(1)

Classical series of Lie groups

realized as matrix groups. For $G = GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$, $n \in \mathbb{N}$,

$\mathfrak{g} = Mat(n, \mathbb{R})$ or $Mat(n, \mathbb{C})$ the Lie algebras

with $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $[A, B] = A \circ B - B \circ A$, $\exp(A) = e^A$, $Ad(g)B = ABA^{-1}$

for $A, B \in \mathfrak{g}$, $e^A, g \in G$.

These properties descend to any (Lie) subgroup resp. (Lie) subalgebra.

<u>Orthogonal groups</u>:

V ... n-dim $\mathbb{R}$-vector space, $\langle\ ,\ \rangle : V \times V \to V$

pos. def. bilinear form (inner product)

The orthogonal group of V:
(or, special-ortogonal)

$$O(V) := \{ A \in GL(V) \mid \langle Au, Av \rangle = \langle u, v \rangle \quad \forall\, u, v \in V \}$$

consisting of isometries of V

$$SO(V) = \{ A \in O(V) \mid \det A = 1 \}$$

consisting of special (= oriented) isometries of V.

Denoting $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ the rotation matrix by angle θ $\left(R(\theta) \in SO(2) \right)$, there $\exists$ ON-basis of V such that $A \in O(V)$ is $A = \mathrm{diag}\left(R(\theta_1), \dots, R(\theta_m), \pm 1, \dots, \pm 1 \right)$ if $n = 2m+2$ or $A = \mathrm{diag}\left(R(\theta_1), \dots, R(\theta_m), \pm 1 \right)$ if $n = 2m+1$.

The Lie algebras $\mathfrak{o}(V)$ and $\mathfrak{so}(V)$ are

$$\mathfrak{o}(V) \simeq \mathfrak{so}(V) \simeq \{ A \in \mathfrak{g} \mid \langle Au, v \rangle + \langle u, Av \rangle = 0 \quad \forall\, u, v \in V \},$$

is a skew adjoint endomorphism of V, as one can see by differentiating of $\langle e^{tA} u, e^{tA} v \rangle$ by $\frac{d}{dt}\Big|_{t=0}$.

For V of signature (p,q) and $\langle u,v\rangle = \sum_{i=1}^{p} a_i b_i - \sum_{i=p+1}^{q} a_i b_i$

in an ON-basis $u = \sum a_i u_i$, $v = \sum b_i u_i$,

$O(V) \simeq O(p,q) := \{A \in M(n,\mathbb{R}) \mid A^T I_{p,q} A = I_{p,q}\}$

with Lie algebra

$o(V) \simeq o(p,q) = \{A \in M(n,\mathbb{R}) \mid A^T I_{p,q} + I_{p,q} A = 0\}$,

$I_{p,q} = \mathrm{diag}(\underbrace{1,\dots,1}_{p}, \underbrace{-1,\dots,-1}_{q})$.

Let $\mathrm{Sym}(n,\mathbb{R})$ be the set of $n\times n$ real symmetric matrices and $\mathrm{Sym}^+(n,\mathbb{R}) \subseteq \mathrm{Sym}(n,\mathbb{R})$ be the positive definite ones.

<u>Lemma 1</u>: For any $A \in GL(n,\mathbb{R})$ there exists a unique decomposition $A = Re^S$, where $R \in O(n)$ and $S \in \mathrm{Sym}(n,\mathbb{R})$.

<u>Pf</u>: There is a unique decomposition $A = RL$, $R \in O(n)$ and L symmetric ($L = L^T$) and positive definite ($L > 0$).
$L > 0$ is characterized by $\langle Lu, u\rangle > 0$ for all $u \neq 0$, i.e. the eigenvalues of L are positive.
If true, then $A^T A = L^T \underbrace{R^T R}_{Id} L = L^2$;
$A^T A$ is symmetric and positive definite, since $\langle A^T A u, u\rangle = \langle Au, Au\rangle > 0$ for all $u \neq 0$,
hence $A^T A$ has a unique positive definite square root, $L := \sqrt{A^T A}$, and set $R := AL^{-1}$.
Then $R \in O(n)$ since $R^T R = L^{-1} A^T A L^{-1} = L^{-1} L^2 L^{-1}$
$= Id$ $\Rightarrow$ the existence & uniqueness.

(3) The second part of the proof is the existence of diffeomorphism

$$\begin{aligned} Sym(n,\mathbb{R}) &\rightarrow Sym^+(n,\mathbb{R}) \\ L &\mapsto e^L := \sum_{i=0}^{\infty} \frac{L^i}{i!} \end{aligned}$$

We have $e^L \in Sym^+(n,\mathbb{R})$ since $(e^L)^T = e^{(L^T)}$ and if $Lv = \lambda v$ for the eigenvector $v \in V$, $e^L v = e^\lambda v$. If $B \in Sym^+(n,\mathbb{R})$, there $\exists$ a basis $\{u_i\}$ of V: $Bu_i = \mu_i u_i$ and $\mu_i > 0$ for all $i = 1, .., \dim V$. Writing $\mu_i = e^{\lambda_i}$, we define A by $A u_i = \lambda_i u_i$ and $e^A = B \Rightarrow$ the map is surjective. Since the eigenvectors of A and e^A are the same, the map is injective and hence diffeomorphism (because $\underset{\overset{\parallel}{e}}{\exp}$ map is differentiable) $\square$

<u>Lemma 2</u>: $GL(n,\mathbb{R})$ is diffeomorphic to $O(n) \times \mathbb{R}^m$
$GL^+(n,\mathbb{R})$ —— " —— $SO(n) \times \mathbb{R}^m$, and
$SL(n,\mathbb{R})$ —— " —— $SO(n) \times \mathbb{R}^{m-1}$
with $m = \frac{n(n+1)}{2}$.

A general result going beyond Lemma 2 states

<u>Lemma 3</u>: If G is a connected Lie group, then there exists a connected compact subgroup $K \leq G$, which is maximal among compact subgroups and unique up to conjugacy.

(4) Unitary groups: V - n-dim $\mathbb{C}$-vector space,
$\langle\ ,\ \rangle$ - Hermitean positive definite (inner product)

$$\langle \lambda u, v\rangle = \bar{\lambda}\langle u, v\rangle$$
$$\langle u, \lambda v\rangle = \lambda\langle u, v\rangle$$
$$\langle u, v\rangle = \overline{\langle v, u\rangle} \qquad \langle u, u\rangle > 0 \text{ for } u \neq 0.$$

The complex analgue of ortogonal group is the unitary group,

$$U(V) := \{A \in GL(V) \mid \langle Av, Aw\rangle = \langle v, w\rangle \quad \forall v, w \in V\}$$

and its Lie algebra

$$u(V) = \{A \in Mat(n, \mathbb{C}) \mid \langle Av, w\rangle + \langle v, Aw\rangle = 0 \quad \forall v, w \in V\}$$

resp. the special unitary group

$$SU(V) := \{A \in U(V) \mid \det A = 1\},$$
$$su(V) = \{A \in u(V) \mid Tr(A) = 0\}.$$

$u(n)$ are complex matrices, but $u(n)$ is not a complex subspace of $gl(n, \mathbb{C})$, i.e. not a complex Lie algebra. But

$$u(n) \otimes_{\mathbb{R}} \mathbb{C} \simeq gl(n, \mathbb{C}), \quad su(n) \otimes_{\mathbb{R}} \mathbb{C} \simeq sl(n, \mathbb{C}),$$

because a complex matrix A can be written as $A = \underbrace{\frac{(A + A^*)}{2}}_{\text{Hermite}} + \underbrace{\frac{(A - A^*)}{2}}_{\text{skew-Herm}}$.

For P Hermitean is iP skew-Hermitean $\Rightarrow$ $A = P + iQ$ for $P, Q \in u(n)$.

A complex analogue of the polar decomposition is

Lemma 4: For any $A \in GL(n, \mathbb{C})$ there $\exists$ a unique decomposition $A = Re^S$ with $R \in U(n)$ and S Hermitean (i.e., $S = S^* = \overline{S^T}$.)

Hence, as a consequence we get

Lemma 5. The Lie group $GL(n,\mathbb{C})$ is diffeomorphic to $U(n)\times\mathbb{R}^m$, and $SL(n,\mathbb{C})$ to $SU(n)\times\mathbb{R}^{m-2}$ for $m=n^2$.

Thus $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$ are connected and non-compact.

Let us describe the relationship between $GL(n,\mathbb{C})$ and $GL(2n,\mathbb{R})$. The $\mathbb{R}$-linear isomorphism $\mathbb{C}^n \xrightarrow{\sim} \mathbb{R}^{2n}$:

$$\mathbb{R}^{2n}\simeq\mathbb{R}^n\oplus\mathbb{R}^n\to\mathbb{C}^n,$$
$$(u,v)\mapsto u+iv,$$

induces embedding (clearly injective)

$$GL(n,\mathbb{C})\hookrightarrow GL(2n,\mathbb{R}),$$
$$A+iB\longmapsto\begin{pmatrix}A & -B\\ B & A\end{pmatrix}$$

since $(A+iB)(u+iv)=(Au-Bv)+i(Av+Bu)$.

This also induces

$$U(n)\subseteq SO(2n),\qquad U(n)=O(2n)\cap GL(n,\mathbb{C})=$$
$$=SO(2n)\cap GL(n,\mathbb{C}).$$

The Hermitean inner product on $\mathbb{C}^n$ is the euclidean scalar product on $\mathbb{R}^{2n}\simeq\mathbb{C}^n$.

Symplectic groups & quaternions

Another important division algebra besides $\mathbb{R},\mathbb{C}$ is $\mathbb{H}$ (the quaternions). It is over $\mathbb{R}$ generated by $1,i,j,k$ such that $i^2=j^2=k^2=-1$, i,j,k anti-commutes with each other (and 1 commutes with all i,j,k), So $q=a+bi+cj+dk$ for $a,b,c,d\in\mathbb{R}$, $\bar q:=a-bi-cj-dk$ is its conjugate. The Euclidean inner product is

$$\langle q,r\rangle=\mathrm{Re}(\bar q r),\quad |q|^2=q\bar q=a^2+b^2+c^2+d^2$$
$\forall\, q,r\in\mathbb{H}$.

(6) Then $|qr| = |q||r| \Rightarrow S^3(1) = \{q \in \mathbb{H} \mid |q| = 1\} \subseteq \mathbb{R}^4$ is a Lie group (together with $S^1(1) := \{q \in \mathbb{C} \mid |q| = 1\}$ the only spheres with structure of Lie group.)

$V/\mathbb{H}$ a fin. dim. $\mathbb{H}$-vector space, $\mathbb{H}$ acting from the right.

$L : V/\mathbb{H} \to W/\mathbb{H}$ a $\mathbb{H}$-lin. map : $L(vq) = L(v)q$

$v \in V,\ q \in \mathbb{H}$.

While the composition of two $\mathbb{H}$-linear maps corresponds to the matrix multiplication (of linear maps in a basis), the formulas $\det(AB) = \det A \det B$, $Tr(AB) = Tr(BA)$ and $\overline{AB} = \overline{B}\overline{A}$ are no longer true. But since ~~the~~ multiplication of matrices corresponds to composition of $\mathbb{H}$-linear maps, $GL(n, \mathbb{H})$ is a Lie group.

A quaternionic inner product is a bilinear form $V/\mathbb{H} \times V/\mathbb{H} \to \mathbb{H}$: $\langle qu, v\rangle = \bar{q}\langle u, v\rangle$, $\langle u, qv\rangle = \langle u, v\rangle q$ and $\langle u, v\rangle = \overline{\langle v, u\rangle}$ as well as $\langle u, u\rangle > 0$ iff $u \neq 0$.

The symplectic group $Sp(V)$ of $V/\mathbb{H}$ is defined as the group of isometries of $\langle\ ,\ \rangle$, i.e. $A \in Sp(V)$ if $\langle Av, Aw\rangle = \langle v, w\rangle$ $\forall\, v, w \in V/\mathbb{H}$. It is isomorphic to ($\dim_{\mathbb{H}} V = n$)

$$Sp(V) = Sp(n) = \{A \in GL(n, \mathbb{H}) \mid A^*A = Id_V\},\quad A^* = \overline{A^T},$$

$$sp(V) = sp(n) = \{X \in gl(n, \mathbb{H}) \mid A + A^* = 0\},$$

$\dim_{\mathbb{R}} Sp(n) = 2n^2 + n$, and $Sp(n)$ is compact.

Because $\mathbb{H}^n \cong \mathbb{C}^{2n}$, $\mathbb{C}^{2n} \simeq \mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{H}^n$, $u, v \in \mathbb{C}^n$,

$(u, v) \mapsto u + jv$,

there is the Lie group embedding

$$GL(n,\mathbb{H}) \hookrightarrow GL(2n,\mathbb{C})$$

$$A + jB \longmapsto \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix},$$ since

$u, v \in \mathbb{C}^n$

$$(A+jB)(u+jv) = Au + jBjBv + Ajv + jBu =$$
$$= Au - \bar{B}v + j(\bar{A}v + Bu) \quad (\text{note } jA = \bar{A}j \text{ for } A \in gl(n,\mathbb{C}))$$

Consequently, $Sp(n) = U(2n) \cap GL(n,\mathbb{H}) = SU(2n) \cap GL(n,\mathbb{H})$, and the complex part of the quaternion inner product on $\mathbb{H}^n$ is the Hermitean inner product on $\mathbb{C}^{2n}$:

$$\langle u+jv, u'+jv' \rangle_{\mathbb{H}^n} = \langle u, u' \rangle_{\mathbb{C}^n} + \langle v, v' \rangle_{\mathbb{C}^n} + j \cdots$$

Under the identification $\mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{H}^n$, the right multiplication by j corresponds to $\bar{J}(u,v) = (-\bar{v}, \bar{u})$, $v, u \in \mathbb{C}^n$. So $\mathbb{C}$-lin. complex anti-linear endomorphism endomorphism of $\mathbb{H}^n$ is $\mathbb{H}$-linear iff it commutes with $\bar{J}$:

$$GL(n,\mathbb{H}) = \{A \in GL(2n,\mathbb{C}) \mid A\bar{J} = \bar{J}A\},$$
$$Sp(n) = \{A \in U(2n) \mid A\bar{J} = \bar{J}A\},$$ and

analogously for $SL(n,\mathbb{H})$. Then

<u>Lemma 5</u>: For $A \in GL(n,\mathbb{H})$, there exists a unique decomposition $A = Re^S$, with $R \in Sp(n)$ and $S = S^* = \overline{S^T}$.

Thus $GL(n,\mathbb{H}) \simeq Sp(n) \times \mathbb{R}^m$, $m = 2n^2 - n$.

(8) Symplectic groups - non-compact real form

V - fin. dim. $\mathbb{R},\mathbb{C}$ - vector space, $\omega: V\times V \to \mathbb{R},\mathbb{C}$ non-degenerate skew-symmetric bilinear form ($\Rightarrow \dim V$ is even)

The (non-compact form of the) symplectic group is

$$Sp(V,\omega) := \{A\in GL(V) \mid \omega(Av,Aw)=\omega(v,w)\ \forall v,w\in V\},$$

$$sp(V,\omega) := \{A\in gl(V) \mid \omega(Av,w)+\omega(v,Aw)=0 \ \text{—"—}\}.$$

Taking a symplectic basis $e_1,\dots,e_n, f_1,\dots,f_n$: $\omega(e_i,e_j)=\omega(f_i,f_j)$, $=0$, $\omega(e_i,f_j)=\delta_{ij}$,

$$\omega=-\sum_{j=1}^n f_j\wedge e_j;$$

The matrix of ω in this basis is $J=\begin{pmatrix}0 & Id_n\\ -Id_n & 0\end{pmatrix}$, $\omega(u,v)=u^TJv$ $\forall u,v\in V$,

and $Sp(\mathbb{R}^{2n},\omega)$ can be identified with a matrix group

$$Sp(n,\mathbb{R})=\{A\in GL(2n,\mathbb{R}) \mid A^TJA=J\};$$

there is an embedding $GL(n,\mathbb{R})\hookrightarrow Sp(n,\mathbb{R})$

$$B\mapsto \begin{pmatrix}B & 0\\ 0 & (B^T)^{-1}\end{pmatrix}$$

and $Sp(n,\mathbb{R})$ is non-compact. The Lie algebra of $Sp(n,\mathbb{R})$ has block decomposition

$$sp(n,\mathbb{R})=\{X\in gl(2n,\mathbb{R}) \mid XJ+JX^T=0\}$$

$$=\left\{X=\begin{pmatrix}B & S_1\\ S_2 & -B^T\end{pmatrix} \;\middle|\; B\in gl(n,\mathbb{R}),\ S_i\in Sym(n,\mathbb{R}),\ i=1,2\right\}.$$

($\Rightarrow \dim Sp(n,\mathbb{R})=2n^2+n$.)

⑨ $U(n) \subseteq Sp(n,\mathbb{R})$, because $(\mathbb{R}^{2n} \simeq \mathbb{C}^n)$

$$\langle u+iv, u'+iv' \rangle_{\mathbb{C}^n} = \langle u,u' \rangle_{\mathbb{R}^n} + \langle v,v' \rangle_{\mathbb{R}^n} + i\left(\langle v,v' \rangle_{\mathbb{R}^n} - \langle v,u' \rangle_{\mathbb{R}^n} \right)$$
$$= |\langle u,v \rangle|^2 + i\,\omega(u,v).$$

~~Thus~~ and so $A \in U(n)$ iff it preserves both $\langle\,,\rangle_{\mathbb{R}^n}$ and ω:

$U(n) = O(2n) \cap Sp(n,\mathbb{R}) = SO(2n) \cap Sp(n,\mathbb{R})$,

$Sp(n) = U(2n) \cap Sp(n,\mathbb{C}) = SU(2n) \cap Sp(n,\mathbb{C})$.

Lemma 6: 1/ For $A \in Sp(n,\mathbb{R})$, there $\exists$ a unique decomposition $A = Re^S$ with $R \in U(n)$ and $S \in sp(n,\mathbb{R}) \cap Sym(2n,\mathbb{R})$.

2/ —"— $Sp(n,\mathbb{C})$, —"—

—"— $R \in Sp(n)$ and $S \in sp(n,\mathbb{C}) \cap Sym(2n,\mathbb{C})$.

(Polar decomposition for (non-compact) symplectic groups.

$\boxed{\text{Compact Lie groups \& algebras}}$

Def 7: A Lie algebra is termed compact if it is the Lie algebra of a compact Lie group.

Prop 8: 1/ If $\mathfrak{g}$ is a compact Lie algebra, then there exists an inner product on $\mathfrak{g}$ such that $ad_X : \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric $\forall X \in \mathfrak{g}$.

2/ If G is a compact Lie group, then there $\exists$ a biinvariant metric on G, i.e. a Riemannian metric such that L_g, R_g (left/right action by $g \in G$) act by isometries for all $g \in G$.

Pf: G ... compact Lie group, $\mathfrak{g}$ ... Lie algebra of G

$\langle\ ,\ \rangle_0 : \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ ... an inner product on $\mathfrak{g}$,

define a new inner product by

$$\langle X, Y\rangle := \int_G \langle Ad(g)X,\ Ad(g)Y\rangle_0\ \omega,$$

where ω is a G-biinvariant volume form (i.e., $L_g^*\omega = \omega = R_g^*\omega$.) Its existence follows by choosing a volume form ω_0 on $\mathfrak{g}$ and define it on G by $\omega|_{g\in G} := L_g^*\omega_0$. Define a function $f: G\to\mathbb{R}$ by $R_g^*\omega = f(g)\omega$; f is constant since both ω and $R_g^*\omega$ are left invariant. We have $f(gh) = f(g)f(h)$, thus $f(G)$ is a compact subgroup of $\mathbb{R}^*$. This implies $f(g) = 1$ $\forall g\in G$, hence ω is right invariant. Then $Ad(h)$, $h\in G$, is an isometry of $\langle\ ,\ \rangle$:

$$\langle Ad(h)X, Ad(h)Y\rangle = \int_G \langle Ad(hg)X,\ Ad(gh)Y\rangle_0\ \omega$$

$$= \int_G \langle Ad(g)X, Ad(g)Y\rangle_0 \left(R^*_{h^{-1}}\omega\right)$$

$$= \langle X, Y\rangle.$$

Since $Ad(G)$ acts by isometries, $dAd = ad$, $ad(X)$ is skew-symmetric $\forall X\in\mathfrak{g}$.

The proof of 2/ goes along the same lines as in 1/. $\square$

There are standard results on the relation of Killing-Cartan form for $\mathfrak{g}$ and structural properties of $\mathfrak{g}$ (simplicity, reductiveness, nilpotence, solvability, etc.) In particular, a compact Lie group with finite center is semisimple.

<u>Prop (Weyl) 9:</u> If G is a compact Lie group with finite center, then $\pi_1(G)$ is finite and hence $\forall$ Lie group with Lie algebra $\mathfrak{g}$ is compact.

Pf: The geometrical proof is based on the Bonnet-Myer's theorem: for (M,g) a complete Riemannian manifold whose Ricci curvature satisfies $Ric(X_p, X_p) \geq (m-1)c$, where $\dim M = m$ and $c \in \mathbb{R}_+$. Then M is compact with bounded diameter $diam(M) = \sup_{p,q \in M} dist(p,q) \leq \frac{\pi}{\sqrt{c}}$, and consequently, $\pi_1(M)$ is finite.

We take $(G, -B = -\langle\,,\rangle)$ as a Riemannian manifold, where L_g, R_g acts by isometries, and show $Ric \geq \frac{1}{4}$.

The formulas for the Levi-Civita and the curvature tensor are (see <u>Exercises</u>)

$$\nabla_X Y = \frac{1}{2}[X,Y], \quad R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$$

for $X,Y,Z \in \mathfrak{g}$. For example, for the curvature tensor we use the Jacobi identity:

$$\begin{aligned} R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] \\ &\quad - \frac{1}{2}[[X,Y],Z] \\ &= -\frac{1}{4}[Z,[X,Y]] - \frac{1}{2}[[X,Y],Z] \\ &= -\frac{1}{4}[[X,Y],Z], \end{aligned}$$

and so the Ricci tensor is

$$\begin{aligned} Ric(X,Y) &= Tr(Z \to R(Z,X)X) = \\ &= -\frac{1}{4} Tr(Z \to ad_X\, ad_X Z) = -\frac{1}{4} B(X,X). \end{aligned}$$

The proof is complete.

Symmetric spaces

The intro will be geometric, so familiarity with Riemannian geom. (Levi-Civita connection, geodesic, exponential maps, Jacobi fields, isometry, curvature) are assumed.

We denote $B_r(p) := \{q \in M \mid d(p,q) < r\}$ a ball of radius r. It is called a normal ball if it is diffeomorphic image of a ball in the tangent space. We denote $I(M)$ the isometry group of M (a Riemannian manifold), $I_0(M)$ its identity component.

Let us mention a non-trivial result

<u>Theorem 1</u> : Assume (M, g) is a complete Riemannian manifold. Then

1/ $I(M)$ is a Lie group and the stabilizer $I_p(M)$ is compact for all $p \in M$.

2/ If M is compact, then $I(M)$ is compact.

G acts on M : $G_p = \{g \in G \mid gp = p\}$ is the stabilizer (or, isotropy group), of $p \in M$. If G acts transitively on M, $H = G_p$, then

$$\left.\begin{array}{l} G/H \to M \\ gH \mapsto gp \end{array}\right\}$$ is a diffeomorphism. On G/H,

$$L_g : G/H \to G/H$$
$$L_g(aH) = g(aH) = (ga)H$$

so that $L_g(aH) = g(ap) = (ga)p$ (the action of G on M.)

There is also induced action of H on T_pM,

$$h \to d(L_h)_p = (L_h)_*|_p \in \mathrm{End}(T_pM)$$

G acts by isometries on M, the isotropy action $d(L_h)$ at p is effective:

$$d(L_h)|_p = Id_{T_pM} \Rightarrow L_h = Id \quad \text{for } h \in H.$$

(the reason is that isometries are determined by their derivative at one point.)

Def 2 : let $(M, \langle , \rangle)$ be a Riemannian manifold.

1/ M is called (global) symmetric if for $\forall p \in M$ there $\exists$ an isometry $s_p : M \to M$ with $s_p(p) = p$ and $d(s_p)_p = -Id$.

2/ M is called locally symmetric if for $\forall p \in M$ there $\exists$ a radius $r \in \mathbb{R}_+$ and an isometry $s_p : B_r(p) \to B_r(p)$ with $s_p(p) = p$, $d(s_p)_p = -Id$.

$\left(\overset{\text{Def 2}}{\Rightarrow} \right)$

Lemma 3 : Let $(M, \langle , \rangle)$ be a symmetric space.

1/ If γ is a geodesic with $\gamma(0) = p$, then $\sigma_p(\gamma(t)) = \gamma(-t)$,

2/ M is homogeneous,

3/ M is complete,

4/ If M is homogeneous and there exists a symmetry at one point, M is symmetric.

Pf: 1/ Any isometry transforms geodesic to geodesic, and since the geodesic $c(t) = s_p(\gamma(t))$ satisfies $c'(0) = (d s_p)_p (\gamma'(0)) = -\gamma'(0)$, the uniqueness of geodesics implies $c(t) = \gamma(-t)$.

3/ By definition, this means that geodesics $\gamma : \cancel{\mathbb{R}}\ (-\epsilon, \epsilon) \to M$ are defined for all $t \in \mathbb{R}$. Assume M is not complete, and $\gamma : [0, t_0) \to M$ the maximal domain of definition of γ. The the application $s_{\gamma(t_0 - \epsilon)}$ to γ (and by 1/) allows to extend the domain to $[0, 2t_0 - 2\epsilon)$ and clearly $2t_0 - 2\epsilon > t_0$ for $\epsilon > 0$ sufficiently small.

2/ Let $p, q \in M$ be two points, so we need to show $\exists$ of an isometry f : $f(p) = q$. By completeness in 3/, Hopf - Rinow theorem $\Rightarrow \exists\ \gamma : [0, 1] \to M$, $\gamma(0) = p$, $\gamma(1) = q$. Thus by 1/,

$s_{\gamma(\frac{1}{2})}(\gamma(0)) = \gamma(1)$.

 4/ If s_p is given at $p \in M$, we get $s_{gp} = L_g \circ s_p \circ L_{g^{-1}}$. $\square$

On a general manifold M, s_p is defined on a normal ball in T_pM as (the exponential of an open subset of)

$s_p(\exp(tX)) = \exp(-tX)$, $X \in T_pM$, $t \in (-\epsilon, \epsilon)$ in $\mathbb{R}$.

Then we have to impose a strong condition on s_p to be an isometry, and moreover one has to globalize this statement (two geodesics between p, q going in the opposite direction have to end at the same point.)

<u>Example</u>: Manifolds of constant curvature are locally symmetric, and simply connected ones are (globally) symmetric. For $\mathbb{R}^n$ with a flat metric the reflection around p is given by $s_p(p+v) = p - v$. For a sphere of radius 1, the reflection along the line $\mathbb{R}p$:

$s_p(v) = -v + 2\langle v, p\rangle p$, $\|p\| = 1 = \|v\|$, is an isometry preserving $p \in S^{n-1}$ and the tangent map $ds_p = s_p$ acts linearly on $T_pS^{n-1} = \{v \in \mathbb{R}^n \mid \langle v, p\rangle = 0\}$, $ds_p = -Id_{T_pS^{n-1}}$.

For the hyperbolic space, we use the Lorentz space model

$$\{v \in \mathbb{R}^{n+1} \mid \langle v, v\rangle = -1, \ x_{n+1} > 0\}$$

with inner product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i - x_{n+1}y_{n+1} \qquad x, y \in \mathbb{R}^{n+1}$$

A compact Lie group G with a bi-invariant inner product is a symmetric space. We claim $s_e(g) = g^{-1}$ is the required involution at $e \in G$. Because $s_e(e) = e$ and $s_p(\exp(tX)) = \exp(-tX)$ $\forall X \in \mathfrak{g}$, we also have $(ds_e)_e = -Id$.

This is an isometry: 1/ true for $(ds_e)_e$,

2/ $s_e \circ L_g = R_{g^{-1}} \circ s_e \Rightarrow (ds_e)_g \circ d(L_g)_e = d(R_{g^{-1}})_e \circ d(s_e)_e$,

and because L_g, R_g are isometries $\Rightarrow d(s_e)_g$ is isometry as well.

The Grassmanians of k-planes : $G_k(\mathbb{R}^n)$, $G_k(\mathbb{C}^n)$, $G_k(\mathbb{H}^n)$.

Consider the vector space $V := \{P \in Mat(n,n,\mathbb{R}) \mid P = P^T\}$, equipped with the ~~bilinear~~ inner product $\langle P, Q\rangle := Tr(PQ)$.

There is an embedding
$$G_k(\mathbb{R}^n) \hookrightarrow Mat(n,n,\mathbb{R})$$
$$E \mapsto P = P_E \in V \text{ (ON-projector onto } E \subseteq \mathbb{R}^n)$$
$$P^2 = P, \; Im(P) = E.$$

Conversely, any $P \in V$, $P^2 = P$, is an ON-projection onto $Im(P)$, since $\mathbb{R}^n$ is an ON-sum of its $0, 1$-eigenspaces; and the fact that $\dim E = k$, we require $Tr(P) = k$. So
$$G_k(\mathbb{R}^n) = \{P \in V \mid P^2 = P, \; Tr\, P = k\}.$$
The group $O(n)$ acts transitively on $G_k(\mathbb{R}^n)$ by $P \mapsto APA^T = APA^{-1}$
$$Im(P) \mapsto A\, Im(P)$$
The isotropy group at $E_0 = \mathbb{C}\langle e_1, \dots, e_k\rangle$ is equal to $O(k) \times O(n-k) = \begin{pmatrix} O(k) & 0 \\ 0 & O(n-k) \end{pmatrix}$,
so
$$G_k(\mathbb{R}^n) = O(n)/O(k) \times O(n-k).$$

In particular, $\dim_{\mathbb{R}} G_k(\mathbb{R}^n) = k(n-k)$, and $G_k(\mathbb{R}^n)$ is an embedded submanifold of V (since it is the orbit of $O(n)$).

The inner product on $V \Rightarrow$ Riemann. metric on $G_k(\mathbb{R}^n)$. Let $r_E : \mathbb{R}^n \to \mathbb{R}^n$ be the reflection in E: $r_E|_E = Id$, $r_E|_{E^\perp} = -Id$.

We claim $s_E(Q) = r_E Q r_E$ is the required symmetry.

Because $r_E^T = r_E = r_E^{-1}$, the map is the base change given by r_E.

Observe that s_E takes V to V, $d(s_E) = s_E$ preserves the inner product
$$\langle d(s_E)(P), d(s_E)(Q)\rangle = Tr(r_E P \underbrace{r_E r_E}_{Id_{\mathbb{R}^n}} Q r_E) = Tr(PQ)$$
$$= \langle P, Q\rangle.$$

Moreover, s_E takes projectors to projectors, and preserves the condition

(16) A] $s_E(P_E) = P_E$ (since s_E takes P_E to P_E and is 0 on $E^\perp$)

B] By diff. the curve $P_0 + tQ + \dots \in G_k(\mathbb{R}^n)$, i.e.

$$(P_0 + tQ + \dots)^2 = P_E + tQ + \dots, \text{ we see}$$

$$T_E(G_k(\mathbb{R}^n)) = \{ Q \in V \mid P_E Q + Q P_E = Q,\ Tr(Q) = 0 \}.$$

If $Q \in T_E(G_k(\mathbb{R}^n))$, $v \in E$, then

$$P_E Q(v) + Q P_E(v) = Q(v) \implies P_E Q(v) = 0,$$

i.e. $Q(v) \in E^\perp$ and so $(r_E Q r_E)(v) = (r_E Q)(v) = -v$.

Similarly, for $v \in E^\perp$ the formula $P_E Q(v) + Q P_E(v) = Q(v)$ implies $P_E Q(v) = Q(v) \implies Q(v) \in E$ and so

$$(r_E Q r_E)(v) = (r_E Q)(-v) = -v \implies (d s_E)_E = -Id_n.$$

Finally, since $\langle P, P \rangle = Tr(P^2) = Tr(P) = k$, the image of embedded $G_k(\mathbb{R}^n)$ lies in a round sphere of dimension $\frac{n(n+1)}{2} - 2$ (corresponds to Veronese embedding.)

Analogously, $G_k(\mathbb{C}^n) = U(n) / (U(k) \times U(n-k)) = SU(n) / S(U(k) \times U(n-k))$

$$G_k(\mathbb{H}^n) = Sp(n) / (Sp(k) \times Sp(n-k)).$$

The case $k=1$ corresponds to $\mathbb{RP}^n(S^n)$, $\mathbb{CP}^n$, $\mathbb{HP}^n$.

Let us define the concept of transvection; for $\gamma: \mathbb{R} \to M$ a geodesic, $T_t := s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}$.

<u>Prop 4</u>: Let M be a symmetric space, γ a geodesic on M.

1/ T_t translates the geodesic: $T_t(\gamma(s)) = \gamma(t+s)$,

2/ $(dT_t)_{\gamma(s)}$ is given by parallel transport from $\gamma(s)$ to $\gamma(s+t)$ along γ.

3/ T_t is a 1-parameter group of isometries:

Pf: 1/ $s_{\gamma(r)}$ takes $\gamma(t)$ to $\gamma(2r-t)$, so

$$T_t(\gamma(s)) = s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}(\gamma(s)) = s_{\gamma(\frac{t}{2})}(\gamma(-s)) = \gamma(s+t).$$

2/ Since s_* acts by isometries, it takes // vector to // vector fields.
"parallel

Let X be a // vector field along the geodesic γ; then $(s_{\gamma(0)})_* X$ is //, and since $(d\, s_{\gamma(0)})_{\gamma(0)}(X) = -X$, we have $(s_{\gamma(0)})_*(X) = -X$ $\forall t$. Applying the symmetry s_* twice, the sign is positive and by 1/ $(dT_t)_{\gamma(s)}(X(\gamma(s))) = X(\gamma(s+t))$, and in particular $(dT_t)_{\gamma(s)}$ is // - translation.

3/ A basic property of isometries is that they are determined by their value and the derivative at a point. We have

$$T_{t+s}(\gamma(0)) = \gamma(t+s) = (T_t \circ T_s)(\gamma(0)),$$

and by 2/ $(dT_{t+s})_{\gamma(0)}$ is given by parallel translation from $\gamma(0)$ to $\gamma(t+s)$. But $d(T_t \circ T_s)_{\gamma(0)} = d(T_t)_{\gamma(s)} \circ (dT_s)_{\gamma(0)}$, so we first translate from $\gamma(0)$ to $\gamma(s)$ and then from $\gamma(s)$ to $\gamma(t+s)$. This is equal to LHS, hence T_{s+t} and $T_t \circ T_s$ agree.

Corollary: Let M be a symmetric space.

1/ Geodesics in M are images of 1-parameter groups of isometries.

2/ $I_0(M)$ acts transitively on M.

1/ is very special and is not true for $\forall$ homogeneous spaces.

2/ $\Leftarrow$ $T_t \in I_0(M)$ and since $\forall\, p, q \in M$ can be connected by geodesic.

18

$p \in M$ (a point on a Riem. man.), $Hol_p := \{ P_\gamma \mid \gamma(0) = \gamma(1) = p \}$

the holonomy group at $p \in M$, P_γ ... the operator of $\parallel$ transport along piece-wise curve γ

Hol_p° = identity component of Hol_p

$q \in M$ another point, γ a path from p to q, then $Hol_q = P_\gamma Hol_p P_\gamma^{-1}$ and so $Hol_p \simeq Hol_q$ (though non-canonically)

Theorem 5 Assume M is complete. Then

1/ Hol is a Lie group and Hol° is compact,

2/ Hol° is given by $\parallel$ transport along 0-homotopy curves,

3/ If M is simply connected, Hol_p is connected.

There is a surjective homomorphism $\pi_1(M) \twoheadrightarrow Hol_p / Hol_p^\circ$ (well-defined by 2/)
$[\gamma] \mapsto P_\gamma$

Since $G = I(M)$ acts transitively on M, $M = G/K$ for $K = G_p$ the isotropy group at p. By definition, $Hol_p \leq O(T_pM)$, which is just K $(= O(T_pM))$ via the isotropy representation.

Corollary: If $M = G/K$ is a symmetric space, $G = I(M)$. Then $Hol_p \leq K$.

This works for locally symmetric spaces as well. Important property is that symmetric spaces have small holonomy groups. Generically, $Hol_p = O(n)$ on a manifold of dimension n. For symmetric spaces, $Hol_p^\circ = K_0$ in most cases.

M is called decomposable if M is a product $M = M_1 \times M_2$, and the metric is of product type. Otherwise, if that is not possible, M is called indecomposable.

Theorem 6: Let M be a simply connected Riemannian manifold, $p \in M$ and Hol_p the holonomy group. Let

$T_pM \simeq V_0 \oplus V_1 \oplus \dots \oplus V_k$ be a decomposition into Hol_p-irreducible subspaces with $V_0 = \{v \in T_pM \mid hv = v \ \forall h \in Hol_p\}$. Then M is a Riemannian product $M = M_0 \times \dots \times M_k$, where M_0 is isometric to flat $\mathbb{R}^n$. If $p = (p_0, p_1, \dots, p_k)$, then $T_{p_i} M_i \simeq V_i$ and M_i is indecomposable if $i > 0$. The decomposition is unique and $Hol_p \simeq Hol_{p_1} \times \dots \times Hol_{p_k}$, with Hol_{p_i} the holonomy of M_i at p_i, and $I_0(M) \simeq I_0(M_0) \times \dots \times I_0(M_k)$.

Since $Hol_p \subseteq K$ for symmetric space $\Rightarrow$ if $M = G/K$ is a simply connected symmetric space and M is indecomposable, then K acts irreducibly on the tangent space.

Def 7: A symmetric space G/K, $G = I(M)$ and $K = G_p$, is called irreducible if K_0 acts irreducibly on T_pM, and reducible otherwise.

Notice we do not require K acts irreducibly; otherwise $S^2(1) \times S^2(1)$ would be irreducible symmetric space.

sphere of radius 1

Irreducible $\not\Rightarrow$ indecomposable (for the flat $\mathbb{R}^n$ we have $K = O(n)$ acting irreducibly, but this is the only exception.)

If $M = G/K$ is a simply connected symmetric space, then M is isometric to $M_1 \times \dots \times M_k$ with M_i irreducible symmetric spaces. A simply connected symmetric space $M = G/K$ with G simple is irreducible.

Now a geometrical consequence:

Lemma 8: Let B_1, B_2 be two symmetric bilinear forms on a vector space V, and assume B_1 is positive definite. If a compact Lie group K acts irreducibly on V such that B_1, B_2 are invariant under K. Then $B_2 = \lambda B_1$ for some constant $\lambda \in \mathbb{R}$.

Pf: B_1 is non-degenerate $\Rightarrow \exists\ L \in \mathrm{End}(V)$: $B_2(u,v) = B_1(Lu, v)$ for all $u, v \in V$. Since K acts by isometries, $\forall k \in K$:

$$B_1(kLu, v) = B_1(Lu, k^{-1}v) = B_2(u, k^{-1}v) = B_2(ku, v) = B_1(Lku, kv)$$

$$\Rightarrow Lk = kL \quad \forall k \in K.$$

In addition,

$$B_1(Lu, v) = B_2(u,v) = B_2(v,u) = B_1(Lv, u) = B_1(u, Lv)$$

$\Rightarrow L$ is symmetric for B_1 and so its eigenvalues are real.

If $E \subseteq V$ is an eigenspace of L with eigenvalue λ, then $kL = Lk \Rightarrow E$ is K-invariant. Since K acts irreducibly on V, $E = V$ or $E = \{0\}$. Thus $L = \lambda \, \mathrm{Id}_V$ for some $\lambda \in \mathbb{R} \Rightarrow B_2 = \lambda B_1$. Notice that $\lambda \neq 0$ otherwise $B_2 = 0$. ∎

A consequence of Lemma 8 is

Corollary: An irreducible symmetric space is Einstein, i.e., $\mathrm{Ric} = \lambda \langle\, ,\, \rangle$ for some $\lambda \in \mathbb{R}$. Furthermore, the metric is unique up to a multiple.

Firstly, the metric is unique up to a multiple ($B_1 = g_1$, $B_2 = g_2$). Since isometries preserve the curvature, Ric is also symmetric bilin. form invariant under K, hence the Colloray.

(21) Importance of holonomy groups: holonomy principle. If S_p is a tensor on T_pM invariant under Hol_p, we can define a tensor S on M by $\parallel$ transport of S_p along any path. This is independent of any path, since $\parallel$ transport preserves S_p. Then S is smooth and $\parallel$ $\cdot$ $\nabla S = 0$, since $\nabla_X S = \frac{d}{dt}\Big|_{t=0} P_\gamma^*(S(\gamma(t)))$ with γ a path $\gamma'(0) = X$, $\gamma(0) = p$. E.g., if the repr. of Hol_p is complex, the complex structure extends to $\parallel$ complex structure on M (it is integrable and so the metric is Kähler.)

If R is the curvature tensor, (∇R) is defined as

$$(\nabla_X R)(Y,Z)W = \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y,\nabla_X Z)W - R(Y,Z)\nabla_X W.$$

So $\nabla R = 0$ iff $\forall$ $\parallel$ $Y, Z, W \in TM$ along a geodesic γ, $R(Y,Z)W$ is $\parallel$ along γ as well.

Lemma 9: ~~This~~ Let M be a Riemannian manifold.

1/ M is locally symmetric iff $\nabla R = 0$.

2/ If M is locally symmetric and simply connected, M is globally symmetric.

3/ Let M_1, M_2 be two symmetric spaces with M_1 simply connected and $p_i \in M_i$ $(i = 1,2)$ fixed. Given an isometry $A: T_{p_1}M_1 \to T_{p_2}M_2$ with $A^*R_2 = R_1$, there is an isometry (covering) $f: M_1 \to M_2$ with $(df)_{p_1} = A$.

Pf: 1/ M is locally symmetric, s_p the symmetry at $p \in M$.

~~Sx~~ Then for $(ds_p)_p$ we have

$$-(\nabla_X R)(Y, Z)W = (ds_p)_p\big((\nabla_X R)(Y,Z)W\big)$$
$$= \big(\nabla_{(ds_p)_p X} R\big)\big((ds_p)_p Y, (ds_p)_p Z\big)\big((ds_p)_p W\big)$$
$$= (\nabla_X R)(Y, Z)(W)$$

since $(ds_p)_p$ preserves the curvature (it is an isometry)

The opposite implication as well as 2/, 3/ will not be proved. □

$\boxed{\text{Cartan involutions}}$

$M = G/H$, $H = G_{p_0}$, $p_0 \in M$ a homogeneous space

$(dL_h)_{p_0} : T_{p_0}M \to T_{p_0}M$ ... the isotropy representation

The long exact sequence of homotopy groups associated to SES $H \to G \to G/H$ $\Rightarrow$ K is connected iff M is simply connected and G connected.

<u>Proposition 10</u>: Let $M = G/K$, $G = I_0(M)$, $K = G_p$ be a symmetric space.

1/ The symmetry s_p gives rise to an involutive automorphism
$$\sigma = \sigma_p : G \to G, \quad g \mapsto s_p g s_p.$$

2/ If $G^\sigma = \{g \in G \mid \sigma(g) = g\}$ is the fixed point set of σ, then $G_0^\sigma \subset K \subset G^\sigma$.

<u>Pf</u>: 1/ $s_p \in I(M)$ and $s_p^{-1} = s_p$, σ is a conjugation and hence an automorphism preserving $I_0(M)$. Since s_p is involutive, so is σ.

2/

2/ First $K \subseteq G^\sigma$: for $h \in K$, $\sigma(h)p = s_p h s_p \cdot p = h p$.
Moreover, $d\sigma(h)_p = (ds_p)_p (dh)_p (ds_p)_p = (dh)_p$. Since isometries are determined by their derivatives, $\sigma(h) = h$ and so $K \subseteq G^\sigma$.

Secondly $G_0^\sigma \subseteq K$: let $\exp(tX) \subseteq G_0^\sigma$ be a 1-parameter subgroup. Since $\sigma(\exp(tX)) = \exp(tX)$, it follows $s_p \exp(tX) s_p = \exp(tX)$, so $s_p (\exp tX)p = \exp(tX)\cdot p$.
But s_p preserves only p in a normal neighbourhood of p since $(ds_p)_p = -Id$, and so $\exp tX \cdot p = p$ for all $t \in \mathbb{R}$.
Thus $\exp(tX) \in K$ and since G_0^σ is generated by a neighborhood of $e \in G$, the claim follows. $\square$

σ = Cartan involution of the symmetric space.

Riemannian homogeneous spaces: G acts by isometries on Riem. man. M,
$X \in \mathfrak{g} \mapsto X^* \in TM = \mathfrak{X}(M) = Vect(M)$ by

$$X^*(p) := \frac{d}{dt}\Big|_{t=0} (\exp(tX)\cdot p).$$

Since the flow along these vector fields acts by isometries, X^* are Killing vector fields.

G/H, $H = G_{p_0}$, is a Riemann homog. space if L_g is an isometry $\forall g \in G$. The hom. space is reductive if $\exists\, \mathfrak{p} \subseteq \mathfrak{g}$: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and $Ad_H(\mathfrak{p}) \subset \mathfrak{p}$. We identify $\mathfrak{p} \simeq T_{p_0}M$

$$X \mapsto X^*(p_0),$$

$X^*(p_0) = 0$ iff $X \in \mathfrak{h}$.

Lemma 11: G/H a homogeneous space, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ a reductive decomposition.

1/ If $\sigma \in Aut(G)$ with $\sigma(H) = H$, then in the identification $\mathfrak{p} \xrightarrow{\sim} T_{p_0}M$, $X \mapsto X^*(p_0)$

we have $d\sigma = d\bar{\sigma}_{p_0}$, where $\bar{\sigma} : G/H \to G/H$ is defined by $\bar{\sigma}(gH) = \sigma(g)H$.

2/ In the identification $\mathfrak{p} \xrightarrow{\sim} T_{p_0}M$, the isotropy representation of G/H is given by $(dL_h)_{p_0} = Ad(h)\big|_{\mathfrak{p}}$.

3/ Any Ad_H-invariant inner product on $\mathfrak{p}$ can be uniquely extended to a homogeneous metric on G/H.

4/ A homogeneous metric on G/H, restricted to $T_{p_0}M$, induces an inner product on $\mathfrak{p}$ invariant under Ad_H.

Pf: 1/ $H = G_0$, then

$$(d\sigma(X))^*(p_0) = \frac{d}{dt}\Big|_{t=0} \left(\exp(t\, d\sigma(X)) \cdot p_0\right) =$$

$$= \frac{d}{dt}\Big|_{t=0} \left(\sigma(\exp(tX)) \cdot p_0\right)$$

$$= \frac{d}{dt}\Big|_{t=0} \bar{\sigma}\left(\exp(tX)H\right)$$

$$= d\bar{\sigma}_{p_0}(X^*(p_0))$$

2/ 1/ => 2/ by $\sigma := C_h$, the conjugation by $h \in H$, and observing $\bar{\sigma}(gH) = hgh^{-1}H = hgH = L_h(gH)$.

3/ 2/ => 3/

4/ Inner product on $\mathfrak{p}$ => inner product on $T_{p_0}M$ => inner product at gp_0: $(L_g)_{p_0} : T_{p_0}M \to T_{gp_0}M$.

Remark: If there is no reductive decomposition on G/H, 1/, 2/ in the previous lemma hold with $T_{p_0}M \simeq \mathfrak{g}/\mathfrak{h}$.

25

A Riemannian symmetric space G/H is always reductive.

Proposition 12 (Natural reductive decomposition for symmetric spaces = Cartan decomposition)

$M = G/K$, $G_0^\sigma \leq K < G^\sigma$ for an involutive automorph. of G.

Let $\mathfrak{k}, \mathfrak{p}$ be the $+1$ and -1 eigenspaces of $d\sigma$, respectively. Then $\mathfrak{k}$ is the Lie algebra of K and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k},$$

and $Ad_K(\mathfrak{p}) \subseteq \mathfrak{p}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$.

Pf: Since σ respects Lie bracket on $\mathfrak{g}$, $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$ follows. The last bracket $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$ follows as well, but $Ad_K(\mathfrak{p}) \subseteq \mathfrak{p}$ is stronger condition if K is not connected. To prove this fact, we observe that for any automorphism $\alpha: G \to G$ holds $d\alpha \circ Ad(g) = Ad(\alpha(g)) \circ d\alpha$.

If $h \in K$, $X \in \mathfrak{p}$, so that $\sigma(h) = h$ and $d\sigma(X) = -X$, this implies $d\sigma(Ad(h)X) = Ad(h)\, d\sigma(X) = -Ad(h)X$, i.e. $Ad(h)X \in \mathfrak{p}$. □

The symmetric space is irreducible iff the $\mathfrak{k}$-module $\mathfrak{p}$ is irreducible. In the case when G_0^σ is compact, for any compact subgroup K with $G_0^\sigma \leq K < G^\sigma$, the homogeneous space G/K with any G-invariant metric is a symmetric space (and such metrics exist.) Up to a minor point, there is a bijection between symmetric spaces and Cartan involutions.

Proposition 13

1/ If M is a symmetric space and $N \subset M$ is a submanifold such that for all $p \in M$, $s_p(N) = N$, then N is totally geodesic and symetric space.

2/ Let $\sigma : G \to G$ be an involutive automorphism, and G/K the symmetric space for σ. If $L \subset G$ with $\sigma(L) \subset L$, then $L/(L \cap K)$ is a symmetric space such that $L/(L\cap K)$ is totally geodesic submanifold.

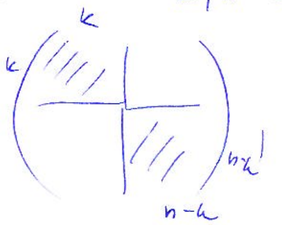
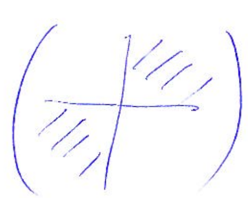
Pf: 1/ $N \subseteq M$ is totally geodesic if $\forall$ geodesic of N is a geodesic of M, i.e., the second fundamental form of embedding $B : T_pN \times T_pN \to (T_pN)^\perp$ vanishes. The isometry s_p preserves T_pN, hence $(T_pN)^\perp$ and so B as well. Since B has degree 3, it vanishes.

2/ By 1/, the symmetry $\bar\sigma$ at eK is $\bar\sigma(gK) = \sigma(g)K$ and if $N = L/(L\cap K) \subseteq G/K$, $\sigma(L) \subset L$ implies $\bar\sigma(N) \subset N$. $\square$

Examples:

1/ $$I_{p,q} = \left.\overbrace{\begin{pmatrix} \begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \end{pmatrix}}^{p+q}\right\} p+q$$ p diagonal entries -1, q —— " —— $+1$

$G = SO(n)$

$\sigma(A) := I_{k,n-k}\, A\, I_{k,n-k}$ is an automorphism of G, $+Id$ on  $-Id$ on 

$G^\sigma = S(O(k) \times O(n-k))$

This corresponds $G_k^o(\mathbb{R}^n) = SO(n)/S(O(k)\times O(n-k))$ embedded diagonally isotropy group of $p_0 = \langle e_1, \dots, e_k\rangle$:

$$G_{p_0} = S(O(k)O(n-k)) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \,\middle|\, A \in O(k), B \in O(n-k) \right\}$$

The -1-eigenspace of $d\sigma$ (and the isotropy representation) are given by

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & X \\ -X^T & 0 \end{pmatrix} \;\middle|\; X \in \mathrm{Mat}(k, n-k, \mathbb{R}) \right\},$$

$$\mathrm{Ad}(A,B)X = AXB^T, \quad (A,B) \in S(O(k)O(n-k)).$$

The isotropy repr. is irreducible iff $(k, n-k) \neq (2,2)$, so the Grassmanians are (except $G_2(\mathbb{R}^4)$) irreducible.

<u>Example:</u> K ... a compact Lie group, $K \times K$ acts on K by

$(a,b)\cdot h = a h b^{-1} \quad \forall a,b,h \in K$,

the isotropy subgroup preserving $e \in K$ $\Delta K = \{(a,a) \mid a \in K\}$ "diagonal subgroup"

So $K = (K \times K)/\Delta K$, and $K \times K$ acts by isometries in the bi-invariant metric on K. The involutive automorph.

$\sigma : (a,b) \mapsto (b,a)$ $\forall a,b \in K$ with $G^\sigma = \Delta K$ makes $(K\times K)/\Delta K$ into a symmetric space. Then

$\mathfrak{p} = \{(X, -X) \mid X \in \mathfrak{g}\}$ with isotropy representation the adjoint action $\mathrm{Ad}(k)(X,-X) = (\mathrm{Ad}(k)X, -\mathrm{Ad}(k)X)$,

so $\mathfrak{p} \xrightarrow{\sim} T_eK \cong \mathfrak{k}$, $(X,-X) \mapsto 2X$, and the symmetric space is irreducible iff K is simple.

<u>Example</u> (related to $SO(2n)/U(n)$)

Let $J = \begin{pmatrix} 0_n & -Id_n \\ Id_n & 0_n \end{pmatrix}$, $J^2 = -Id_{2n}$ be a complex structure on $2n$-dimensional vector space $\mathbb{R}^{2n}$, and consider the embedding $U(n) \hookrightarrow SO(2n)$

$$A + iB \longmapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

The involutive automorphism of $SO(2n)$,

$$\sigma(A) = JAJ.$$

Then $G^\sigma = U(n)$, so $SO(2n)/U(n)$ is a symmetric space; G^σ is connected in this case. The embedding implies

$$\mathfrak{h} = \mathfrak{u}(n) = \left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \middle| \; X, Y \in \mathfrak{gl}(n, \mathbb{R}),\ X = -X^T,\ Y = Y^T \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \middle| \; X, Y \in \mathfrak{gl}(n, \mathbb{R}),\ X = -X^T,\ Y = -Y^T \right\}.$$

Because $\mathfrak{p} \simeq \Lambda^2 \mathbb{R}^{2n}$, which is irreducible $\mathfrak{u}(n)$-mod $\Rightarrow$ the symmetric space is irreducible. Because $\pi_1(U(n)) \twoheadrightarrow \pi_1(SO(2n))$, it is simply connected.

Low-dimensional cases: $SO(4)/U(2) \simeq S^2$, because $SU(2) \hookrightarrow SO(4)$ is a normal subgroup, so $SO(4)/U(2) \simeq {SO(4)/SU(2)} \big/ {SU(2)/U(2)} \simeq$

$$\simeq SO(3)/\underset{\substack{\shortparallel \\ U(1)}}{SO(2)} \simeq S^2;$$

$$SO(6)/U(3) \simeq SU(4)/S(U(3) \times U(1)) \simeq \mathbb{CP}^3$$

<u>Example</u>: $A \in U(n)$, realized in $U(n) \subseteq SO(2n)$.

Define the automorphism $\sigma: U(n) \to U(n)$

$$A \mapsto \sigma(A) = \overline{A}.$$

We have

$$\mathfrak{h} = \left\{ \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \middle| \; X \in \mathfrak{gl}(n, \mathbb{R}),\ X = -X^T \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix} \middle| \; Y \in \mathfrak{gl}(n, \mathbb{R}),\ Y = Y^T \right\}.$$

and the isotropy representation $Ad(\mathrm{diag}(X,X))Y = XYX^T$, so $\mathfrak{p} \simeq S^2\mathbb{R}_n$ as $O(n)$-module. This repr. is not irreducible $\Rightarrow$ $U(n)/O(n)$ is not an irreducible symmetric space. Since σ preserves $SU(n)$, $SU(n)/SO(n) \subseteq U(n)/O(n)$ is totally geodesic submanifold. The isotropy repr. of $SU(n)/SO(n)$ is irreducible

Low-dimensional examples: $SU(2)/SO(2) \simeq S^2 \simeq \mathbb{C}P^1$, the set of Lagrangian subspaces of $\mathbb{R}^4 \simeq \mathbb{C}^2$.

$\boxed{\text{Geodesics and curvature of symmetric spaces}}$

Def. 14: (G, K, σ) is called a symmetric pair if K is compact, σ is an involution $\sigma: G \to G$ with $G_0^\sigma \subseteq K \subseteq G^\sigma$, and G acts $\underbrace{\text{almost effectively}}_{}$ on G/K.

$\mathfrak{g}$ and $\mathfrak{p}$ have no common ideal

Remark: A Riemannian homog. space G/H is a symmetric space $\not\Rightarrow$ (G, H) may not be a symmetric pair unless $G = I_0(M)$. E.g., $S^n = S^n(1) = SO(n+1)/SO(n)$ is a symmetric pair, but $SU(n) \subseteq SO(2n)$ also acts transitively on $S^{2n-1}(1)$ with the isotropy group $SU(n-1)$: there is no automorphism σ of $SU(n)$ with $G^\sigma = SU(n-1)$.

Proposition 15: Let (G, K) be a symmetric pair with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. If $X \in \mathfrak{p}$, then $\gamma(t) = \exp(tX) \cdot p_0$ is the geodesic in M with $\gamma(0) = p_0$, $\gamma'(0) = X \in \mathfrak{p} \simeq T_{p_0}M$.

(30) Pf: σ induces the symmetry s at p_0, $s(gK) = \sigma(g)K$.

Let (G', K') be the symmetric pair with $G' = I_0(M)$ for the symmetric metric on G/K and with Cartan involution $\sigma'(g) = sgs$ and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$. We first prove the claim for (G', K').

Let γ be the geodesic in M with $\gamma(0) = p_0$, $\gamma'(0) \in T_{p_0}M$. Then the transvection $T_t = s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}$ is the flow of a Killing vector field $X \in \mathfrak{g}'$. Since $\gamma(t) = T_t \cdot p_0$, it follows $\gamma'(0) = X^*(p_0)$. Furthermore,

$$\sigma'(T_t) = s_{\gamma(0)}\, s_{\gamma(\frac{t}{2})}\, \underbrace{s_{\gamma(0)}\, s_{\gamma(0)}}_{Id} = s_{\gamma(0)}\, s_{\gamma(\frac{t}{2})} =$$

$$= \left(s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}\right)^{-1} = \left(T_t\right)^{-1} = T_{-t},$$

and differentiating gives $d\sigma'(X) = -X \Rightarrow X \in \mathfrak{p}'$.

Now look back at the symmetric pair (G, K), assume it is effective (without changing Lie algebras.) We first show that $\sigma'|_G = \sigma$. Indeed, if $g \in G$ then $L_{sgs} = L_{\sigma(g)}$ since $L_{sgs}(hK) = L_{sg}(\sigma(h)K) = L_s(g\sigma(h)K)$ $= \sigma(g)hK = L_{\sigma(g)}(hK)$ (L is the left action on G).

Thus efectiveness of the action $\Rightarrow$ $\sigma(g) = sgs = \sigma'(g)$

Next we prove that $\mathfrak{p} = \mathfrak{p}'$, which completes the claim. Since they have the same dimension, it is sufficient to show $\mathfrak{p} \subset \mathfrak{p}'$. If $u \in \mathfrak{p}$, i.e. $d\sigma(u) = -u$, then $\sigma'|_G = \sigma$ implies $d\sigma'(u) = -u$, i.e. $u \in \mathfrak{p}'$. $\square$

Proposition 16 (Connection & Curvature on symmetric spaces)

Let (G, K) be a symmetric ~~space~~ pair with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

1/ $\forall\, Y \in \mathfrak{X}(M = G/K)$ vector field, $X \in \mathfrak{p}$, we have
$$(\nabla_{X^*} Y)(p_0) = [X^*, Y](p_0).$$

2/ If $X, Y, Z \in \mathfrak{p}$, then $\underbrace{\left(R(X^*, Y^*) Z^*\right)(p_0)}_{\text{curvature of } \nabla \,\equiv\, [\,,\,]\text{-invariant metric}} = -[[X, Y], Z]^*(p_0)$

Recall $\mathfrak{p} \xrightarrow{\sim} T_{p_0}M$
$X \mapsto X^*(p_0)$ is used in 1/, 2/.

Pf: 1/ $X \in \mathfrak{p}$, $\gamma(t) = \exp(tX) \cdot p_0$ is geodesic in M. The flow of X^* is the transvection $T_t = s_{\gamma(\frac{t}{2})} s_{\gamma(0)} = L_{\exp tX}$.
Also $(dT_t)_{\gamma(s)}$ is $\parallel$ transport along γ, so if $Y \in \mathfrak{X}(M)$, we have
$$\nabla_{X^*} Y = \frac{d}{dt}\Big|_{t=0} \left((P_t^{-1} Y)(\gamma(t)\right) = \frac{d}{dt}\Big|_{t=0} (dT_t^{-1})_{\gamma(t)} Y(\gamma(t))$$
$$= [X^*, Y].$$

2/ Compute ~~at~~ $p_0 \in M$:
$$\nabla_{X^*} \nabla_{Y^*} Z^* = [X^*, \nabla_{Y^*} Z^*] = -\nabla_{Y^*} [X, Z]^* =$$
$$= -[Y^*, [X, Z]^*] = [Y, [X, Z]]^* \Rightarrow$$
$$R(X^*, Y^*) Z^* = \nabla_{X^*} \nabla_{Y^*} Z^* - \nabla_{Y^*} \nabla_{X^*} Z^* - \nabla_{[X^*, Y^*]} Z^*$$
$$= [Y, [X, Z]]^* - [X, [Y, Z]]^*$$
$$\overset{\text{by Jacobi identity}}{=} -[[X, Y], Z]^*,$$
which follows by
$$\nabla_{X^*} \nabla_{Y^*} Z^* = [X^*, \nabla_{Y^*} Z^*] = \nabla_{[X^*, Y^*]} Z^* + \nabla_{Y^*} [X^*, Z^*]$$
since isometries preserve the connection and the flow of X^* is generated by isometries. Moreover, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$ implies $[X, Y]^*(p_0) = 0$. $\square$

(32) We usually write $\nabla_X Y = [X, Y]$, $R(X,Y)Z = -[[X,Y],Z]$.

1/ $\Rightarrow$ Geometric interpretation of the Cartan decomposition in terms of Killing vector fields, assuming $G = I_0(M)$:

$\mathfrak{k} = \{ X \in \mathfrak{g} \mid X^*(p_0) = 0 \}$,

$\mathfrak{p} = \{ X \in \mathfrak{g} \mid (\nabla_v X^*)(p_0) = 0 \ \forall v \in T_{p_0}M \}$.

this follows by 1/ : $(\nabla_{X^*} Y^*)(p_0) = [X,Y]^*(p_0) = 0$ for $X, Y \in \mathfrak{p}$, since $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.

<u>Proposition 17</u>: Let G/H be a symmetric space corresponding to the Cartan involution σ, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ a Cartan decomposition. If $\mathfrak{a} \subseteq \mathfrak{p}$ is a subspace with $[[\mathfrak{a}, \mathfrak{a}], \mathfrak{a}] \subseteq \mathfrak{a}$, called a Lie triple system, then $\exp(\mathfrak{a})$ is a totally geodesic submanifold.

<u>Pf</u>: First of all, $\mathfrak{h}' := [\mathfrak{a}, \mathfrak{a}] \leq \mathfrak{h}$, is a subalgebra of $\mathfrak{h}$.
By Jacobi identity, $[\mathfrak{h}', \mathfrak{h}'] = [[\mathfrak{a},\mathfrak{a}],[\mathfrak{a},\mathfrak{a}]] =$
$= [[[\mathfrak{a},\mathfrak{a}],\mathfrak{a}],\mathfrak{a}] = [\mathfrak{a},\mathfrak{a}]$.
since $\mathfrak{a}$ is a Lie triple system

Because $[\mathfrak{h}', \mathfrak{a}] = [[\mathfrak{a},\mathfrak{a}],\mathfrak{a}] \subseteq \mathfrak{a}$ and $[\mathfrak{a},\mathfrak{a}] \subset \mathfrak{h}'$,
$\mathfrak{g}' := [\mathfrak{a},\mathfrak{a}] \oplus \mathfrak{a}$ is a subalgebra of $\mathfrak{g}$.

Let $G' \subset G$ connected subgroup
$H' \subset H$ } the Lie algebras $\mathfrak{g}'$, $\mathfrak{h}'$. Since $d\sigma$ preserves $\mathfrak{g}'$, σ preserves G, the proof follows by one of the previous Propositions. $\square$

It seems that Proposition 17 helps to classify totally geodesic submanifolds of a symmetric space - however, even for Grassmannians they are not classified.

Types of symmetric spaces and their dualities

Def 18: Let (G,K) be a symmetric pair with B the Killing form of $\mathfrak{g}$.
The symmetric pair is called to be of

1/ compact type if $B|_{\mathfrak{p}} < 0$,

2/ non-compact —"— $B|_{\mathfrak{p}} > 0$,

3/ Euclidean —"— $B|_{\mathfrak{p}} = 0$.

Proposition 19: Let (G,K) be a symmetric pair.

1/ If (G,K) is irreducible, it is either compact, non-compact or euclidean type.

2/ If $M = G/K$ is simply connected, then M is isometric to a Riemannian product $M = \underset{\substack{\text{Eucl.}\\ \text{type}}}{M_0} \times \underset{\substack{\text{Compact}\\ \text{type}}}{M_1} \times \underset{\substack{\text{non-compact}\\ \text{type}}}{M_2}$.

3/ If (G,K) is of compact type $\Rightarrow$ G is semi-simple and G, M are compact

If (G,K) —"— non-compact —"— $\Rightarrow$ G is semi-simple and G, M are non-compact

If (G,K) is of euclidean type iff $[\mathfrak{p},\mathfrak{p}] = 0$; if G/K is simply connected, it is isometric to $\mathbb{R}^n$.

Pf:

1/ G/K irreducible $\overset{\text{Schur's lemma}}{\Longrightarrow}$ $B|_{\mathfrak{p}} = \lambda \langle\, , \rangle$, where $\langle\, , \rangle$ is $\mathfrak{g}$-invariant metric on $\mathfrak{p}$. So

$$\left.\begin{array}{l}\text{compact}\\ \text{non-compact type}\\ \text{Euclidean}\end{array}\right\} \begin{array}{l}\lambda<0\\ \lambda>0\\ \lambda=0\end{array}$$

2/ M is isometric to $M_1 \times \dots \times M_k$, $M_i \equiv$ irred. symm. spaces, so the claim follows by 1/.

3/

(34) σ is the automorphism for (G, K), $d\sigma$ preserves B $\Rightarrow$
$B(\mathfrak{k}, \mathfrak{p}) = 0$ since $d\sigma|_{\mathfrak{k}} = Id_{\mathfrak{k}}$, $d\sigma|_{\mathfrak{p}} = -Id_{\mathfrak{p}}$. The next claim is $B|_{\mathfrak{k}} < 0$. Indeed, since K is compact, $\exists$ an inner product on $\mathfrak{g}$ such that $ad_X : \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric if $X \in \mathfrak{k}$.
Thus $B(X, X) = Tr(ad_X^2) \leq 0$, $B(X, X) = 0$ iff $X \in Z(\mathfrak{g})$ (the center of $\mathfrak{g}$.) But $Z(\mathfrak{g}) \cap \mathfrak{k}$ since $\mathfrak{g}$ and $\mathfrak{k}$ have no ideals in common. Hence, if $B(X, X) = 0 \Rightarrow X = 0 \Rightarrow B|_{\mathfrak{k}} < 0$.
In particular, G is semi-simple if (G, K) is compact / non-compact type.
If it is compact type, then $B < 0$ and hence G (and so M as well), is compact. Similarly for the non-compact case.
If $B|_{\mathfrak{p}} = 0$, then $B(\mathfrak{k}, \mathfrak{p}) = 0$ and $B|_{\mathfrak{k}} < 0 \Rightarrow$ the radical of B is $\mathfrak{p}$. But this is an ideal in $\mathfrak{g}$, and this together with $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$ implies $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{p} \cap \mathfrak{k} = 0$. Conversely, $[\mathfrak{p}, \mathfrak{p}] = 0$ & $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ $\Rightarrow B|_{\mathfrak{p}} = 0$.
If $[\mathfrak{p}, \mathfrak{p}] = 0$, the sectional curvature is 0, and for simply connected $M \Rightarrow M \underset{\text{isometric}}{\simeq} \mathbb{R}^n$. $\square$

So we usually consider symmetric spaces without Euclidean factors.

<u>Proposition 20</u>: Let (G, K) be a symmetric pair.

1/ If (G, K) is of compact type, $sec \geq 0$. (sectional curvature)

2/ — " — non- — " —, $sec \leq 0$.

3/ A 2-plane spanned by $u, v \in \mathfrak{p}$ has trivial curvature iff $[u, v] = 0$.

4/ If (G, K) is irreducible, $\langle\, ,\rangle = \pm B|_{\mathfrak{p}}$, then $sec(X, Y) = \pm \| [X, Y] \|^2$.

Pf: Assume the irreducibility of (G,K). In this case $B|_{\mathfrak{p}} = \lambda \langle\, , \rangle$ for some $\lambda \neq 0$, where $\langle\, , \rangle$ is invariant metric on $\mathfrak{p}$. If $u, v \in \mathfrak{p} \simeq T_pM$ is an ON-base of a 2-plane, then

$$\lambda\, sec(u,v) = \lambda \langle R(u,v)v, u\rangle = -\lambda \langle [[u,v],v], u\rangle$$
$$= -B([[u,v],v],u) = B([u,v],[u,v]).$$

We used the fact ad_u is skew-symmetric for B. Since $[u,v] \in \mathbb{k}$ and since $B|_{\mathbb{k}} < 0$ we have $B([u,v],[u,v]) < 0$
$\Rightarrow$ sec is determined by the sign of λ. $\square$

Proposition 21: Let (G,K) be an irreducible symmetric pair which is not of Euclidean type. Then either G is simple, or $(G,K) = (K \times K, \Delta K)$ and G/K is isometric to a compact simple Lie group with bi-invariant metric.

Pf: By previous Proposition 19 $\mathfrak{g}$ is semi-simple, $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r$ with $\mathfrak{g}_i$ simple. This decomposition into simple ideals is unique up to order $\Rightarrow \sigma$ permutes $\mathfrak{g}_i$. Hence
$\mathfrak{g} = \tilde{\mathfrak{g}}_1 \oplus \dots \oplus \tilde{\mathfrak{g}}_s$ is a sum of ideals such that $\tilde{\mathfrak{g}}_i$ is either $\mathfrak{g}_k$ (for some $k = 1, \dots, r$) and $\sigma(\mathfrak{g}_k) = \mathfrak{g}_k$, or $\tilde{\mathfrak{g}}_i = \mathfrak{g}_k \oplus \mathfrak{g}_l$ and $\sigma(\mathfrak{g}_k) = \mathfrak{g}_l$, $\sigma(\mathfrak{g}_l) = \mathfrak{g}_k$ for some $k, l = 1, \dots, r$.

Then $\sigma|_{\tilde{\mathfrak{g}}_i}$ decomposes as $\mathbb{k}_i \oplus \mathfrak{p}_i$, ± 1-eigenspaces, so $\mathbb{k} = \mathbb{k}_1 \oplus \dots \oplus \mathbb{k}_s$, $\mathfrak{p} = \mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_s$. Notice $\mathfrak{p}_i \neq 0$ $\forall i$ since otherwise $\sigma|_{\mathbb{k}_i} = Id \Rightarrow \mathbb{k}_i$ is an ideal common to $\mathfrak{g}$ an $\mathbb{k}$, which contradicts efectiveness. Because $[\mathbb{k}_i, \mathfrak{p}_j] = 0$ for $i \neq j$, we have $[\mathbb{k}, \mathfrak{p}_i] \subseteq \mathfrak{p}_i \overset{\text{irreducibility}}{\Longrightarrow} s = 1$.
If G is not simple $\Rightarrow \mathfrak{g} = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}$ with $\sigma(a,b) = (b,a) \Rightarrow$

 $\mathfrak{g}^\sigma = \Delta\tilde{\mathfrak{g}}$. This is the symmetric pair $K\times K/\Delta K$, where K is any compact simple Lie group with Lie algebra $\mathfrak{k}$. $\square$

<u>Proposition 22</u>: Let (G,K) be a symmetric pair with no euclidean factor and Cartan decomposition. Then

1/ $[\mathfrak{p},\mathfrak{p}] = \mathfrak{k}$,

2/ $Hol_p^0 = K_0$, where Hol_p is the holonomy group,

3/ If G/K is effective, then $G = I_0(M)$.

<u>Pf</u>: 1/ By previous results $\Rightarrow$ G/K is irreducible, so that $B|_{\mathfrak{p}}$ is non-degenerate. If $a\in\mathfrak{k}$ with $B(a,[u,v])=0$ $\forall\, u,v\in\mathfrak{p}$, then $0 = B(a,[u,v]) = -B(u,[a,v])$ $\forall\, w\in\mathfrak{p}$ $\Rightarrow$ $[a,v]=0$ $\forall\, v\in\mathfrak{p}$. Since
$$Ad(\exp(ta)) = e^{t\,ad a},\ \text{we have } Ad(\exp(ta))|_{\mathfrak{p}} = Id.$$
Because G/K is almost effective $\Rightarrow$ isotropy representation has finite kernel. So $\exp(ta)=e$ $\forall t\in\mathbb{R}$ $\Rightarrow$ $a=0$. This implies $\mathfrak{k} = \langle [u,v] \mid u,v\in\mathfrak{p}\rangle$.

2/, 3/ are similar/analogues to 1/. $\square$

Duality for symmetric spaces

Let (G,K) be a symmetric ~~space~~ pair, $\pi_1(G/K)=0$. Let $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$; we have $\mathfrak{g}\subseteq \mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$ and define a new subalgebra $\mathfrak{g}^*\subseteq\mathfrak{g}\otimes\mathbb{C}$ by $\mathfrak{g} = \mathfrak{k}\oplus i\mathfrak{p}$. This is indeed a subalgebra:

$[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}$, $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$ $\Rightarrow$

$[\mathfrak{k}, i\mathfrak{p}]\subset i\mathfrak{p}$, $[i\mathfrak{p}, i\mathfrak{p}] = -[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$.

G^* - simply connected Lie group, $Lie(G^*) = \mathfrak{g}^*$, $K^*\subseteq G^*$ connected subgroup, $Lie(K^*) = \mathfrak{k}\subseteq\mathfrak{g}^*$.

(37) Since $\mathfrak{g}$, $\mathfrak{k}$ and consequently $\mathfrak{g}^*$, $\mathfrak{k}^*$ have no ideal in common, G^*/K^* is simply connected and almost effective.

The terminology is: (G^*, K^*) is the dual (symmetric pair) of (G, K). K, K^* have the same Lie algebras, but need not be isomorphic as Lie groups.

Proposition 23: Let (G,K) be a symmetric pair with (G^*, K^*) dual symmetric pair.

1/ If (G,K) is of compact type, then (G^*, K^*) is of non-compact type (and vice-versa)

2/ If (G,K) is of Euclidean type, (G^*, K^*) is of Euclidean type as well.

3/ The pairs (G,K), (G^*, K^*) have the same infinitesimal isotropy repr., hence (G,K) is irred. iff (G^*, K^*) is irred.

4/ If (G,K) and (G^*, K^*) are effective and simply connected without Euclidean factors, $K = K^*$.

Pf: 1/ $\mathfrak{g}$ ... a real Lie algebra, then $B_{\mathfrak{g}} = B_{\mathfrak{g}_{\mathbb{C}}}|_{\mathfrak{g}}$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$.

By construction, $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{g}^*_{\mathbb{C}}$. If (G,K) is of compact type, $B_{\mathfrak{g}}(u,u) < 0$ for $\forall\, u \in \mathfrak{p}$, then $B_{\mathfrak{g}^*}(iu, iu) = -B(u,u) > 0$, i.e. (G^*, K^*) is of non-compact type.

3/ $\Leftarrow$ the action of $\mathfrak{k}$ on $\mathfrak{p}$ and $i\mathfrak{p}$ is the same.

4/ (G,K) has no Euclidean factors $\Rightarrow G = I_0(M)$, K is connected.

R is the curvature tensor of $(G,K) \Rightarrow K = K_0 = \{ A \in GL(T_{p_0} M) \mid A^* R = R \}$.

since G/K is effective and simply connected

R^* — " — (G^*, K^*) $\Rightarrow K^* = K_0^* = \; \ldots$

Because

$$R^*(iX,iY)(iZ) = -[[iX,iY],iZ] = [[X,Y],Z] = -R(X,Y)Z$$

for $\forall\, X,Y,Z \in \mathfrak{p} \Rightarrow K = K^*$. $\square$

Here are several examples of this duality:

<u>Example</u>: $S^n \leftrightarrow \mathbb{H}^n$ (sphere $\leftrightarrow$ hyperbolic space)

Multiplication by i on $T_p M$ illustrates why $\sin x, \cos x$
$\overset{\shortparallel}{S^n, \mathbb{H}^n}$
in the spherical geometry is replaced by $\sinh x = \sin(ix)$, $\cosh x = \cos(ix)$.

This corresponds to the duality between

$$G/K = SO(p+q)/(SO(p)\times SO(q)), \quad G^*/K^* = SO(p,q)/(SO(p)\times SO(q)).$$

In both cases is the involution given by $\sigma(A) = Ad(I_{d_{p,q}})$.

Writing the Lie algebras in a block-decomposition, we have

$$o(p+q) = \{A \in Mat(p+q, p+q, \mathbb{R}) \mid A + A^T = 0_{p+q,p+q}\},$$

$$o(p,q) = \{A \in Mat(p+q,p+q,\mathbb{R}) \mid A I_{p,q} + I_{p,q} A^T = 0_{p+q,p+q}\}.$$

The Cartan decomposition is then

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & X \\ -X^T & 0 \end{pmatrix} \,\middle|\, X \in Mat(p,q,\mathbb{R}) \right\},$$

$$\mathfrak{p}^* = \left\{ \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix} \,\middle|\, X \in Mat(p,q,\mathbb{R}) \right\}.$$

It is not true $\mathfrak{p}^* = i\mathfrak{p}$, but the inner automorphism

$$Ad(\mathrm{diag}(\underbrace{i,\dots,i}_{p}, \underbrace{-1,\dots,-1}_{q})) \text{ of } so(n,\mathbb{C}) \simeq o(p+q) \otimes_{\mathbb{R}} \mathbb{C} \simeq o(p,q) \otimes_{\mathbb{R}} \mathbb{C}$$

takes $i\mathfrak{p}$ to $\mathfrak{p}^*$ and preserves $\mathfrak{k}$ $\Rightarrow$ it conjugates $\mathfrak{g}^*$ into a new Lie algebra $\mathfrak{g}'$ that satisfies the setup of duality

(39) Analogously for ($\mathbb{R} \to \mathbb{C} \to \mathbb{H}$)

$\mathbb{C}P^n \leftrightarrow \mathbb{C}H^n$, $\mathbb{H}P^n \leftrightarrow \mathbb{H}H^n$.

Example: There is a duality between $SU(n)/SO(n)$ and $SL(n,\mathbb{R})/SO(n)$.

Since the involutions are given by $d\sigma(A) = \bar{A}$ and $d\sigma(A) = -A^T$, respectively, we have in both cases

$$\mathbb{k} = \{A \in Mat(n,n,\mathbb{R}) \mid A = -A^T\}$$

and

$$\mathfrak{p} = \{A \in su(n) \mid \bar{A} = -A\},\ \mathfrak{p}^* = \{A \in Mat(n,n,\mathbb{R}) \mid A = A^T\}.$$

But $\mathfrak{p}$ can be also written as

$$\mathfrak{p} = \{iA \mid A \in M(n,n,\mathbb{R}) \text{ and } A = A^T\} \text{ and so } \mathfrak{p}^* = i\mathfrak{p} \subseteq sl(n,\mathbb{C}).$$

Example: Let K be a compact Lie group, and consider symmetric pair $(G,K) \simeq (K \times K, \Delta K)$. The claim is that the dual

(ΔK: diagonal subgroup in $K \times K$)

of $(K\times K, \Delta K)$ is $\underbrace{((\mathbb{k} \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}}, \mathbb{k})}_{\text{on the Lie algebra level}}$ with Cartan involution $\sigma(A) = \bar{A}$ (here $(\mathbb{k}_{\mathbb{C}})_{\mathbb{R}} := (\mathbb{k} \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}}$ is $\mathbb{k} \otimes_{\mathbb{R}} \mathbb{C}$ regarded as a real Lie algebra.)

To prove this, recall $\mathfrak{g} = \tilde{\mathbb{k}} \oplus \tilde{\mathbb{k}}$ with $\sigma(X,Y) = (Y,X)$, so $\mathbb{k} = \{(X,X) \mid X \in \tilde{\mathbb{k}}\}$, $\mathfrak{p} = \{(X,-X) \mid X \in \tilde{\mathbb{k}}\}$. We now describe $\mathfrak{g}^* = \mathbb{k} \oplus i\mathfrak{p} \subseteq \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \tilde{\mathbb{k}}_{\mathbb{C}} \oplus \tilde{\mathbb{k}}_{\mathbb{C}}$.
Let $\mathfrak{p}' = \{(iX, iX) \mid X \in \tilde{\mathbb{k}}\} \subset \tilde{\mathbb{k}}_{\mathbb{C}} \oplus \tilde{\mathbb{k}}_{\mathbb{C}}$, $\mathfrak{g}' = \mathbb{k} \oplus \mathfrak{p}'$.
The $\mathbb{R}$-linear isomorphism $\tilde{\mathbb{k}}_{\mathbb{C}} \oplus \tilde{\mathbb{k}}_{\mathbb{C}} \to \mathbb{k}_{\mathbb{C}} \oplus \tilde{\mathbb{k}}_{\mathbb{C}}$:

$$(X_1 + iY_1, X_2 + iY_2) \longmapsto (X_1 + iY_1, X_2 - iY_2)$$

is an isomorphism of Lie algebras and takes $\mathbb{k} \to \mathbb{k}$,
$i\mathfrak{p} \to \mathfrak{p}'$.

Thus the dual pair is isomorphic to $(\mathfrak{g}^*, \mathbb{k}) \simeq (\mathfrak{g}', \mathbb{k})$. But $\mathfrak{g}' = \{(X+iY, X+iY) \mid X, Y \in \tilde{\mathbb{k}}\} = \Delta \tilde{\mathbb{k}}_{\mathbb{C}} \subseteq \tilde{\mathbb{k}}_{\mathbb{C}} \oplus \tilde{\mathbb{k}}_{\mathbb{C}}$ and $\mathbb{k} = \Delta \tilde{\mathbb{k}} \subset \tilde{\mathbb{k}}_{\mathbb{C}} \oplus \tilde{\mathbb{k}}_{\mathbb{C}}$, so $(\mathfrak{g}', \mathbb{k})$ is isomorphic to $(\tilde{\mathbb{k}}_{\mathbb{C}}, \tilde{\mathbb{k}}) \simeq (\tilde{\mathbb{k}}_{\mathbb{C}}, \mathbb{k})$.
The Cartan involution is given by conjugation.

Symmetric spaces of non-compact type

Though it is easier to study the classification of symmetric pairs for compact types, their non-compact duals have several special geometric properties.
For (G, K) of compact type, there is positive definite inner product on $\mathfrak{g}$ given by $-B$. In the non-compact case we have

Lemma 24: If (G, K) is a symmetric space of non-compact type, then the inner product $B^*(X, Y) := -B(\sigma(X), Y)$ on $\mathfrak{g}$ has the following properties:
1/ B^* is positive definite,
2/ If $X \in \mathbb{k}$, then $ad_X : \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric,
3/ If $X \in \mathfrak{p}$, then $ad_X : \mathfrak{g} \to \mathfrak{g}$ is symmetric.

Pf: 1/ $B|_{\mathbb{k}} < 0$, $B|_{\mathfrak{p}} > 0$, $B(\mathbb{k}, \mathfrak{p}) = 0$ and $\sigma|_{\mathbb{k}} = Id_{\mathbb{k}}$, $\sigma|_{\mathfrak{p}} = -Id_{\mathfrak{p}}$.

2/, 3/ follows from the fact that ad_X, $X \in \mathfrak{g}$, commutes with σ and ad_X is always skew-symmetric for B.

$\square$

41 Proposition 25: Let (G, K) be a symmetric pair of non-compact type with Cartan involution σ. Then

1/ G is non-compact and semisimple, G^σ and K are connected.
2/ K is a maximal compact subgroup of G.
3/ $Z(G) \subset K$, or equivalently, if G/K is effective, $Z(G) = \{e\}$.
4/ G is diffeomorphic to $K \times \mathbb{R}^n$, G/K is diffeomorphic to $\mathbb{R}^n$ (and simply connected.)

The proof is based on the careful study of the exponential map for symmetric metric on G/K of non-positive curvature,

$$\exp_M : \mathfrak{p} \to G/K,$$
$$X \mapsto \exp(X)K,$$

and use the Hadamard Theorem ($\exp_M$ is a local diff.) and Hopf - Rinow theorem (it is onto), respectively.

Hermitean symmetric spaces

A class of symmetric spaces preserving complex structure.

(M, J) - almost complex manifold if J is a complex structure $J_p \in \mathrm{End}(T_pM)$,
$$J_p^2 = -Id_p, \quad \forall\, T_pM,\ p \in M.$$

(M, J) - complex manifold if $\exists$ charts covering M, which are open subsets in $\mathbb{C}^n$ such that transition functions are holomorphic (standard complex structure on $\mathbb{C}^n \Rightarrow$ complex structure on M (or, TM)).

Almost complex structure induced from local charts in $\mathbb{C}^n \Rightarrow$ complex structure on M; the measure of integrability is the Nijenhuis tensor N:

$$\tfrac{1}{2} N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY], \quad X, Y \in TM.$$

Theorem 42 (Newlander – Nirenberg): If J is an almost complex structure, J is integrable iff $N=0$.

$(M, \langle\,,\,\rangle, J)$ is called almost Hermitean if $g=\langle\,,\,\rangle$ and J are compatible, $\langle JX, JY\rangle = \langle X, Y\rangle$. Notice that since $J^2=-Id$, this is equivalent to J being skew-adjoint: $\langle JX, Y\rangle = -\langle X, JY\rangle$ $\forall X, Y \in TM = \mathfrak{X}(M)$. If J integrable, M is Hermitean manifold.

To M we can associate a 2-form, $\omega \in \Lambda^2 T^*M : \omega(X,Y) = \langle JX, Y\rangle$:

$$\omega(X,Y) = \langle JX, Y\rangle = -\langle X, JY\rangle = -\langle JY, X\rangle = -\omega(Y,X) \;\forall X,Y \in TM.$$

Moreover, $\omega^n \neq 0$ since $\exists$ ON-basis u_i, v_i, $i=1,\dots,n$ with $Ju_i = v_i$ and $Jv_i = -u_i$, hence $\omega = \sum_{i=1}^{n} du_i \wedge dv_i$. M is almost Kähler if (M, J) is almost Hermitean and $d\omega = 0$, and Kähler if J is in addition integrable. Almost Kähler manifolds are symplectic, so $H^{2i}_{dR} \neq 0$ since $[\omega^i] \neq 0$.

Recall that $\nabla J = 0$ iff JX is $\parallel$ ($\nabla(JX)=0$) for $X \parallel$ ($\nabla X = 0$).

Proposition 26: Let J be an almost complex structure on M, g a J-compatible metric.

1/ If (M, g, J) is almost Hermitean and $\nabla J = 0$, then M is Kähler.

2/ If — " — is Hermitean, then $d\omega = 0$ iff $\nabla J = 0$.

Pf: This is based on the identity

$$4g(\nabla_X Y, Z) = 6d\omega(X, JY, JZ) - 6d\omega(X, Y, Z) + g(N(Y,Z), JX)$$

On diff. forms, $d\omega$ is skew-symmetrization of $\nabla\omega$. Since the metric is $\parallel$, $\nabla J = 0$ iff $\nabla\omega = 0$. $\square$

We remark that being Kähler manifold is equivalent to the fact that the holonomy group at a point is contained in $U(n) \subseteq SO(2n)$ (equivalent to $\nabla J = 0$).

<u>Definition 27</u>: A symmetric space M is called Hermitean symmetric space if it is a Hermitean manifold and the symmetries s_p are holomorphic.

Here we could replace Hermitean by almost Hermitean (the Nijenhuis tensor vanishes if s_p is complex linear.) The reason is that s_p being complex linear implies $(s_p)_*(N) = N$ an since N has odd order $(TM \times TM \longrightarrow TM)$, $(ds_p)_p = -Id_{T_pM}$ implies $N = 0$.

<u>Proposition 28</u>: Let (G, K) be a symmetric pair with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. If $J: \mathfrak{p} \to \mathfrak{p}$ satisfies

1/ J is OG-transformation and $J^2 = -Id$,

2/ $J \circ Ad(k) = Ad(k) \circ J \quad \forall k \in K$,

then M is a Hermitean symmetric space (in fact, a Kähler manifold.)

<u>Pf</u>: By definition, $J_{gp} = (L_g)_* J_p$, i.e., $J_{gp} = (dL_g)_p \circ J_p \circ (dL_{g^{-1}})_{gp}$.
Since Ad_K preserves J_p, this is well-defined. Thus we obtain an almost complex structure on G/K and L_g preserves it.
We now claim that s_p preserve J as well, i.e., $(s_p)_* J = J$.
Recall $s_{gp} \circ L_g = L_g \circ s_p \Rightarrow (s_p)_* J$ is another complex structure which is G-invariant:

$$(L_g)_* \circ (s_p)_* (J) = (L_g \circ s_p)_* (J) = (s_{gp} \circ L_g)_* (J) =$$
$$= (s_{gp})_* \circ (L_g)_* (J) = (s_{gp})_* (J).$$

But J and $(s_p)_* J$ agree at p, hence everywhere. By above, this implies that J is integrable and hence M is Hermitean symmetric. It is also Kähler: ∇J is odd degree tensor, and because it is preserved by s_p, it vanishes. $\square$

Any symmetric space with complex-linear isotropy representation is Hermitean symmetric

Korollary 44: Let (G,K) be a symmetric pair. Then

1/ (G,K) is Hermitean symmetric $\Leftrightarrow$ (G^*,K^*) is Hermitean symmetr.

2/ If (G,K) is irreducible and Hermitean symmetric, it is Kähler-Einstein.

Some other characterization of Hermitean symmetric spaces.

Proposition 45: Let (G,K) be an irreducible symmetric pair.

1/ The complex structure J is unique up to a sign.

2/ (G,K) is Hermitean symmetric iff K is not semi-simple. (i.e., K is simple)

3/ If (G,K) is of compact type, it is Hermitean symmetric iff $H^2_{dR}(M) \neq 0$.

Pf: 3/ Hermitean symmetric $\Rightarrow$ Kähler $\Rightarrow$ $H^2_{dR}(M) \neq 0$.
The other implication follows from the observation that the cohomology representative can be chosen to be G-invariant: G acts trivially on cohomology and hence $\tilde{\omega}_p := \int_G \omega_{gp}\, dg$, and $\tilde{\omega}$ lies in the same cohomogy class as ω.
Now define $J: \mathfrak{p} \to \mathfrak{p}$ by $\omega_p(X,Y) = \langle JX, Y\rangle$ $\forall X,Y \in \mathfrak{p}$.
Then $\langle JX,Y\rangle = \omega(X,Y) = -\omega(Y,X) = -\langle JY, X\rangle$, i.e.
J is skew-adjoint. Since ω is G-invariant and well-defined on $M = G/K$, ω_p is Ad_K-invariant; $Ad_K \circ J = J \circ Ad_K$. But Ad_K acts irreducibly on $\mathfrak{p}$, and because J^2 is self-adjoint and commutes with Ad_K as well $\Rightarrow$ $J^2 = \lambda\, Id$ for some $\lambda < 0$ real number. So $J^2 = -\mu^2\, Id$ for some positive real number μ, hence $J' := \frac{1}{\mu} J$ fulfills $(J')^2 = -Id$.
Since J' is skew adjoint and $(J')^2 = -Id$, J' is ortogonal and Proposition 28 $\Rightarrow$ G/K is Hermitean symmetr.

2/ Assume (G,K) is of compact type. If G/K is Hermitean symmetric, we showed $H_{dR}(M) \neq 0$. We have
$$\underset{\text{''}}{} G/K \qquad \Leftarrow \begin{cases} \cdot\ G \dots \text{connected}, \\ \cdot\ K \dots \text{fin many components}, \\ \cdot\ \# \pi_1(G) < \infty \text{ since } G \text{ is semi-simple} \end{cases}$$

$M = G/K$ is such that $\pi_1(M)$ is finite.

$\tilde{M}$ ... universal cover of M, then $H_{dR}(\tilde{M}) \neq 0$ as well.

Hurewitz theorem $\Rightarrow$ $\mathbb{Z} \subseteq \pi_2(\tilde{M}) = \pi_2(M)$; because $\pi_2(G) = 0$ for G semi-simple (in particular, compact). The long exact homotopy sequence $\Rightarrow$ $\mathbb{Z} \subseteq \pi_1(K)$ $\Rightarrow$ K cannot be semi-simple.
(one can reverse the series of arguments to prove the opposite implication.)

3/ J_1, J_2 - orthogonal invariant complex structures, then
$\omega_i(X,Y) = \langle J_i X, Y\rangle$ are two non-degenerate symplectic forms.
$i = 1,2$, $X, Y \in \mathfrak{X}(M)$

As for inner products, we have $\omega_1 = \lambda \omega_2$ for some $0 \neq \lambda \in \mathbb{R}$.
But $J_i^2 = -Id \Rightarrow \lambda = \pm 1$. $\square$

Yet another characterization of non-Euclidean Hermitean symmetric spaces:

<u>Proposition 45</u>: Let (G,K) be an effective irreducible Hermitean symmetric space not of Euclidean type.

1/ K is connected and $\pi_1(G/K) = 0$,

2/ $Z(G) = \{e\}$ and $rk\, K = rk\, G$,

3/ $Z(K) \simeq SO(2) \simeq S^1$ and K is the centralizer of $Z(K)$ in G,

4/ The complex structure J is given by $Ad(i) = J$ for $i \in S^1 \simeq Z(K)$,

5/ $\forall$ isometry in $I_0(M)$ is holomorphic.

($Z(A)$ denotes the centrum of a group A.)

<u>Pf</u>: Let us prove the crucial property $Z(K) \simeq S^1$. Recall that K acts irreducibly and effectively on $\mathfrak{p}$ and so $Z(K)$ acts as an intertwining operator (for all $k \in Z(K)$) of the isotropic representation.

 The algebra of intertwining operators of an irreducible repr. on $\mathbb{R}$-vector space is either $\underbrace{\mathbb{R}, \mathbb{C} \text{ or } \mathbb{H}}$. Since the action is

depending on wheather the repr. is complex or quaternion

orthogonal, complex (for J) and $Z(K)$ is abelian and not finite $\Rightarrow Z(K) \simeq S^1$. This acts via complex multipl. on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, hence $J = Ad(i)$ satisfies $J^2 = -Id$ and commutes with $Ad(K)$ $\underset{\text{uniqueness}}{\Longrightarrow}$ it is the complex structure on G/K. $\square$

Proposition 46: (G, K)... simply connected irreducible Hermitean Symmetric space. Then for G a classical Lie group, G/K is one of the possibilities:

1/ $U(m+n)/(U(n) \times U(m))$,

2/ $SO(2n)/U(n)$,

3/ $Sp(n)/U(n)$,

4/ $SO(n+2)/(SO(n) \times SO(2))$.

Isomorphisms and coverings among (small rank) Lie groups

~~Propo~~

<u>Lemma 1</u>: 1/ $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geq 3$ and $\pi_1(SO(2)) = \mathbb{Z}$.

2/ $\pi_1(U(n)) = \mathbb{Z}$ and $\pi_1(SU(n)) = 0$ for $n \geq 1$.

3/ $\pi_1(Sp(n)) = 0$ for $n \geq 1$.

<u>Pf</u>: 1/ $\pi_1(SO(2)) = \pi_1(S^1) = \mathbb{Z}$. In general, $SO(n+1)$ acts on S^n (n-dimensional sphere, $S^n \subseteq \mathbb{R}^{n+1}: \|x\|^2 = 1\}$), $p \mapsto Ap$ $\forall p \in S^n$, $A \in SO(n+1)$. The isotropy subgroup at $e_1 \in S^n$ is $\{A \in SO(n+1) \mid Ae_1 = e_1\} =$
$= \{\mathrm{diag}(1, B) \mid B \in SO(n)\} \simeq SO(n)$. Since the action is transitive, $S^n = SO(n+1)/SO(n)$, so the short exact sequence $SO(n) \to SO(n+1) \to S^n$ induces long exact sequence of homotopy groups

$$\cdots \to \pi_i(SO(n)) \to \pi_i(SO(n+1)) \to \pi_i(S^n) \to \pi_{i-1}(SO(n)) \to$$

Since $\pi_i(S^n) = 0$ when $i = 1, \dots, n-1$, it follows the inclusion $\pi_1(SO(n)) \to \pi_1(SO(n+1))$ is onto for $n \geq 2$ and injective for $n \geq 3$. Thus
$\mathbb{Z}_2 \simeq \pi_1(\mathbb{RP}^3) \simeq \pi_1(SO(3)) = \pi_1(SO(n))$ for $n \geq 4$, which proves the claim.

2/ Similarly to 1/, $U(n+1)$ and $SU(n+1)$ acts transitively on $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ with isotropy $U(n)$ resp. $SU(n)$.

3/ $Sp(n+1)$ acts transitively on $S^{4n+3} \subseteq \mathbb{H}^{n+1}$ with isotropy $Sp(n)$ and $\pi_1(Sp(1)) = \pi_1(S^3) = 0$. $\square$

The polar decomposition then implies

Lemma 2: 1/ $\pi_1(GL^+(n,\mathbb{R})) = \pi_1(SL(n,\mathbb{R})) = \mathbb{Z}_2$ for $n \geq 3$

$= \mathbb{Z}$ for $n = 2$.

2/ $\pi_1(GL(n,\mathbb{C})) = \mathbb{Z}$ and $\pi_1(SL(n,\mathbb{C})) = 0$ for $n \geq 2$.

3/ $\pi_1(GL(n,\mathbb{H})) = \pi_1(SL(n,\mathbb{H})) = 0$ for $n \geq 2$.

4/ $\pi_1(Sp(n,\mathbb{R})) = \mathbb{Z}$ for $n \geq 1$.

related to Maslov index for a loop in symplectic matrices.

Lemma 3: 1/ $Sp(1)$ is a 2-folded cover of $SO(3) \simeq \mathbb{R}P^3$.

2/ $Sp(1) \times Sp(1)$ is a 2-folded cover of $SO(4)$.

Pf: 1/ The adjoint representation of $Sp(1) = \{q \in \mathbb{H} \mid |q| = 1\}$ as a 2-fold cover:

$$\varphi : Sp(1) \to SO(3)$$
$$q \mapsto \{v \to q v \bar{q}\} \in SO(\mathrm{Im}\,\mathbb{H}) \simeq SO(3),$$

because: $v \to q v \bar{q}$ is an isometry of $\mathbb{H}$, since $|q v \bar{q}| = |v|$. Moreover, $\varphi(q)(1) = 1$, hence $\varphi(q)$ preserves $(\mathbb{R}\cdot 1)^\perp = \mathrm{Im}(\mathbb{H})$ and lies in $SO(3)$ (since $Sp(1)$ is connected.) The center of $Sp(1)$ is $\{\pm 1\}$, so $\mathrm{Ker}(\varphi) = \{\pm 1\}$. Since both groups are 3-dimensional, φ is covering $\Rightarrow SO(3) = Sp(1)/\{\pm 1\}$.

2/ Analogously,

$$\psi : Sp(1) \times Sp(1) \to SO(4)$$
$$(q, r) \mapsto \{v \to q v \bar{r}\} \in SO(\mathbb{H}) \simeq SO(4),$$

and $\mathrm{Ker}(\psi) = \{\pm(1,1)\}$. $\square$

③ More difficult to describe are the following 2-fold covers.

<u>Lemma 4</u>: 1/ $SL(4,\mathbb{C})$ is a 2-fold cover of $SO(6,\mathbb{C})$.
2/ $Sp(2)$ — " — $SO(5)$.
3/ $SU(4)$ — " — $SO(6)$.

<u>Pf</u>: $\mathbb{C}^4$, standard Hermitean inner product $\langle -,- \rangle$.
It induces Hermitean inner product on $\Lambda^2\mathbb{C}^4 \simeq \mathbb{C}^6$:

$$\langle v_1\wedge v_2, w_1\wedge w_2\rangle := \det\left(\langle v_i, w_j\rangle_{i,j=1,2}\right)$$

If $A\in GL(4,\mathbb{C})$, we define

$$\Lambda^2 A : \Lambda^2\mathbb{C}^4 \to \Lambda^2\mathbb{C}^4$$
$$(\Lambda^2 A)(v\wedge w) := (Av)\wedge(Aw).$$

If $A\in U(4)$, $\Lambda^2 A\in U(6)$.

There is a bilinear form α on $\Lambda^2\mathbb{C}^4\simeq\mathbb{C}^6$.

$$\alpha : \Lambda^2\mathbb{C}^4\times\Lambda^2\mathbb{C}^4 \to \Lambda^4\mathbb{C}^4\simeq\mathbb{C}$$
$$(u, v)\mapsto u\wedge v,$$

which is symmetric. It is clearly non-degenerate and so the matrices preserving α are in $SO(6,\mathbb{C})$. If $A\in SL(4,\mathbb{C})$, then

$$\alpha\left((\Lambda^2A)u, (\Lambda^2A)v\right) = (\Lambda^2A)u\wedge(\Lambda^2A)v = (\Lambda^4A)(u,v)$$
$$= (\det A)\, u\wedge v = u\wedge v = \alpha(u,v),$$

as claimed. We have the map

$$\psi : SL(4,\mathbb{C}) \to SO(6,\mathbb{C})$$
$$A\mapsto \Lambda^2 A,$$

which is a homomorphism since

$$\Lambda^2(AB)(v\wedge w) = ABv\wedge ABw = (\Lambda^2A)(Bv\wedge Bw)$$
$$= (\Lambda^2A)(\Lambda^2B)(v\wedge w).$$

If $A\in \operatorname{Ker}(\psi)$, then $Au\wedge Av = u\wedge v \quad \forall u,v\in\mathbb{C}^4$,

(4)

which implies that A preserves planes and hence lines as well. Thus $Ae_i = \pm 1 \cdot e_i$ $\forall i = 1, \dots, 4$, so $A = \pm Id$. Then $Ker(\Psi) = \{\pm Id\}$ $\Rightarrow$ Ψ is 2-fold cover since both have the same dimension.

The proof of 2/, 3/ is analogous. $\square$

1 $\boxed{\text{Exercise : Lie group with bi-invariant metrics}}$

Motivating question: $G \ldots$ a Lie group, $\exp : T_e G \to G$

$$X \mapsto \exp(X) = \rho(1),$$

for $\rho\ \mathbb{R} \to G$ the unique 1-parameter subgroup of G (i.e., $\rho(s+t) = \rho(s)\rho(t)$, $\rho(0) = e$, $\rho'(0) = X$.)

On the other hand, if G is equipped with a Riem. metric $\langle\ ,\ \rangle$, then for $\forall p \in G$ $\exists$ Riemannian exponential map $\mathrm{Exp}_p : T_p G \to G$

$$X \mapsto \mathrm{Exp}_p(X) = \gamma(1),$$

where $\gamma : \mathbb{R} \to G$ is the unique geodesic with $\gamma(0) = p$, $\gamma'(0) = X$.

The basic result says that once G is a Lie group with a bi-invariant metric (e.g., G is compact), then $\exp(X) = \mathrm{Exp}_e(X)$ for all $X \in T_e G$, $e \in G$. There is 1:1 correspondence between Ad-invariant inner products on $T_e G$ and bi-invariant Riemann. metrics on G.

<u>Lemma</u>: For a Lie group with a bi-invariant metric, the map $i : G \to G$, $g \mapsto g^{-1}$ is an isometry. We have $\underset{(i_e)_*}{\underset{\parallel}{di_e}}(X) = -X$

<u>Pf</u>: $X \in T_e G$, at $T_{g^{-1}}G$ have the equality

$$di(X) = -(dL_g)^{-1}(dR_g)^{-1}X.$$

This holds because for 1-par. subgroup γ of G, given by $\gamma'(0) = X$, $e = \gamma(t)\gamma(t)^{-1}$, the differentiation gives

$$0 = dR_{g^{-1}}(X) + dL_g(di(X)),$$

(2) and as $dR_{g^{-1}} = (dR_g)^{-1}$, the claim is proved. Now

$$\langle di(X), di(X')\rangle_{g^{-1}} = \langle -dL_g^{-1} dR_g^{-1} X, -dL_g^{-1} dR_g^{-1} X'\rangle_{g^{-1}}$$

for all $X, X' \in T_e G$. $\qquad = \langle X, X'\rangle_g$

<u>Lemma</u>: If G is a Lie group with a bi-invariant metric, ∇ its Levi-Civita connection. Then for $\forall$ left-invariant vector fields X, Y on G, $\nabla_X Y = \frac{1}{2}[X, Y]$, where $[\ ,\ ]$ is a Lie bracket.

<u>Pf</u>: Firstly, $\nabla_X X = 0$ at $e \in G$, because the integral curve of X is $\gamma(t) = \exp(tX)$, and this is just a geodesic (by previous exposition), so $\nabla_X X = \nabla_{\gamma'}\gamma' = 0$.

For $g \in G$, L_g is an isometry and so

$$\nabla_{X_g} X = \nabla_{(dL_g)X} X = (dL_g)\nabla_{X_e} X = 0.$$

For left-invariant X, Y,

$$0 = \nabla_{X+Y}(X+Y) = \nabla_X Y + \nabla_Y X,$$

and this together with the torsion-free property,

$$0 = T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$ the result follows. $\square$

Recall that $Ad_g \in Aut(T_e G)$ is defined by

$$Ad_g = (dL_g)(dR_{g^{-1}}),$$

and the tangent map $ad_g := d(Ad_g)$ is a linear map

$$ad : T_e G \to End(T_e G)$$
$$X \mapsto ad_X$$

$$ad_X : Y \mapsto [X, Y] = ad_X(Y).$$

③ The Jacobi identity for $X, Y, Z \in \mathfrak{X}(M)$ three vector fields on M states

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

means that the Lie derivative $L_X Y = [X,Y]$ is a derivative:

$$L_X [Y,Z] = [L_X Y, Z] + [Y, L_X Z].$$

The curvature of bi-invariant metric is

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$= \nabla^2 Z\,(X,Y) - \nabla^2 Z(Y,X).$$

<u>Lemma</u>: For X, Y, Z invariant vector fields on a Lie group G with biinvariant metric, we have

$$R(X,Y)Z = -\tfrac{1}{4}[[X,Y],Z].$$

<u>Pf</u>: By invariance of the Lie bracket, $[X,Y]$ is invariant as well. We compute by previous lemma:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$= \tfrac{1}{2}\nabla_X [Y,Z] - \tfrac{1}{2}\nabla_Y [X,Z] - \tfrac{1}{2}[[X,Y],Z]$$

$$= \left(\tfrac{1}{4}[X,[Y,Z]] - \tfrac{1}{4}[Y,[X,Z]]\right) - \tfrac{1}{2}[[X,Y],Z]$$

$$= \tfrac{1}{4}[[X,Y],Z] - \tfrac{1}{2}[[X,Y],Z]$$

$$= -\tfrac{1}{4}[[X,Y],Z],$$

where we used the Jacobi identity. $\square$

(4)

For G with bi-invariant metric, for ON vectors $X, Y \in T_e G$, by the left invariance of the metric the uniquely extended vector fields to G (by left-invariance) are still ON at each point of G and span a 2-plane Π_{XY} in $T_g G$; the sectional curvature is

$$K(X, Y) = \langle R(X,Y)Y, X \rangle.$$

Lemma: G is assumed to fulfill the same conditions as in the last Lemma. Let $X, Y \in T_e G$ ON-vectors, extended to G as left-invariant vector fields. Then at $\forall$ point:

$$K(X, Y) = \frac{1}{4} |[X, Y]|^2.$$

Pf: We know from the previous Lemma:

$$\langle R(X,Y)Y, X \rangle = -\frac{1}{4} \langle [[X,Y],Y], X \rangle.$$

But

$$\langle [[X,Y],Y], X \rangle = \langle [Y,X], [X,Y] \rangle = -|[X,Y]|^2;$$

and the proof is complete. □

Examples

Example: (The orthogonal group):

Let $S = O(n) = \{g \in Mat(n,n,\mathbb{R}) \mid g^T g = Id_n\}$. Since Id_n is the regular value of the smooth map $x \mapsto x^T x$ from $Mat(n,n,\mathbb{R})$ to $S(n) = Sym\,Mat(n,n,\mathbb{R})$ (symmetric matrices of rank n.) The Riemannian metric on $O(n)$ is induced by scalar product on $Mat(n,n,\mathbb{R})$:

$$\langle x, y \rangle = Tr(x^T y) = \sum_{i,j} x_{ij} y_{ij},$$

which is bi-invariant for $O(n)$:

$$\langle gx, gy \rangle = Tr(x^T g^T g y) = Tr(x^T y) = \langle x, y \rangle,$$

$$\langle xg, yg \rangle = Tr(g^T x^T y g) = Tr(g^{-1} x^T y g) = Tr(x^T y) = \langle x, y \rangle.$$

The right/left multiplication with $g \in O(n)$ acts as isometries of $O(n)$ making it a homogeneous space.

Consider the linear map $s_I(x) = x^T$,

$$s_I : Mat(n,n,\mathbb{R}) \to Mat(n,n,\mathbb{R}).$$

This map fulfills
$$\cancel{Id}_n \longrightarrow Id_n$$
$$X \mapsto -X \quad \forall X \in \underset{\{X \in Mat(n,n,\mathbb{R}) \mid X + X^T = 0_n\}}{\underset{\shortparallel}{T_{Id_n} O(n)}}.$$

The symmetry at $g \in O(n)$ is given by

$$s_g(X) = g(s_I(g^T X)) = g X^T g,$$

because

$$s_g(g) = g, \quad \forall g_* X \in T_g O(n) = g_* T_{Id_n} O(n) \Rightarrow$$
$$ds_g(g_* X) = s_g(g_* X) = g X^T = -g_* X.$$

Analogously for
$$\left.\begin{aligned} U(n) &= O(2n) \cap Mat(n,n,\mathbb{C}) \\ Sp(n) &= O(4n) \cap Mat(n,n,\mathbb{H}) \end{aligned}\right\} \; x^T \ldots \; x^* = \overline{x}^T!$$

Example (Complex stuctures on $\mathbb{R}^n$):

Let S be the set of orthogonal complex structures on $\mathbb{R}^n$, $n=2m$.
So $S = \{ J \in O(n),\ J^2 = -Id_n \Leftrightarrow J^{-1} = -J \}$. From the orthogonality we also have $J^{-1} = J^T$, so $S = \{ J \in T_{Id_n} O(n),\ J^2 = -Id_n \}$.

Since any $J \in S$ is an orthogonal matrix with eigenvalues $+i, -i$ of multiplicity $m = \frac{n}{2}$, all J's are conjugate and S is a conjugate class $=$ the orbit under $O(n)$ by conjugation. The isotropy group of the standard complex structure $J = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ & & \ddots \end{pmatrix}$ of $\mathbb{R}^{2m} = \mathbb{C}^m$ is $U(m) \subseteq O(n)$.
It is characterized as a fixed point set of an isometry on $Mat(n,n,\mathbb{R})$, $X \mapsto -X^T$, preserving $O(n)$. The differential of the defining map $F: J \mapsto J^2 + Id_n$ is not surjective (by dimension reason), so S is not a regular preimage. We have $T_J S = Ker(dF_J) = \{ X \in T_{Id_n} O(n) \mid XJ + JX = 0_n \}$, which is complementary to the subspace $T_{Id_n} U(m) \subseteq T_{Id_n} SO(n)$ (containing the elements commuting with J.) The symmetry s_J is the conjugation with J: it preserves J and maps $X \in T_J S$ to $-X$ since $JXJ^{-1} = -XJJ^{-1} = -X$.

The space of quaternion structures on $\mathbb{C}^{2m}$ is treated analogously – it is embedded to the space of anti-hermitean complex $n \times n$ matrices, and is an orbit of $U(2m)$ under conjugation. Its isotropy subgroup is $Sp(m) \subseteq U(2m)$.

Example (Real structures on $\mathbb{C}^n$)

Now S denotes the set of real structures on $\mathbb{C}^n$. A real structure on $\mathbb{C}^n = \mathbb{R}^{2n}$ is a reflection $\underline{r}$ at a totally real subspace $E \subseteq \mathbb{C}^n$ of half-dimension (totally real means $iE \perp E$ in the Hermitean scalar product on $\mathbb{C}^n$) In other words, the reflection $\underline{r}$ is complex anti-linear, i.e. anti-commutes with the (standard) complex structure i. If a reflection anti-commutes with i then i interchanges its $+1$ and -1 eigenspaces E_+ and E_-. Since any reflection is a symmetric operator, S is a subset of $\mathrm{Sym\,Mat}(2n, 2n; \mathbb{R})$ given by intersection with anti-linear maps on $\mathbb{C}^n$.

An ON-basis of E is a unitary basis of $\mathbb{C}^n$ and the real span of a unitary basis is an n-dimensional totally real subspace of $\mathbb{C}^n$. Thus $U(n)$ acts transitively on S and S is the orbit of the (standard) complex conjugation $X \longrightarrow \overline{X}$ in $\mathbb{C}^n$ (corresponding to standard $\mathbb{R}^n \subseteq \mathbb{C}^n$.) The isotropy group of (standard) complex conjugation is $O(n) \subset U(n)$, hence $S = U(n)/O(n)$.

Denote $\mathrm{Sym\,Mat}(2n, 2n, \mathbb{R})_-$ the intersection of $\mathrm{Sym\,Mat}(2n, 2n, \mathbb{R})$ with the space of anti-linear maps on $\mathbb{C}^n$.

The map F: $\mathrm{Sym\,Mat}(2n, 2n, \mathbb{R})_- \to \mathrm{Sym\,Mat}(2n, 2n, \mathbb{R})_-$

$$X \mapsto F(X) = X^T X - \mathrm{Id}_{2n},$$

which defines S, fulfills

$$\mathrm{Ker}(dF_{\underline{r}}) = \{ X \in \mathrm{Sym\,Mat}(2n, 2n, \mathbb{R})_- \mid X\underline{r} + \underline{r}X = 0 \}.$$

Thus $X \in \mathrm{Ker}(dF_{\underline{r}})$ iff the $\mathbb{C}$-linear map $\underline{r}X$ is anti-symmetric (as a real matrix of rank $2n$), $(\underline{r}X)^T = X\underline{r} = -\underline{r}X$, hence $\underline{r}X \in T_{\mathrm{Id}_{2n}} O(2n) \cap \mathrm{Mat}(n, n, \mathbb{C})$.

Moreover, $\underline{r}X$ anti-commutes with $\underline{r}$, so it is purely imaginary with respect to the real structure $\underline{r}$. The purely imaginary matrices in $T_{Id_{2n}}U(n)$ form a complement to $T_{Id_{2n}}O(n)$, so $Ker(dF_{\underline{r}}) = T_{\underline{r}}S$ for dimensional reasons. The symmetry $s_{\underline{r}}$ is given by $s_{\underline{r}}(X) = \underline{r}X\underline{r}$ (it preserves $\underline{r}$ and acts by $-Id_{2n}$ on $T_{\underline{r}}S$.)

The space of complex structures on $\mathbb{H}^n$ is treated analogously, as a quotient it is $Sp(n)/U(n)$.

<u>Example (Positive definite matrices)</u>:

Let $S = SymMat(n,n,\mathbb{R})_{PD}$ be the set of positive definite real symmetric $n\times n$-matrices, an open subset of $SymMat(n,n,\mathbb{R})$. There is a Riemannian metric on S: $X, Y \in T_p S \cong S$,

$$\langle X, Y\rangle_p = Trace(Xp^{-1}Yp^{-1}) = Trace(p^{-1}Xp^{-1}Y).$$

The group $G = GL(n,\mathbb{R})$ acts on S by $g\cdot p := gpg^T$, and this action is isometric for $\langle\, ,\rangle$:

$dg_p(X) = gXg^T$, hence

$$\begin{aligned}\langle dg_p(X), dg_p(Y)\rangle_{g(p)} &= Tr\left(gXg^T(gpg^T)^{-1}gYg^T(gpg^T)^{-1}\right)\\ &= Tr\left(gXp^{-1}Yp^{-1}g^{-1}\right)\\ &= \langle X, Y\rangle_p.\end{aligned}$$

Since any $p\in S$ can be written as $p = g^Tg = g\cdot I$ for some $g\in G$, this action is transitive and the isotropy group of $Id_n \in S$ is $O(n)$.

(5)

The inversion $s_{Id_n}(p) = p^{-1}$ is an isometry of S. Since $(ds_{Id_n})(X) = -p^{-1}Xp^{-1}$, we have

$$\langle (ds_{Id_n})_p(X), (ds_{Id_n})(Y)\rangle_{p^{-1}} = Tr(p^{-1}Xp^{-1}pp^{-1}Yp^{-1}p)$$
$$= Tr(p^{-1}Xp^{-1}Y)$$
$$= \langle X, Y\rangle_p$$

$\forall X, Y \in T_pS$. Since s_{Id_n} preserves Id_n and acts as $-Id_n$ on $T_{Id_n}S$, it is the symmetry at Id_n. The symmetry at $p \in S$ is $s_p(q) = pq^{-1}p$.

⑥ <u>Example</u>:

Another way how to describe the decomposition $\mathfrak{g} \simeq \mathfrak{k} \oplus \mathfrak{p}$ (the Cartan decomposition) is captured by the structure of Lie triple systems.
A Lie triple system is an euclidean vector space with a triple product $(X,Y,Z) \mapsto R(X,Y)Z$, skew-symmetric in X, Y and satisfies Bianchi identity for which $R(X,Y)$ are skew-adjoint derivations. The example we have in mind is for $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ given by $\mathfrak{p}$ (an euclidean space) and $R(X,Y)Z := [Z,[X,Y]]$.
Bianchi identity is the Jacobi identity for $\mathfrak{g}$, and $X, Y, Z \in \mathfrak{p}$.

Since $Ad(K)$ acts by automorphisms on the Lie algebra $\mathfrak{g}$ preserving $\mathfrak{p}$, $ad(\mathfrak{k})$ acts by skew-adjoint derivations, hence $R(X,Y) := -ad(\underbrace{[X,Y]}_{\in \mathfrak{k}}) = -ad(\quad)$ is a skew-adjoint derivation.

Consider the complex projective space,

$$S = \mathbb{CP}^n = G_1(\mathbb{C}^{n+1}) = U(n+1) / (U(1) \times U(n)).$$

The Lie algebra of $G = U(n+1)$ is the space of anti-hermitean complex $(n+1) \times (n+1)$ matrices. The Cartan involution on $\mathfrak{g}$ is the conjugation with $\begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}$, hence

$$\mathfrak{p} = \left\{ \tilde{X} \mid \tilde{X} = \begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix}, X \in \mathbb{C}^n, X^* = \overline{X^T} \right\}.$$

$\forall\, \tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{p}$ (regarded as matrices via ( )), we obtain $R(\tilde{X},\tilde{Y})\tilde{Z} \in \mathfrak{p}$ is $R(\tilde{X},\tilde{Y})\tilde{Z} = [\tilde{Z}, \underbrace{[\tilde{X},\tilde{Y}]}_{\in \mathfrak{p}}]$

(7)

$r(X,Y)Z \in \mathbb{C}^n$ corresponding to $R(\tilde{X},\tilde{Y})\tilde{Z}$ is

$$r(X,Y)Z = Z(Y^*X - X^*Y) + X(Y^*Z) - Y(X^*Z) \in \mathbb{C}^n$$

where $Y^*Z \in \mathbb{C}$ is the Hermitean scalar product on $\mathbb{C}^n$.

In particular, for $Y = Z$ we get

$$r(X,Y)Y = Y(Y^*X - 2X^*Y) + X(Y^*Y),$$

and for Y a unit vector, $Y^*Y = 1$, and X perpendicular to Y with respect to the real scalar product $\langle X, Y\rangle = \mathrm{Re}(X^*Y)$, we have $Y^*X = -X^*Y = i\langle X, iY\rangle$ $\Rightarrow$

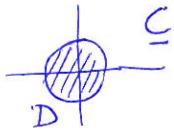
$$r(X,Y)Y = X + 3\langle X, iY\rangle iY.$$

Thus the eigenvalues of $r(-,Y)Y$ on $Y^\perp$ are

- 1 on $\langle Y, iY\rangle^\perp$,
- 4 on $\mathbb{R}iY$,

and so the sectional curvature of $\mathbb{C}P^n$ varies between 1 and 4.

Example / Problem:



$\subseteq \mathbb{C}$ the unit disk in $\mathbb{C}$,

$D := \{ w := x + iy \in \mathbb{C} \mid w\bar{w} \leq 1 \}$.

D equipped with a metric $\Rightarrow$ the realization of hyperbolic space

$$D \to D$$

$$w \mapsto w' := \frac{\alpha w + \beta}{\bar{\beta} w + \bar{\alpha}}, \quad \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \in SU(1,1),$$

$$\bar{\alpha}\alpha - \bar{\beta}\beta = 1.$$

Show

1/ $M, -M \in SU(1,1)$ induce identical mapping on D,

2/ $w = e^{i\varphi} \mapsto w' = e^{i\psi}$ (compute $\psi = \psi(\varphi)$),

3/ The metric

$$g = \frac{1}{(1-\bar{w}w)^2} dx^2 + \frac{1}{(1-\bar{w}w)^2} dy^2$$

is invariant under $SU(1,1)$.

4/ The distance between $w_1, w_2 \in D$ is

$$d(w_1, w_2) = \tanh^{-1}\left(\frac{|w_1 - w_2|}{|1 - w_1 \bar{w_2}|} \right).$$

~~5/ The map $\mathbb{H} \to D$ $\mathbb{H}$ .. upper half space~~

~~$z \mapsto w = e^{i\varphi} \frac{z - z_0}{z - \bar{z}_0}$, $z_0 \in \mathbb{H}$ fixed~~

5/ Invariant metric g induces invariant measure on D – – write it.

① Exercises / Problems:

1/ G ... a Lie group, $K \leq G$ a closed (Lie) subgroup, $Lie(G) = \mathfrak{g}$, $Lie(K) = \mathfrak{k}$. If $\mathfrak{m}$ is a linear complement of $\mathfrak{k}$ in $\mathfrak{g}$, $\mathfrak{g} \simeq \mathfrak{k} \oplus \mathfrak{m}$, and giving to G/K the quotient topology, let $\pi : G \to G/K$ the projection. Then $\pi \circ \exp : \mathfrak{m} \to G/K$ is a local homeomorphism at $0 \in \mathfrak{m}$. Prove it.

2/ For $G = SO(n+1)$, $K = SO(n)$ $\Rightarrow$ $SO(n+1)/SO(n)$ has the manifold structure for which

$$SO(n+1) \times SO(n+1)/SO(n) \longrightarrow SO(n+1)/SO(n)$$

$$(A_1, A_2 K) \longmapsto A_1 A_2 K \quad , \quad A_1, A_2 \in SO(n+1)$$

is smooth. Find the linear complement $\mathfrak{m}$ (see 1/).

3/ Consider the complex projective space, $\mathbb{CP}^n \simeq U(n+1)/(U(1) \times U(n))$. Find the linear complement $\mathfrak{m}$.

4/ At $p_0 \in \mathbb{R}^n$, define the involution

$$\sigma_{p_0} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$x \longmapsto -(x - p_0) + p_0$$

Consider a geodesic passing through p_0 and given by $v \in T_{p_0}\mathbb{R}^n \simeq \mathbb{R}^n$. Compute transvection and verify its translation property.

5/ Consider the sphere $S^n_r = \{ x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = r \}$ in $\mathbb{R}^{n+1}$. Prove that

$$\sigma_{p_0} : S^n_r \longrightarrow S^n_r$$

$$x \longmapsto 2 \frac{\langle p_0, x \rangle}{r} p_0 - x$$

is the symmetry at $p_0 \in S^n_r \subseteq \mathbb{R}^{n+1}$.

② Exercises/Problems:

In general, for a submanifold $M \subseteq (\mathbb{R}^n, g)$, $n \in \mathbb{N}$ and $M = \{x \in \mathbb{R}^n \mid g(x,x) = r\}$, we have

$$\sigma_{p_0} : M \to M$$
$$x \mapsto \frac{2g(p_0, x)}{r} p_0 - x.$$

6/ $G = SO(3)$, $K = SO(2)$, $M = G/K \simeq S^2$. Consider

$p_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in S^2$, $X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{p}$, $K = \begin{pmatrix} 1 & 0 \\ 0 & (SO(2)) \end{pmatrix}$.

Compute: a/ geodesic starting at p_0 in the direction X,

b/ —"— in arbitrary direction.

7/ Let $G = O(n)$, $K = O(k) \times O(n-k)$, $J = \begin{array}{c} \begin{matrix} k & n-k \end{matrix} \\ \begin{pmatrix} Id_k & 0 \\ 0^T & -Id_{n-k} \end{pmatrix} \end{array} \begin{matrix} k \\ n-k \end{matrix}$

Prove a/ $J \in Z(K)$,

b/ the Cartan decomposition $\mathfrak{g} \simeq \mathfrak{k} \oplus \mathfrak{p}$.

8/ (G, K) the symmetric pair $(SO(p+q), SO(p) \times SO(q))$, involutive automorph. $S: A \to SAS^{-1}$, $S = \begin{pmatrix} Id_p & 0 \\ 0 & -Id_q \end{pmatrix}$

on G/K with Ad_G-invariant metric $B(X,Y) = -\frac{1}{2}\operatorname{Trace}(XY)$ at the origin.

on (G^*, K^*), the dual symm. pair $(SO_0(p,q), SO(p) \times SO(q))$, $K^* = K$,

we take the involutive autom. $J \in Z(K)$, $J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$,

on $G^*/K^* = M^*$ take the metric $B^*(X,Y) = \frac{1}{2}\operatorname{Trace}(XY)$ at $e \in G^*$,

B^* is pos. def. on $\mathfrak{p}^*$, $\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*$.

Let

③ Exercises / Problems:

$\varphi : \mathfrak{k} \to \mathfrak{k}^*$ be $Id_{\mathfrak{k}=\mathfrak{k}^*}$, fulfilling $B(X,Y) = -\frac{1}{2}\operatorname{Trace}(XY)$

Lie alg. isom.

$= -(+\frac{1}{2}\operatorname{Trace} XY)$

$\forall X, Y \in \mathfrak{k} = \mathfrak{k}^*$. $= -B^*(X,Y)$

Prove: $\hat{\varphi} : \mathfrak{p} \to \mathfrak{p}^*$

$$\begin{pmatrix} 0 & X \\ -X^T & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}$$

is an isometry fulfilling $[\hat{\varphi}(X), \hat{\varphi}(Y)]^* = -\varphi([X,Y])$
$\forall X, Y \in \mathfrak{p}$.