Pochhammer symbol.

<u>Th</u>: $p,q\in\mathcal{P}_n^d$, define a bilinear form

$$\langle p,q\rangle_\partial := p(\partial)q,$$

where $p(\partial)$ is the diff. operator given by $x^\alpha\mapsto\partial^\alpha$ in $p(x)$ (the "Fourier transform" of p.) Then

1/ $\langle p,q\rangle_\partial$ is an inner product on $\mathcal{P}_n^d$,

2/ The reproducing kernel for $\langle\ ,\ \rangle_\partial$ is $k_n(x,y) = \langle x,y\rangle^n/n!$, that is,

$$\langle k_n(x,-),p\rangle_\partial = p(x), \quad \forall p\in\mathcal{P}_n^d$$

3/ For $p\in\mathcal{P}_n^d$, $q\in\mathcal{H}_n^d$,

$$\langle p,q\rangle_\partial = 2^n\left(\tfrac{d}{2}\right)_n\langle p,q\rangle_{S^{d-1}}$$

<u>Pf</u>: $p,q\in\mathcal{P}_n^d$, $p(x)=\sum_{|\alpha|=n}a_\alpha x^\alpha$, $q(x)=\sum_{|\alpha|=n}b_\alpha x^\alpha$, $a_\alpha, b_\alpha\in\mathbb{R}$.

Then $$\langle p,q\rangle_\partial = \sum_{|\alpha|=n}a_\alpha\partial^\alpha\sum_{|\beta|=n}b_\beta x^\beta = \sum_{|\alpha|=n}\alpha!\,a_\alpha b_\alpha,$$

and so $\langle p,p\rangle_\partial>0$ for $p\neq 0$. Consequently, $\langle\ ,\ \rangle_\partial$ is an inner product on $\mathcal{P}_n^d$.

By the multi-binomial formula, for $q_\alpha(x)=x^\alpha$, $|\alpha|=n$,

$$\langle k_n(x,-),q_\alpha\rangle_\partial = \frac{1}{n!}\sum_{|\beta|=n}\binom{n}{\beta}x^\beta\frac{\partial^\beta}{\partial y^\beta}y^\alpha = q_\alpha(x)$$

$\Longrightarrow$ $k_n(x,y)$ is the reproducing kernel w.r.t. $\langle -,-\rangle_\partial$.

(5) Note: multi-nomial formula:

$$\langle k_n(x,y), q_\alpha(y)\rangle_\partial = \frac{1}{n!}\langle x,\partial\rangle^n q_\alpha(y) = \frac{1}{n!}\left(\sum_{j=1}^{n} x_j \partial_{y_j}\right)^n q_\alpha(y),$$

and for $n \geq j_1 \geq j_2 \geq \dots \geq j_k \geq 0$, $j = (j_1, j_2, \dots, j_k)$

$$\binom{n}{j} = \frac{n!}{(n-j_1)!\,(j_1-j_2)!\cdots(j_k)!}$$

$$\Sigma = n - j_1 + (j_1 - j_2) + \dots + j_k = n.$$

This is a generalization of binomial formula, e.g.

$$(a+b+c)^n = \sum_{j=0}^{n}\binom{n}{j}(a+b)^j c^{n-j} = \sum_{j=0}^{n}\binom{n}{j}\binom{j}{k}a^k b^{j-k} c^{n-j}$$

$$= \sum_{j,k}\frac{n!}{(n-j)!\,(j-k)!\,k!}\ \cdots$$

To prove 3/, we notice

$$\int_{\mathbb{R}^d}(\partial_i f)(x)\, g(x)\, e^{-\frac{\|x\|^2}{2}} = -\int_{\mathbb{R}^d} f(x)\big((\partial_i g)(x) - x_i g(x)\big) e^{-\frac{\|x\|^2}{2}}\,dx.$$

Since $p(\partial)q$ is a constant, the integration by parts shows

$$\langle p, q\rangle_\partial = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} p(\partial)q(x)\, e^{-\frac{\|x\|^2}{2}}\,dx$$

$$= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} q(x)\big(p(x) + s(x)\big) e^{-\frac{\|x\|^2}{2}}\,dx,$$

for some $s \in \Pi^d_{n-1}$. Since $q \in \mathcal{H}^d_n$, $p \in \mathcal{P}^d_n$, substitution to polar coordinates & orthogonality of $\mathcal{H}^d_n$, we obtain

$$\langle p, q\rangle_\partial = \frac{1}{(2\pi)^{d/2}}\int_0^\infty r^{2n+d-1} e^{-\frac{r^2}{2}}\,dr \int_{S^{d-1}} q(x')\,p(x')\,d\sigma(x').$$

Evaluating the radial integral gives the proof. ☒

Spherical harmonic polynomials can be constructed by differentiation.

<u>Theorem</u>: Let $d > 2$. For $\alpha \in \mathbb{N}_0^d$, $n = |\alpha|$, define

$$p_\alpha(x) := \frac{(-1)^n}{2^n\left(\frac{d-2}{2}\right)_n}\|x\|^{2|\alpha|+d-2}\,\partial^\alpha\{\|x\|^{-d+2}\}.$$

⑥ Then

1/ $p_\alpha \in \mathcal{H}^d_n$ and p_α is the monic spherical harmonic of the form

$$p_\alpha(x) = x^\alpha + \|x\|^2 q_\alpha(x), \qquad q_\alpha \in \mathcal{P}^d_{n-2}.$$

2/ p_α satisfies the recurrence relation

$$p_{\alpha+e_i}(x) = x_i\, p_\alpha(x) - \frac{1}{2n+d-2}\|x\|^2 \partial_i p_\alpha(x).$$

3/ $\{p_\alpha \mid |\alpha| = n,\ \alpha_d = 0 \text{ or } 1\}$ is a basis of $\mathcal{H}^d_n$.

Pf: Taking ∂_i - derivative of $p_\alpha(x)$ gives 2/.

We have $p_0(x) = 1$, by induction it follows from the recurrence relation that p_α is a homog. pol. of degree n is of the form $x^\alpha + \|x\|^2 q_\alpha(x)$.

We show that p_α is a spherical harmonic. For $g \in \mathcal{P}^d_n$ and $\rho \in \mathbb{R}$, $\sum_{i=1}^d x_i \partial_{x_i} g(x) = n g(x)$ implies

$$\Delta\left(\|x\|^\rho g(x)\right) = \rho(2n+\rho+d-2)\|x\|^{\rho-2} g(x) + \|x\|^\rho \Delta g(x).$$

(Laplace op. in sph. coordinates $\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{d-1}}$ acts on $r^{\frac{\rho}{2}} g(x)$)

In particular, for $n=0$ and $g(x)=1 \Rightarrow \Delta\left(\|x\|^{2-d}\right) = 0$.

For $g = p_\alpha$, $\rho = -2n-d+2$, this gives (substitute $p_\alpha(x) = \frac{(-1)^n}{\ldots}\|x\|^{2|\alpha|+d-2}\partial^\alpha(\|x\|\ldots$

$$\boxed{\Delta p_\alpha(x) =} \ \|x\|^{-\rho}\Delta\left(\|x\|^\rho p_\alpha\right) = \frac{(-1)^n}{2^n\left(\frac{d}{2}-1\right)_n}\|x\|^{2|\alpha|+d-2}\partial^\alpha \underbrace{\Delta\left\{\|x\|^{-d+2}\right\}}_{= 0} = 0$$

$\Rightarrow p_\alpha \in \mathcal{H}^d_n$

Now since $p_\alpha(x) = x^\alpha + \|x\|^2 q_\alpha(x)$ and $\|x\|^2 q(x)$ is a lin. comb. of monomials x^β, $\beta_d \geq 2$, the linear independence of $\{x^\alpha \mid |\alpha| = n, \alpha_d = 0, 1\}$ implies lin. independence of $\{p_\alpha \mid |\alpha| = n,\ \alpha_d = 0 \text{ or } 1\}$. The card. of this set is

$$\underset{\alpha_d = 0}{\dim \mathcal{P}^{d-1}_n} + \underset{\alpha_d = 1}{\dim \mathcal{P}^{d-1}_{n-1}} = \binom{n+d-2}{d-2} + \binom{n+d-3}{d-2}$$

which is by simple binomial identity $= \dim \mathcal{H}^d_n$. ▨

⑦ The complete set of $\{p_\alpha \mid |\alpha|=n\}$ is linearly dependent. By previous,

$$p_{\alpha+2e_1}+\dots+p_{\alpha+2e_d} = \frac{(-1)^n}{2^n\left(\frac{d}{2}\right)_n}\|x\|^{2|\alpha|+d-2}\,\partial^\alpha\Delta\{\|x\|^{-d+2}\}=0,$$

which gives $\dim P^d_{n-2}$ lin. ind. relations among $\{p_\alpha \mid |\alpha|=n\}$.
The set $\{p_\alpha : |\alpha|=n\}$ contains many bases of $\mathcal{H}^d_n$, the one in 3/ is just an example = choice.
The proof of Theorem relies on the fact that $\|x\|^{-d+2}$ is a harmonic fction in $\mathbb{R}^d\setminus\{0\}$ for $d>2$. In the case $d=2$, replace by fction $\log\|x\|$.
The basis $\{p_\alpha \mid |\alpha|=n,\ \alpha_d=0 \text{ or } 1\}$ of $\mathcal{H}^d_n$ is not ONmal, have to use G-S ONmazion process.

Projekční operátory a harmonické funkce

$L^2(S^{d-1})$: $\langle f,f\rangle_{S^{d-1}}<\infty$. Let $\mathrm{proj}_n : L^2(S^{d-1})\to\mathcal{H}^d_n$, denote the ON-projection $L^2(S^{d-1})$ onto $\mathcal{H}^d_n$. If $P\in P^d_n$, then $P=P_n+\|x\|^2Q_n$, where $P_n\in\mathcal{H}^d_n$, $Q\in P^d_{n-2}$, so that $\mathrm{proj}_n P=P_n$.
In particular, the last Theorem shows that $p_\alpha(x)=\frac{(-1)^n}{2^n\left(\frac{d-2}{2}\right)_n}\|x\|^{2|\alpha|-\dots}\partial^\alpha(\|x\|\dots$ is the orthogonal projection of $q_\alpha(x)=x^\alpha$, i.e., $p_\alpha=\mathrm{proj}_n q_\alpha$.
This leads to the

<u>Lemma</u>: Let $p\in P^d_n$. Then

$$\mathrm{proj}_n p=\sum_{j=0}^{[\frac{n}{2}]}\frac{1}{4^j j!\left(-n+2-\frac{d}{2}\right)_j}\|x\|^{2j}\Delta^j p.$$

<u>Pf</u>: By linearity, it is sufficient to discuss $p=q_\alpha(x)=x^\alpha$. By previous Theorem, $\mathrm{proj}_n q_\alpha(x)=p_\alpha(x)$, and so have to show that $p_\alpha(x)$ defined above can be expanded by this formula.
Use induction on n, $n=0$ works from trivial reasons.

Assume the formula works for $m=0,1,\dots,n \overset{?}{\Rightarrow} m=n+1$.
Now we apply this formula for proj_n to $q_\alpha(x)$, $|\alpha|=n$:

(8)

$$\partial^\alpha \left[\|x\|^{-d+2} \right] = (-1)^n 2^n \left(\tfrac{d}{2}-1\right)_n \|x\|^{-2n-d+2} \times \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{4^j j! \,(-n+2-d/2)_j} \|x\|^{2j} \Delta^j (x^\alpha).$$

Application of ∂_i to this identity gives

$$\partial_i \partial^\alpha \left[\|x\|^{-d+2} \right] = (-1)^n 2^n \left(\tfrac{d}{2}-1\right)_n (-2n-d+2) \|x\|^{-2n-d+2} \times \sum_{j=0}^{[\frac{n+1}{2}]} \frac{1}{4^j j! \,(-n+1-d/2)_j} \|x\|^{2j} \underbrace{\left[x_i \Delta^j (x^\alpha) + 2j \Delta^{j-1} \partial_i (x^\alpha) \right]}_{\text{this is } \Delta^j (x_i x^\alpha)},$$

because $[\Delta, x_i] = 2\partial_i$,

$[\Delta^j, x_i] = 2j \Delta^{j-1} \partial_i$.

This means that the fomula is true for $p(x) = x_i x^\alpha$, which completes induction step. $\boxed{}$

<u>Def</u>: The reproducing kernel $Z_n(-,-)$ of $\mathcal{H}_n^d$ is uniquely determined by

$$\frac{1}{\omega_d} \int_{S^{d-1}} Z_n(x,y)\, p(y)\, d\sigma(y) = p(x),$$

$\forall p \in \mathcal{H}_n^d$, $x \in S^{d-1}$

(Consequently, $Z_n(x,-)$ is an element of $\mathcal{H}_n^d$ $\forall x$.)

This ($\exists$ + uniqueness) is the consequence of Riesz repr. theorem applied to lin. fional $L_x(Y) := Y(x)$, $Y \in \mathcal{H}_n^d$, for fixed $x \in S^{d-1}$.

<u>Lemma</u>: let $\{ Y_j \mid 1 \le j \le \dim \mathcal{H}_n^d \}$ be an ON-basis of $\mathcal{H}_n^d$. Then

$$Z_n(x,y) = \sum_{k=1}^{\dim \mathcal{H}_n^d} Y_k(x) Y_k(y), \quad x, y \in S^{d-1}$$

and $Z_n(x,y)$ is independent on the choice of the basis of $\mathcal{H}_n^d$.

<u>Pf</u> $Z_n(x,-) \in \mathcal{H}_n^d$ (the linear form on $\mathcal{H}_n^d$ is represented by an element of $\mathcal{H}_n^d$), so can be expressed as $Z_n(x,y) = \sum_k c_k Y_k(y)$, where the coeff are determ. by scalar product in Def ↑ as

(9)

The uniqueness $\Rightarrow$ independence on basis; a direct proof is:

$\{Y_1, \dots, Y_{\dim \mathcal{H}^d_n}\}$ ... a basis, then $Z_n(x,y) = (Y_1, \dots, Y_{\dim \mathcal{H}^d_n}) \begin{pmatrix} Y_1 \\ \vdots \\ Y_{\dim \mathcal{H}^d_n} \end{pmatrix}$

("scalar product")

If $\{Y'_1, \dots, Y'_{\dim \mathcal{H}^d_n}\}$ ... another basis, then

$(Y'_1) = Q(Y_1, \dots, Y_{\dim \mathcal{H}^d_n})$ and ON-lity $= \frac{1}{\omega_d} \int_{S^{d-1}} \sum_i Y_i(x) Y_i(x)\, d\sigma(x)$ is identity matrix $\Rightarrow$ Q is OG-matrix $\Rightarrow$ $Z_n(x,y)$ is basis independent. ▨

The reproducing kernel is the kernel for projection operator.

Lemma: The projection operator proj_n is given by

$$(\text{proj}_n f)(x) = \frac{1}{\omega_d} \int_{S^{d-1}} f(y) Z_n(x,y)\, d\sigma(y)$$

Pf: Since $\text{proj}_n f \in \mathcal{H}^d_n$, it can be expanded in the ON-basis Y_j, $j = 1, \dots, N_n$, $N_n = \dim \mathcal{H}^d_n$, of $\mathcal{H}^d_n$; the coefficients are (by ON-mality)

$$(\text{proj}_n f)(x) = \sum_{j=1}^{N_n} c_j Y_j(x), \quad c_j = \frac{1}{\omega_d} \int_{S^{d-1}} f(y) Y_j(y)\, d\sigma(y).$$

Finite sum commutes with integration, the proof follows. ▨

Lemma: The kernel $Z_n(-,-)$ satisfies:

1/ $\forall \xi, \eta \in S^{d-1}$:

$$\frac{1}{\omega_d} \int_{S^{d-1}} Z_n(\xi, y) Z_n(\eta, y)\, d\sigma(y) = Z_n(\xi, \eta),$$

2/ $Z_n(x,y)$ depends on $\langle x, y\rangle$ only.

Pf: We know by uniqueness of $Z_n(x,y)$ that $Z_n(Qx, Qy) = Z_n(x,y)$ $\forall Q \in O(d)$. Since for $x, y \in S^{d-1}$ $\exists Q \in SO(d)$ such that

$$\left.\begin{aligned} Qx &= (0, \dots, 0, 1) \\ Qy &= (0, \dots, 0, \sqrt{1-\langle x,y\rangle^2}, \langle x,y\rangle) \end{aligned}\right\} \Rightarrow \text{shows that } Z_n(x,y) \text{ depends on } \langle x,y\rangle.$$

▨

⑩

The hypergeometric function ${}_2F_1$ is defined for $|z|<1$ by power series

$${}_2F_1(a, b, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad c \notin -\mathbb{N}_0$$

$$(a)_n = \begin{cases} 1 & n=0, \\ a(a+1)\cdots(a+n-1) & n>0. \end{cases}$$

which is a polynomial if either $\underline{a}$ or $\underline{b}$ are non-positive integers. Then

$${}_2F_1(-m, b, c; z) = \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{(b)_n}{(c)_n} z^n$$

and it is analytically continued to $|z| \geq 1$ (avoiding the points $z=0, 1$)

Gegenbauer polynomials are orthogonal polynomials on $[-1,1]$ with respect to the weight function $(1-x^2)^{\alpha - 1/2}$, given by

$$C_n^{(\alpha)}(z) = \frac{(2\alpha)_n}{n!} {}_2F_1\left(-n, 2\alpha+n, \alpha+\tfrac{1}{2}, \frac{1-z}{2}\right).$$

First few are

$$C_0^{\alpha}(x) = 1,$$
$$C_1^{\alpha}(x) = 2\alpha x,$$
$$C_n^{\alpha}(x) = \frac{1}{n}\left[2x(n+\alpha-1)C_{n-1}^{\alpha}(x) - (n+2\alpha-2)C_{n-2}^{\alpha}(x)\right]$$

(recursion relation)

and solve hypergeometric differential equation

$$(1-x^2)y'' - (2\alpha+1)xy' + n(n+2\alpha)y = 0.$$

The explicit form of $C_n^{\alpha}(z)$ is

$$C_n^{(\alpha)}(z) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)\, k!\,(n-2k)!} (2z)^{n-2k}.$$

(11) The last Lemma $\Rightarrow$ $Z_n(x,y) = F_n(\langle x,y\rangle)$ is
- harmonic,
- depends on $\langle x,y\rangle$.

Th: For $n\in\mathbb{N}_0$, $x,y\in S^{d-1}$, $d\geq 3$,
$$Z_n(x,y) = \frac{n+\lambda}{\lambda} C_n^{\lambda}(\langle x,y\rangle), \quad \lambda = \frac{d-2}{2}.$$

Pf: Let $p\in\mathcal{H}_n^d$, then $p(x) = \langle k_n(x,-), p\rangle_\partial$. For fixed x, it follows from the same Theorem

$$p(x) = \langle k_n(x,-),p\rangle_\partial = \langle \operatorname{proj}_n(k_n(x,-)), p\rangle_\partial$$
$$= \frac{2^n\left(\frac{d}{2}\right)_n}{\omega_d}\int_{S^d} \operatorname{proj}_n(k_n(x,-))(y)\,p(y)\,d\sigma(y).$$

Since the kernel is uniquely determined by reproducing property, $Z_n(x,y) = 2^n\left(\frac{d}{2}\right)_n \operatorname{proj}_n[k_n(x,-)](y)$. Because $\partial_i k_n(x,y) = x_i k_{n-1}(x,y)$, $\Delta^j k_n(x,y) = \|x\|^{2j} k_{n-2j}(x,y)$, the lemma on realization of proj shows for $x,y\in S^{d-1}$:

$$Z_n(x,y) = 2^n\left(\frac{d}{2}\right)_n \operatorname{proj}_n(k_n(x,-))(y) = \sum_{j=0}^{\left[\frac{n}{2}\right]} \frac{\left(\frac{d}{2}\right)_n 2^{n-2j}}{j!\,(1-n-\lambda)_j} k_{n-2j}(x,y)$$

and by $\frac{1}{(n-2j)!} = \frac{(-n)_{2j}}{n!} = \frac{2^{2j}\left(-\frac{n}{2}\right)_j\left(\frac{-n+1}{2}\right)_j}{n!}$ $\boxed{\lambda = \frac{d-2}{2}}$

we conclude

$$Z_n(x,y) = \frac{n+\lambda}{\lambda}\frac{(\lambda)_n 2^n}{n!}\sum_{j=0}^{\left[\frac{n}{2}\right]} \frac{\left(-\frac{n}{2}\right)_j\left(\frac{-n+1}{2}\right)_j}{j!\,(1-n-\lambda)_j}\langle x,y\rangle^{n-2j}$$
$$= \frac{n+\lambda}{\lambda}\frac{(\lambda)_n 2^n}{n!}\langle x,y\rangle^n\, {}_2F_1\left(\begin{matrix}-\frac{n}{2}, \frac{1-n}{2}\\ 1-n-\lambda\end{matrix}; \frac{1}{\langle x,y\rangle^2}\right). \quad \boxtimes$$

Let $\{Y_i \mid i\in\{1,\dots,\dim\mathcal{H}_n^d\}\}$ be an ON-basis of $\mathcal{H}_n^d$. Then
$$\sum_{j=1}^{\dim\mathcal{H}_n^d} Y_j(x)Y_j(y) = \frac{n+\lambda}{\lambda}C_n^\lambda(\langle x,y\rangle), \quad \lambda = \frac{d-2}{2}.$$

Corollary: $n\in\mathbb{N}_0$, $x,y\in S^{d-1}$, $d\geq 3$. Then
$$|Z_n(x,y)| \leq \dim\mathcal{H}_n^d, \quad Z_n(x,x) = \dim\mathcal{H}_n^d.$$

Pf: Set $F_n(t) := \frac{n+\lambda}{\lambda}C_n^\lambda(t)$. By $Z_n(x,y) = \frac{n+\lambda}{\lambda}C_n^\lambda(\langle x,y\rangle)$

(12)

$\underset{F_n(1)}{Z_n(x,x)}$ is constant $\forall x \in S^{d-1}$. Setting $x=y$ in $Z_n(x,y) = \sum_1^{\dim \mathcal{H}_n^d} Y_j(x) Y_j(y)$

and integrating over S^{d-1}:

$$F_n(1) = \frac{1}{\omega_d} \int_{S^{d-1}} Z_n(\langle x,x\rangle)\, d\sigma(x) = \frac{1}{\omega_d} \int_{S^{d-1}} \sum_{k=1}^{\dim \mathcal{H}_n^d} Y_k^2(x)\, d\sigma(x)$$

$= \dim \mathcal{H}_n^d$.

The inequality follows by application of Schwarz inequality to . ∎

The functions on S^{d-1}, depending on $\langle x,y\rangle$ only, are analogues of radial functions on $\mathbb{R}^d$. The Funk-Hecke theorem states

Theorem: $f \in L_1(\mathbb{R}^d)$: $\int_{-1}^{1} |f(t)|\, (1-t^2)^{\frac{d-3}{2}} dt < \infty$, $d \geq 2$.
Then $\forall\, Y_n \in \mathcal{H}_n^d$:

$$\int_{S^{d-1}} f(\langle x,y\rangle)\, Y_n(y)\, d\sigma(y) = \lambda_n(f)\, Y_n(x), \quad x \in S^{d-1},$$

where $\lambda_n(f)$ is defined by

$$\lambda_n(f) := \omega_{d-1} \int_{-1}^{1} f(t)\, \frac{C_n^{\frac{d-2}{2}}(t)}{C_n^{\frac{d-2}{2}}(t)}\, (1-t^2)^{\frac{d-3}{2}} dt.$$

Pf: f pol. degree m on $\mathbb{R}^d$, then f can be expanded in term of Gegenbauer pol. as
(complete basis of polynomials)

$$f(t) = \sum_{k=0}^{m} \lambda_k \frac{k + \frac{d-2}{2}}{\frac{d-2}{2}}\, C_k^{\frac{d-2}{2}}(t),$$

where λ_k are determined by orthogonality of Gegenbauer polyn.:

$$\lambda_k = \frac{c_d}{C_k^{\frac{d-2}{2}}(1)} \int_{-1}^{1} f(t)\, C_k^{\frac{d-2}{2}}(t)\, (1-t^2)^{\frac{d-3}{2}} dt,$$

$$c_d^{-1} = \int_{-1}^{1} (1-t^2)^{\frac{d-3}{2}} dt = \frac{\omega_d}{\omega_{d-1}}$$

From $Z_n(x,y) = \frac{n+\lambda}{\lambda} C_n^{\lambda}(\langle x,y\rangle)$ and reproducing ~~property~~ of Z_n
it follows for $n \leq m$:

(13)

$$\frac{1}{\omega_d}\int_{S^{d-1}} f(\langle x,y\rangle)\, Y_n(y)\, d\sigma(y) = \lambda_n Y_n(x), \quad x\in S^{d-1}.$$

The explicit formula used for the last statement is

$$\frac{\omega_{d-1}}{\omega_d}\int_{-1}^{1} C_n^\lambda(t)\, C_m^\lambda(t)\, (1-t^2)^{\lambda-\frac{1}{2}}\, dt = h_n^\lambda\, \delta_{m,n}$$

with

$$h_n^\lambda = \frac{\lambda}{n+\lambda}\, C_n^\lambda(1).$$

■

<u>Remark</u>: A collection of points $\{x_1, \dots, x_N\}$ in S^{d-1} is called a fundamental system of degree n on S^{d-1} if

$$\det\left[\, C_n^\lambda(\langle x_i, x_j\rangle)\right]_{i,j=1}^N > 0, \quad \lambda = \frac{d-2}{2}.$$

There exist infinitely many fundamental systems (of given degree), because the previous condition specifies a complement of an algebraic surface in $\mathbb{R}^{(d-1)N}$.

The claim is that if $\{x_1, \dots, x_N\}$ is a fundamental system of points on S^{d-1}, $\{\, C_n^\lambda(\langle -, x_i\rangle) \mid i=1,2,\dots,N\}$, $\lambda = \frac{d-2}{2}$, is a basis of $\mathcal{H}_n^d\big|_{S^{d-1}}$.

$\boxed{\text{Laplace - Beltrami operator}}$

$x\in\mathbb{R}^d$, $x\mapsto u = u(x)$ a change of variables (invertible bijection), $u\mapsto x = x(u)$

Introduce the tensors $\quad g_{ij} = \sum_{k=1}^d \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j}$, $\quad g^{ij} = \sum_{k=1}^d \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}$, $1\le i,j\le d$.

$$g := \det(g_{ij})_{i,j=1}^d, \quad (g_{ij})^{-1} = g^{ij}$$

A general result in tensor analysis shows the formula for Laplace op.

$$\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} = \frac{1}{\sqrt{g}}\sum_{i=1}^d\sum_{j=1}^d \frac{\partial}{\partial u_i}\sqrt{g}\, g^{ij}\frac{\partial}{\partial u_j}$$

in coordinates u_j; the Laplace - Beltrami operator, i.e., the spherical

(14) part of the Laplace operator, is given by $x \to (r, \xi_1, \dots, \xi_{d-1})$, $r \in \mathbb{R}_+$ and $\underbrace{(\xi_1, \dots, \xi_d)}_{\xi} \in S^{d-1}$.

<u>Lemma:</u> In the spherical (polar) coordinates $x = r\xi$, $r > 0$, $\xi \in S^{d-1}$, the Laplace op. satisfies

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0,$$

where

$$\Delta_0 = \sum_{i=1}^{d-1}\frac{\partial^2}{\partial \xi_i^2} - \sum_{i=1}^{d-1}\sum_{j=1}^{d-1}\xi_i\xi_j\frac{\partial^2}{\partial\xi_i\partial\xi_j} - (d-1)\sum_{i=1}^{d-1}\xi_i\frac{\partial}{\partial\xi_i}.$$

<u>Pf.</u> $\xi \in S^{d-1}$, $\xi_1^2 + \dots + \xi_d^2 = 1$. For the change of variables

$$(x_1, \dots, x_d) \mapsto (r, \xi_1, \dots, \xi_{d-1}), \quad x = r\xi,$$

whose inverse is $\xi_1 = \frac{x_1}{\|x\|}, \dots, \xi_{d-1} = \frac{x_{d-1}}{\|x\|}, \xi_d = \frac{x_d}{\|x\|}$, $r = \|x\|$.

The chain rule implies

$$\frac{\partial}{\partial x_i} = \frac{1}{r}\frac{\partial}{\partial\xi_i} - \frac{\xi_i}{r}\sum_{j=1}^{d-1}\xi_j\frac{\partial}{\partial\xi_j} + \xi_i\frac{\partial}{\partial r}, \quad 1 \le i \le d-1,$$

$$\frac{\partial}{\partial x_d} = -\frac{x_d}{r^2}\sum_{j=1}^{d-1}\xi_j\frac{\partial}{\partial\xi_j} + \frac{x_d}{r}\frac{\partial}{\partial r}, \quad \left(x_d = r\xi_d, \ \sum_{i=1}^{d}\xi_i^2 = 1\right)$$

and the substitution back into Δ gives the result. ∎

<u>Lemma:</u> Let $\Delta_{0,d}$ be the Laplace-Beltrami operator for S^{d-1}. For $\xi \in S^{d-1}$ write $\xi = (\sqrt{1-t^2}\,\eta, t)$ with $-1 \le t \le 1$, $\eta \in S^{d-2}$. Then

$$\Delta_{0,d} = \frac{1}{(1-t^2)^{\frac{d-3}{2}}}\frac{\partial}{\partial t}\left((1-t^2)^{\frac{d-1}{2}}\frac{\partial}{\partial t}\right) + \frac{1}{1-t^2}\Delta_{0,d-1}.$$

<u>Pf:</u> Let us make the following change of variables:

$$(\xi_1, \dots, \xi_{d-1}) \longmapsto (\eta_1, \dots, \eta_{d-2}, t)$$

$$\xi_1 = \sqrt{1-t^2}\,\eta_1, \dots, \xi_{d-2} = \sqrt{1-t^2}\,\eta_{d-2}, \ \xi_{d-1} = t$$

$$\eta_1 = \frac{\xi_1}{\sqrt{1-\xi_{d-1}^2}}, \dots, \eta_{d-2} = \frac{\xi_{d-2}}{\sqrt{1-\xi_{d-1}^2}}, \ t = \xi_{d-1},$$

and the chain rule implies

(15) and the substitution into Δ_0 gives

$$\Delta_{0,d} = (1-t^2)\frac{\partial^2}{\partial t^2} - (d-1)t\frac{\partial}{\partial t} + \frac{1}{1-t^2}\Delta_{0,d-1},$$

and the is the claim of the lemma. ▨

Let $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\right)$. Then $\nabla = \frac{1}{r}\nabla_0 + \xi\frac{\partial}{\partial r}$, $x = r\xi$, $\xi \in S^{d-1}$, where ∇_0 is the spherical gradient, i.e., the spherical part of ∇ involving $\frac{\partial}{\partial \xi_i}$'s only. (It is the part orthogonal to $\frac{\partial}{\partial r}$.)

A computation shows $\Delta_0 = \nabla_0 \cdot \nabla_0$, based on

$$\Delta = \nabla\cdot\nabla = \frac{1}{r^2}\nabla_0\cdot\nabla_0 + \frac{1}{r}\nabla_0\left(\xi\frac{\partial}{\partial r}\right) + \xi\frac{\partial}{\partial r}\left(\frac{1}{r}\nabla_0\right) + \frac{\partial^2}{\partial r^2}.$$

<u>Theorem</u>: The spherical harmonics are eigenfunctions of Δ_0,

$$(\Delta_0 Y)(\xi) = -n(n+d-2)Y(\xi) \quad \forall\, Y \in \mathcal{H}_n^d,\ \xi \in S^{d-1}.$$

<u>Pf</u>: $x = r\xi$, $\xi \in S^{d-1}$, and since $Y \in \mathcal{H}_n^d$ is homogeneous, $Y(x) = r^n Y(\xi)$. So

$$0 = (\Delta Y)(\xi) = n(n-1)r^{n-2}Y(\xi) + (d-1)n\, r^{n-2}Y(\xi) + r^{n-2}\Delta_0 Y(\xi),$$

which is the previous equality. ▨

The previous identity implies that Δ_0 is self-adjoint operator.

Spherical harmonics in spherical coordinates

The polar coordinates $(x_1, x_2) \mapsto (r\cos\theta, r\sin\theta)$
$r \in \mathbb{R}_+$, $0 \le \theta \le 2\pi$
give coordinates on S^1 when $r = 1$. Spherical Coordinates

$$x_1 = r\sin\theta_{d-1} \cdots \sin\theta_2 \sin\theta_1,$$
$$x_2 = r\sin\theta_{d-1} \cdots \sin\theta_2 \cos\theta_1,$$
$$\vdots$$
$$x_{d-1} = r\sin\theta_{d-1}\cos\theta_{d-2}$$
$$x_d = r\cos\theta_{d-1}$$

(16) $r \geq 0$, $0 \leq \theta_1 \leq 2\pi$, $0 \leq \theta_i \leq \pi$ for $i = 2, \dots, d-1$.

When $r = 1$, $\theta_1, \dots, \theta_{d-1}$ are coordinates on S^{d-1},

recursively defined by $x = (\xi \sin\theta_{d-1}, \cos\theta_{d-1}) \in S^{d-1}$ for $\xi \in S^{d-2}$.

Then
$$d\sigma = d\sigma_d = \prod_{j=1}^{d-2} (\sin\theta_{d-j})^{d-j-1} \, d\theta_{d-1} \dots d\theta_2 \, d\theta_1$$
is the Lebesgue measure on S^{d-1} $\left(\underbrace{\sqrt{\det J}}_{\text{Jacobian, det computed recursively}} d\theta_1 \dots d\theta_{d-1} \, dr \,/_{r=1} \right)$

Recall the ON-relations for Gegenbauer polynomials:
$$\frac{\omega_{d-1}}{\omega_d} \int_{-1}^{1} C_n^\lambda(t) \, C_m^\lambda(t) \, (1-t^2)^{\lambda - 1/2} \, dt = \overbrace{\frac{\lambda}{n+\lambda} \underbrace{C_n^\lambda(1)}_{\frac{\lambda}{n+\lambda} \dim \mathcal{H}_n^d}}^{h_n^\lambda} \delta_{m,n}, \qquad \lambda = \frac{d-2}{2}$$

which can be written also as $(t = \cos\theta)$
$$\int_0^\pi C_n^\lambda(\cos\theta) \, C_m^\lambda(\cos\theta) \, (\sin\theta)^{d-2} \, d\theta = \sqrt{\pi} \, \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \, h_n^\lambda \, \delta_{m,n} \qquad \lambda = \frac{d-2}{2}$$

We shall write a basis of spherical harmonics in terms of Gegenbauer polyn. in the spherical coordinates.

<u>Theorem</u>: For $d > 2$, $\alpha \in \mathbb{N}_0^d$, define
$$Y_\alpha(x) := h_\alpha^{-1} \, r^{|\alpha|} \, g_\alpha(\theta_1) \prod_{j=1}^{d-2} (\sin\theta_{d-j})^{|\alpha^{j+1}|} \, C_{\alpha_j}^{\lambda_j}(\cos\theta_{d-j})$$
where

- $$g_\alpha(\theta_1) = \begin{cases} \cos \alpha_{d-1}\theta_1 & \alpha_d = 0, \\ \sin \alpha_{d-1}\theta_1 & \alpha_d = 1. \end{cases}$$

- $|\alpha^j| = \alpha_j + \dots + \alpha_{d-1}$,
  $\lambda_j = |\alpha^{j}| + \frac{(d-j-1)}{2}$

- $$h_\alpha^2 = b_\alpha \prod_{j=1}^{d-2} \frac{\alpha_j! \left(\frac{d-j+1}{2}\right)_{|\alpha^{j+1}|} (\alpha_j + \lambda_j)}{(2\lambda_j)_{\alpha_j} \left(\frac{d-j}{2}\right)_{|\alpha^{j+1}|} \lambda_j}$$

  where $b_\alpha = \begin{cases} 2 & \alpha_{d-1} + \alpha_d > 0 \\ 1 & \text{otherwise} \end{cases}$

Then $\{ Y_\alpha \mid |\alpha| = n, \alpha_d = 0, 1 \}$ is an ON-basis of $\mathcal{H}_n^d$,
$\langle Y_\alpha, Y_\beta \rangle_{S^{d-1}} = \delta_{\alpha,\beta}$.

Pf: Y_α is a homogeneous polynomial: because $\cos\theta_k = \dfrac{x_{k+1}}{\sqrt{x_1^2+\dots+x_{k+1}^2}}$, $1 \le k \le d-1$, $Y_\alpha(x)$ can be re-written as

$$Y_\alpha(x) = h_\alpha^{-1}\, g(x) \prod_{j=1}^{d-2} \left(x_1^2+\dots+x_{d-j+1}^2\right)^{\frac{\alpha_j}{2}} C_{\alpha_j}^{\lambda_j}\left(\frac{x_{d-j+1}}{\sqrt{x_1^2+\dots+x_{d-j+1}^2}}\right),$$

where

$$g(x) = \rho^{\alpha_{d-1}} \cos \alpha_{d-1}\theta_1 \quad \text{for } \alpha_d = 0$$
$$= \rho^{\alpha_{d-1}} \sin \alpha_{d-1}\theta_1 \quad \text{for } \alpha_d = 1, \quad \rho = \sqrt{x_1^2+x_2^2}.$$

Since $x_1 = \rho \sin\theta_1$, $x_2 = \rho\cos\theta_1$, $g(x)$ is either real or imaginary part of $(x_2 + i x_1)^{\alpha_{d-1}}$ $\Rightarrow$ it is a homogeneous polynomial of degree α_{d-1} in x. All together, we see $Y_\alpha \in \mathcal{P}_n^d$. (notice that $C_n^\lambda(t)$ is even for n even and odd when n is odd.)

It also follows from the recursive parametrization of S^{d-1}:

$$\int_{S^{d-1}} f(x)\, d\sigma_d(x) = \int_0^\pi \int_{S^{d-2}} f(\xi \sin\theta, \cos\theta)\, d\sigma_{d-1}(\xi) (\sin\theta)^{d-2}\, d\theta,$$

for any smooth (continuous) fn $f \in C^\infty(S^{d-1})$.

Then

$$\langle Y_\alpha, Y_{\alpha'} \rangle_{S^{d-1}} = \frac{h_\alpha^{-1} h_{\alpha'}^{-1}}{\omega_d} \int_0^{2\pi} g_\alpha(\theta_1)\, g_{\alpha'}(\theta_1)\, d\theta_1 \;\times$$
$$\prod_{j=1}^{d-2} \int_0^\pi C_{\alpha_j}^{\lambda_j}(\cos\theta_{d-j})\, C_{\alpha_j'}^{\lambda_j}(\cos\theta_{d-j}) (\sin\theta_{d-j})^{2\lambda_j}\, d\theta_{d-j},$$

from which the ortogonality follows by OG-uality of Gegenbauer polyn. and functions $\{\cos m\theta, \sin m\theta\}_{m\in\mathbb{N}}$ on $(0,2\pi)$. The formula for h_α follows from normalization constants of the Gegenbauer pol.

$\square$

Representation of rotation group

We describe the representation of $SO(d)$ on the space of harmonic polynomials.
$\forall Q \in SO(d) \mapsto T(Q) \in \mathrm{End}(L^2(S^{d-1}))$
cont. invertible map

$$\left(T(Q)f\right)(x) = f(Q^{-1}x), \quad x \in S^{d-1}$$
$$T(Q_1 Q_2) = T(Q_1)\,T(Q_2), \quad \forall\, Q_1, Q_2 \in SO(d).$$

Since $d\sigma$ is invariant under rotations $SO(d)$, $\|T(Q)f\|_{L^2} = \|f\|_{L^2}$ in L^2-norm on S^{d-1} $\Rightarrow$ $T(Q)$ is unitary. The trivial space and the whole space $L^2(S^{d-1})$ are invariant subspaces of $L^2(S^{d-1})$, and also $\mathcal{H}^d_n|_{S^{d-1}}$ are invariant subspaces $\forall n \in \mathbb{N}_0$, due to $SO(d)$-invariance of Δ.
We want to prove $(SO(d), \mathcal{H}^d_n|_{S^{d-1}})$ is an irreducible repr.

<u>Lemma</u>: A spherical harmonic $Y \in \mathcal{H}^d_n$ is invariant under rotations in $SO(d)$ preserving x_d (isomorphic to $SO(d-1) \subseteq SO(d)$) iff

$$Y(x) = c\,\|x\|^n\, C_n^\lambda\left(\frac{x_d}{\|x\|}\right), \quad \lambda = \frac{d-2}{2},$$

for a constant $c \in \mathbb{C}$.

<u>Pf</u>: A polynomial $Y(x) = Y(x_1, \ldots, x_{d-1}, x_d)$ is invariant under $SO(d-1)$ acting by rotations in $\langle x_1, \ldots, x_{d-1}, x_d \rangle$ iff

$$Y(x) = \sum_{0 \le j \le \frac{n}{2}}{}' \; b_j\, x_d^{\,n-2j} \underbrace{(x_1^2 + \ldots + x_{d-1}^2)^j}_{x_1^2 + \ldots + x_{d-1}^2 = \|x\|^2 - x_d^2}$$

$$= \sum_{0 \le j \le \frac{n}{2}} c_j\, x_d^{\,n-2j}\, \|x\|^{2j},$$

for some $\{b_j\}_j, \{c_j\}_j \in \mathbb{C}$. Since $Y(x)$ is harmonic, $\Delta Y(x) = 0$ implies the recurrence relations

$$4(j+1)(n-j-1)\,c_{j+1} + (n-2j)(n-2j-1)\,c_j = 0,$$

which can be solved into

$$Y(x) = c_0 \sum_{0 \le j \le \frac{n}{2}} \frac{\left(-\frac{n}{2}\right)_j \left(\frac{1-n}{2}\right)_j}{j!\,(1-n-d-2)_j}\, x_d^{\,n-2j}\, \|x\|^{2j}.$$

The result follows by comparison with the definition of Gegenbauer polynomials. ▨

Theorem: The representation $T_{n,d}$ of $SO(d)$ on $\mathcal{H}_n^d$ is irreducible.

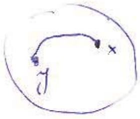
Pf. Assume $\mathcal{U} \subseteq \mathcal{H}_n^d$ is an invariant subspace, $\mathcal{U} \neq \emptyset$. Let $\{Y_j \mid 1 \leq j \leq M\}$, $M \leq \dim \mathcal{H}_n^d$, be an ON-basis of $\mathcal{U}$. There is a polynomial of 1-variable $F = F(t)$, such that $\sum_{j=1}^{M} Y_j(x) Y_j(y) = F(\langle x, y\rangle)$. It is harmonic, and for $y = e_d = (0, \dots, 0, 1)$ shows that $F(\langle x, e_d\rangle)$ is in $\mathcal{H}_n^d$ and evidently invariant under $SO(d-1) \subseteq SO(d)$ (rotations of $\mathbb{R}^d$ preserving vector space $\langle (0, \dots, 0, 1)\rangle$). By previous lemma, $F(\langle x, e_d\rangle) = c\, \|x\|^n C_n^\lambda\left(\frac{x_d}{\|x\|}\right)$ and $\|x\|^n C_n^\lambda\left(\frac{x_d}{\|x\|}\right) \in \mathcal{U}$.

Let $\mathcal{U}^\perp$ denote the OG-complement of $\mathcal{U}$ in $\mathcal{H}_n^d$. If $f \in \mathcal{U}^\perp$, $g \in \mathcal{U}$, $\langle T(\varphi) f, g\rangle_{S^{d-1}} = \langle f, T(\varphi^{-1}) g\rangle_{S^{d-1}} = 0$, which proves that $\mathcal{U}^\perp$ is an invariant subspace of $\mathcal{H}_n^d$ as well. The application of the same argument as for $\mathcal{U}$ shows $\|x\|^n C_n^\lambda\left(\frac{x_d}{\|x\|}\right) \in \mathcal{U}^\perp$, which contradicts to $\mathcal{U} \cap \mathcal{U}^\perp = \{0\}$. Thus $\mathcal{U}$ is trivial and the claim follows.

▨

Convolution operator and spherical harmonic expansion

$x, y \in S^{d-1}$, the distance of x, y is defined to be geodesic distance,



$$d(x,y) = \arccos \langle x, y \rangle.$$

and the reproducing kernel of $\mathcal{H}_n^d$ depends on $\langle x, y \rangle$ only.

We introduce the notation for the weight, $w_\lambda(x) = (1-x^2)^{\lambda - 1/2}$, $\lambda > -\frac{1}{2}$, $x \in (-1, 1)$. We give the definition of convolution operator on the sphere:

<u>Def</u>: For $f \in L^1(S^{d-1})$, $g \in L^1(w_\lambda; \langle -1, 1 \rangle)$ with $\lambda = \frac{d-2}{2}$. We define

$$(f * g)(x) := \frac{1}{\omega_d} \int_{S^{d-1}} f(y)\, g(\langle x, y \rangle)\, d\sigma(y).$$

Denote weighted L^p-space $L^p(w_\lambda; \langle -1, 1 \rangle)$: for $g \in L^p(w_\lambda, \langle -1, 1 \rangle)$ the L^p-norm $\| - \|_{\lambda, p}$ is

$$\| g \|_{\lambda, p} := \left(c_\lambda \int_{-1}^{1} |g(x)|^p w_\lambda(x)\, dx \right)^{\frac{1}{p}}, \quad 1 \le p \le \infty$$

for the normalization constant c_λ: $c_\lambda \int_{-1}^{1} w_\lambda(t)\, dt = 1$, L_p-norm for $p = \infty$ is the maximum norm L_∞ on $\langle -1, 1 \rangle$.

The convolution on the sphere S^{d-1} satisfies Young's inequality:

<u>Theorem</u>: Let $p, q, r \ge 1$ and $p^{-1} = r^{-1} + q^{-1} - 1$. For $f \in L^q(S^{d-1})$ and $g \in L^r(w_\lambda; \langle -1, 1 \rangle)$ with $\lambda = \frac{d-2}{2}$)

$$\| f * g \|_p \le \| f \|_q \, \| g \|_{\lambda, r}.$$

In particular, for $1 \le p \le \infty$ holds

$$\| f * g \|_p \le \| f \|_p \, \| g \|_{\lambda, 1}$$

$$\| f * g \|_p \le \| f \|_1 \, \| g \|_{\lambda, p}.$$

<u>Pf</u>: By Minkowski's inequality,

$$\| f * g \|_q \le \frac{1}{\omega_d} \int_{S^{d-1}} |f(y)| \left(\frac{1}{\omega_d} \int_{S^{d-1}} |g(\langle x, y \rangle)|^q \, d\sigma(y) \right)^{1/q} d\sigma(x)$$

$$= \| f \|_1 \, \| g \|_{\lambda, q},$$

(21) and the standard interpolation argument implies the complete result. $\boxtimes$

By our previous results, the operator of projection on harmonic part is convolution,

$$\operatorname{proj}_n f = f * Z_n, \quad Z_n(t) = \frac{n+\lambda}{\lambda} C_n^{\lambda}(t) \quad \text{for } \lambda = \frac{d-2}{2}.$$

For $g \in L^1(w_\lambda; \langle -1,1\rangle)$, let $\hat{g}_n^{\lambda}$ denote the Fourier coefficients of g with respect to the Gegenbauer polynomials:

$$\hat{g}_n^{\lambda} := c_\lambda \int_{-1}^{1} g(t) \frac{C_n^{\lambda}(t)}{C_n^{\lambda}(1)} (1-t^2)^{\lambda - 1/2} dt.$$

<u>Theorem</u>: For $f \in L^1(S^{d-1})$, $g \in L^1(w_\lambda; \langle -1,1\rangle)$ with $\lambda = \frac{d-2}{2}$, we have

$$\operatorname{proj}_n (f*g) = \hat{g}_n^{\lambda} \operatorname{proj}_n f, \quad n \in \mathbb{N}_0.$$

<u>Pf</u>: By Funk-Hecke formula,

$$\operatorname{proj}_n(f*g)(x) = \frac{1}{\omega_d} \int_{S^{d-1}} (f*g)(\xi) Z_n(x,\xi)\, d\sigma(\xi)$$

$$= \frac{1}{\omega_d} \int_{S^{d-1}} f(y) \left(\frac{1}{\omega_d} \int_{S^{d-1}} g(\langle \xi, y\rangle) Z_n(x,\xi)\, d\sigma(\xi) \right) d\sigma(y)$$

$$= \hat{g}_n^{\lambda} \frac{1}{\omega_d} \int_{S^{d-1}} f(y) Z_n(x,y)\, d\sigma(y) = \hat{g}_n^{\lambda} \operatorname{proj}_n f(x),$$

where we used $c_\lambda = \frac{\omega_{d-1}}{\omega_d}$, $\lambda = \frac{d-2}{2}$. $\boxtimes$

The previous identity can be viewed as an analogue of the fact that Fourier transform of $f*g$ is equal to products of the Fourier transforms of f, g. It justify the terminology convolution for

$$(f*g)(x) := \frac{1}{\omega_d} \int_{S^{d-1}} f(y) g(\langle x,y\rangle)\, d\sigma(y).$$

We define the translation operator T_θ on the sphere can be interpreted in terms of geodesic distance.

<u>Def</u>: For $0 \le \theta \le \pi$, $f \in L^1(S^{d-1})$, define

$$(T_\theta f)(x) = \frac{1}{\omega_{d-1} (\sin\theta)^{d-1}} \int_{\langle x,y\rangle = \cos\theta} f(y)\, d\ell_{x,\theta}(y)$$

where $d\ell_{x,\theta}(y)$ denotes Lebesgue measure on the set $\{y \in S^{d-1} \mid \langle x,y\rangle = \cos\theta\}$

(22)

<u>Prop:</u> Let $0 \leq \theta \leq \pi$, $f \in L^2(S^{d-1})$. Then

1/ Let $S_x^\perp := \{ y \in S^{d-1} \mid \langle x, y \rangle = 0 \}$, the equator in S^{d-1} with respect to x; then

$$(T_\theta f)(x) = \frac{1}{\omega_{d-1}} \int_{S_x^\perp} f(x \cos\theta + u \sin\theta)\, d\sigma(u).$$

In particular, if $f_0(x) := 1$, $T_\theta f_0(x) = 1$.

2/ For a generic $g : \langle -1, 1 \rangle \to \mathbb{R}$,

$$(f * g)(x) = \frac{\omega_{d-1}}{\omega_d} \int_0^\pi g(\cos\theta) (T_\theta f)(x) (\sin\theta)^{d-2}\, d\theta.$$

<u>Pf:</u> 1/ follows from a change of variables $y \mapsto x\cos\theta + u \sin\theta$.

As for 2/, choose ~~the~~ a coordinate system such that x is the "north pole", and set $y = x\cos\theta + u\sin\theta$ to obtain

$$(f*g)(x) = \frac{1}{\omega_d} \int_0^\pi g(\cos\theta) \int_{S_x^\perp} f(x\cos\theta + u \sin\theta)\, d\sigma(u) (\sin\theta)^{d-2}\, d\theta$$

$$= \frac{\omega_{d-1}}{\omega_d} \int_0^\pi g(\cos\theta) (T_\theta f)(x) (\sin\theta)^{d-2}\, d\theta,$$

since $S_x^\perp$ is isomorphic to the sphere S^{d-2}. ☒

<u>Lemma:</u> The operator T_θ maps $\Pi_n(S^{d-1})$ into itself $\forall n \in \mathbb{N}$. For $f \in L^1(S^{d-1})$,

$$\mathrm{proj}_n(T_\theta f) = \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)}\, \mathrm{proj}_n f, \quad \lambda = \frac{d-2}{2}.$$

<u>Pf:</u> For $Y \in \mathcal{H}_n^d$, we denote $\langle f, Y \rangle$ the Fourier coef. of f w.r. to Y.

By previous Theorem,

$$\langle f * g, Y \rangle = \frac{1}{\omega_d} \int_{S^{d-1}} \mathrm{proj}_n (f*g)(x)\, Y(x)\, d\sigma(x)$$

$$= \langle f, Y \rangle \frac{\omega_{d-1}}{\omega_d} \int_0^\pi g(\cos\theta) \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} (\sin\theta)^{d-2}\, d\theta,$$

while

$$\langle f * g, Y \rangle = \frac{\omega_{d-1}}{\omega_d} \int_0^\pi g(\cos\theta) \underbrace{\cancel{\frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)}}}_{\langle T_\theta f, Y \rangle} (\sin\theta)^{d-2}\, d\theta.$$

Since the above holds for a generic g (whenever the integrals make sense), this shows $\langle T_\theta f, Y\rangle = \langle f, Y\rangle \dfrac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)}$, which proves the claim. ▨

<u>Lemma</u>: For $f \in L^p(S^{d-1})$, $1 \le p < \infty$, or $f \in C_0(S^{d-1})$ and $p = \infty$,

$$\|T_\theta f\|_p \le \|f\|_p, \quad \text{and} \quad \lim_{\theta \to 0^+} \|T_\theta f - f\|_p = 0.$$

<u>Pf</u>: $f \in L^1(S^{d-1})$, $\lambda = \frac{d-2}{2}$:

$$\|T_\theta f\|_1 \le \frac{1}{\omega_d} \int_{S^{d-1}} T_\theta(|f|)\, d\sigma(x) = \mathrm{proj}_0(T_\theta |f|)$$

by previous Lemma

$$\frac{C_0^\lambda(\cos\theta)}{C_0^\lambda(1)} \mathrm{proj}_0(|f|) = \frac{1}{\omega_d} \int_{S^{d-1}} |f(x)|\, d\sigma(x) = \|f\|_1.$$

(We used positivity of T_θ in the first inequality.)
We have by definition $\|T_\theta f\|_\infty \le \|f\|_\infty$, and the (Riesz-Thorin) interpolation theorem implies $\|T_\theta f\|_p \le \|f\|_p$, $1 \le p \le \infty$. Furthermore, $\|T_\theta f - f\|_p \le 2\|f - P\|_p + \|T_\theta P - P\|_p$ $\forall$ polynomial P. By previous lemma

$$T_\theta P - P = \sum_{j=0}^{n} \left(\frac{C_j^\lambda(\cos\theta)}{C_j^\lambda(1)} - 1 \right) \mathrm{proj}_j P, \quad P \in \Pi_n(S^{d-1}),$$

so that $T_\theta P - P \to 0$ for $\theta \to 0^+$ and the convergence follows by density of polynomials. ▨

Fourier orthogonal expansion

The space of spherical harmonics has a close relationship to the approximation of the Hilbert space $L^2(S^{d-1})$.
Let us denote by S_n, $n \in \mathbb{N}$, the integral operator

$$(S_n f)(x) = (f * K_n)(x), \quad x \in S^{d-1},$$

where the kernel K_n satisfies $\left(\lambda = \frac{d-2}{2}\right)$

$$K_n(t) = \sum_{k=0}^{n} \frac{k+\lambda}{\lambda} C_k^{\lambda}(t) = \frac{(2\lambda+1)_n}{(\lambda+\frac{1}{2})_n} P_n^{(\lambda+\frac{1}{2}, \lambda-\frac{1}{2})}(t),$$

with $P_n^{(\alpha,\beta)}(t)$ the Jacobi polynomial of degree n and spectral parameters $\alpha, \beta \in \mathbb{C}$.

Theorem: The family of spherical harmonics is dense in $L^2(S^{d-1})$, and

$$L^2(S^{d-1}) = \sum_{n=0}^{\infty} \mathcal{H}_n^d, \quad \text{i.e.,} \quad f = \sum_{n=0}^{\infty} \operatorname{proj}_n f,$$

in the sense that

$$\lim_{n\to\infty} \| f - S_n f \|_2 = 0 \qquad \forall f \in L^2(S^{d-1}).$$

In particular, for $f \in L^2(S^{d-1})$ the Parseval identity holds,

$$\| f \|_2^2 = \sum_{n=0}^{\infty} \| \operatorname{proj}_n f \|_2^2 .$$

Just as in the case of classical Fourier series in several variables, $S_n f$ does not in general converge either pointwise or in L^p for $p \neq 2$.

Def: For $f \in L^1(S^{d-1})$, the Poisson integral of f is defined by

$$(P_r f)(\xi) := (f * P_r)(\xi), \quad \xi \in S^{d-1},$$

where the kernel $P_r(\langle x, -\rangle)$ is given by

$$P_r(t) := \frac{1-r^2}{(1-2rt+r^2)^{d/2}}, \quad 0<r<1.$$

Lemma: For $0<r<1$, the Poisson kernel satisfies the following properties:

1/ $\forall\, x, y \in S^{d-1}$: $P_r(\langle x,y\rangle) = \sum_{n=0}^{\infty} Z_n(x,y)\, r^n$,

2/ $P_r f = \sum_{n=0}^{\infty} r^n \operatorname{proj}_n f$,

3/ $P_r(\langle x,y\rangle) \geq 0$ and $\omega_d^{-1} \int_{S^{d-1}} P_r(\langle x,y\rangle)\, d\sigma(y) = 1$.

(25)

Pf: The proofs of 1/, 2/ are staigtforward, the integration term by term in 2/ (which follows by uniform convergence in r) gives 3/. ☒

<u>Theorem</u>: Let f be a continuous function on S^{d-1}. For $0 \leq r < 1$,

$$u(r\xi) := (P_r f)(\xi)$$

is a hamonic function in $x = r\xi$, and

$$\lim_{r \to 1^-} u(r\xi) = f(\xi) \quad \forall \xi \in S^{d-1}.$$

So this Theorem describes the converse to the restiction of a hamonic polynomial to S^{d-1}, and solves the Dirichlet problem $\Delta u = 0$ in the unit ball of S^{d-1} with the boundary conditions $u = f$ on S^{d-1}. Notice that for $f, g \in L^1(S^{d-1})$ and $\text{proj}_n f = \text{proj}_n g$ for all $n \in \mathbb{N}$, then $f = g$.