7 The first fundamental theorem for Sp_{2m}

Th ● (FFT for Sp_{2m}): The algebra $K[V^p]^{Sp_{2m}}$ is generated by the invariants $\langle i|j\rangle$, $1\leq i<j\leq p$.

Pf: The proof is based on the Weyl theorem, since $Sp_{2m} < SL_{2m}$, the determinants $[v_1, \dots, v_{2m}]$ are contained in $K[\langle i|j\rangle]$. Therefore, it is sufficient to consider the case $p = 2m-1$, and to assume K alg. closed and proceed by induction on m.

(1)

Simultaneous conjugation of 2×2-matrices : $K=\mathbb{C}$ (alg. closed)

The representation of SL_2 on $sl_2 := \{A\in M_2 \mid Tr A = 0\}$ by conjugation leaves invariant the non-degenerate quadratic form $q(A):=\det A - \frac{1}{2}Tr(A$

The $\mathbb{C}$-basis of sl_2 is $E_1 := \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}$, $E_2 := \begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}$, $E_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ gives $q(x_1E_1+x_2E_2+x_3E_3) = x_1^2+x_2^2+x_3^2$, so $Im(SL_2 \to GL(sl_2)=GL_3)$ is contained in SO_3. In fact, $SL_2(\mathbb{C})/\{\pm 1\} \simeq SO_3(\mathbb{C})$. A consequence of FFT for SO_3 is: invariants of $\forall$ number of copies of sl_2 (under simultaneous conjugation) are generated by the quadratic traces $Tr_{ij} : (A_1,\dots,A_p) \mapsto Tr(A_iA_j)$

$$(A_1,\dots,A_p)\in \underbrace{sl_2\times\dots\times sl_2}_{p}$$

and cubic traces

$$Tr_{ijk} : (A_1,\dots,A_p) \longmapsto Tr(A_iA_jA_k) :$$

$$K[sl_2^{\,p}]^{SL_2} = K[Tr_{ij}, Tr_{ijk} \mid 1\le i\le j\le k\le p].$$

The invariants of several copies of 2×2 matrices under simultaneous conjugation are generated by the traces Tr_{ij}, Tr_{ijk} together with standard traces $Tr_i : (A_1,\dots,A_p)\mapsto Tr(A_i)$.

Claim: $K[M_2^{\,p}]^{GL_2} = K[Tr_i, Tr_{ij}, Tr_{ijk} \mid 1\le i\le j\le k\le p]$.

Every trace function $Tr_{i_1,\dots,i_k}$ for $k>2$ can be expressed as a polynomial in the traces Tr_i, Tr_{ij}.

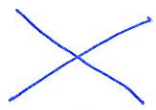
1 Basic examples of classical invariants

Affine/plane conics (quadrics) in $\mathbb{R}^2/\mathbb{C}^2$ ($\mathbb{F}=\mathbb{R},\mathbb{C}$):

A quadric ~~in~~ $\mathbb{R}$ (or, $\mathbb{C}$)-affine space of dimension 2 is the zero locus of

$$ax^2+2bxy+cy^2+2dx+2ey+f=0.$$

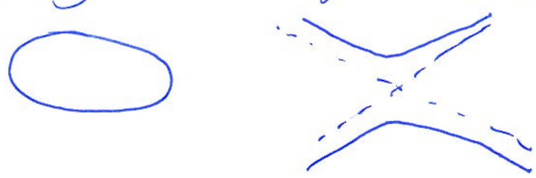
If it is product of linear forms, the curve is degenerate (conic):



line pair

double line

, otherwise they are non-degenerate conics



The previous equation can be rewritten as

$$(x,y,1)\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}=0. \qquad (*)$$

Def 1: Affine quadrics (conics, curves) are equivalence classes of equations (*), where two representatives differ by the action of an element of Euclidean transformation group:

$$X=\begin{pmatrix} p & q & l \\ -q & p & m \\ 0 & 0 & 1 \end{pmatrix}\in \mathrm{Eucl}_3, \quad p,q,l,m\in\mathbb{F}$$

$$p^2+q^2=1$$

$$\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}\longmapsto X^T\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}X.$$

$\mathrm{Eucl}_3 < GL_3(\mathbb{F})$ is not reductive (simple): it contains the normal subgroup of translations $\begin{matrix} x\mapsto x+l \\ y\mapsto y+m\end{matrix}$ (isomorphic to $\mathbb{F}^2$). In fact, $\mathrm{Eucl}_3\simeq SO_2(\mathbb{F})\ltimes\mathbb{F}^2$

② The 6-dim vector space of 3×3 symmetric matrices is a repr. of $Eucl_3$, and the question is its ring of invariants ($\mathbb{F}[S^3\mathbb{F}]^{Eucl_3}$).

$S^3\mathbb{F} \equiv$ symmetric matrices 3×3 over $\mathbb{F}$

$\mathbb{F}[a,b,c,d,e,f]^{Eucl_3}$

$X \in Eucl_3$: $\det X = 1$ $\Rightarrow$ invariant of $? \to X^T ? X$ is

$$D := \det\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix},$$

$D \neq 0$ iff the curve is non-degenerate (D is called discriminant of the quadric)

Moreover, $T := a+c$ and $E := ac - b^2$ are also $Eucl_3$ invariants.

Th 2: The $\mathbb{F}$-algebra $\mathbb{F}[S^3\mathbb{F}]^{Eucl_3}$ is a subalgebra of $\mathbb{F}[a,b,c,d,e,f]$, and generated by D, T, E. Moreover these elements are algebraically independent:
$\mathbb{F}[S^3\mathbb{F}]^{Eucl_3} \simeq \mathbb{F}[D,T,E]$.

Pf: Let $\begin{pmatrix} 1 & 0 & l \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} < Eucl_3$ be the subgroup of translations
$\quad = T_r \qquad Eucl_3/T_r \xrightarrow{\sim} O(2,\mathbb{F})$

It is enough to show that $\mathbb{F}[S^3\mathbb{F}]^{T_r} = \mathbb{C}[a,b,c,D]$, because $\mathbb{F}[a,b,c]^{O(2,\mathbb{F})}$ is generated by T and E (as we already know from the last lecture.)

Claim: $\mathbb{F}[a,b,c,d,e,f,\frac{1}{E}]^{T_r} = \mathbb{F}[a,b,c,D,\frac{1}{E}]$
(this then implies the former claim.)

We have $D = Ef + (2bde - ae^2 - cd^2)$, so that

$$f = \frac{D + ae^2 + cd^2 - 2bde}{E}, \text{ hence}$$

$$\mathbb{F}[a,b,c,d,e,f,\tfrac{1}{E}] = \mathbb{F}[a,b,c,d,e,D,\tfrac{1}{E}].$$

An element in this ring T_r-invariant fulfills

$$p(a,b,c,d+al+bm,e+bl+cm,D) = p(a,b,c,d,e,D)$$

for arbitrary $(l,m) \in \mathbb{F}\times\mathbb{F}$. Taking

- $(l,m) = (-bt, at) \Rightarrow p$ canot have terms involving e,
- $(l,m) = (-ct, bt) \Rightarrow$ —— " —— d.

This proves the claim, as well as Th 2. ▨

<u>Classical binary invariants</u>: (assume $\mathbb{F} = \mathbb{C}$.)

$f(x) \in \mathbb{C}[x]_d \quad f(x) = a_0 x^d + a_1 x^{d-1} + \dots + a_{d-1}x + a_d = a_0 \prod_{i=1}^{d} (x - \lambda_i)$,

$g(x) \in \mathbb{C}[x]_e \quad g(x) = b_0 x^e + b_1 x^{e-1} + \dots + b_{e-1}x + b_e = b_0 \prod_{j=1}^{e} (x - \mu_j)$.

$(\deg f = d,\ \deg g = e)$, $a_0 b_0 \neq 0$

We define $\quad R(f,g) = a_0^e b_0^d \prod_{i,j} (\lambda_i - \mu_j)$, so the vanishing of $R(f,g) \Leftrightarrow f(x) = 0 = g(x)$ have common root

<u>Def / Lemma 3</u> (Resultant) $R(f,g)$ is equal to the $(d+e)\times(d+e)$ determinant

$$R(f,g) = \begin{vmatrix} a_0 & a_1 & \cdots & a_d & 0 & \cdots & \\ 0 & a_0 & a_1 & \cdots & a_d & \cdots & \\ & & \ddots & & & \ddots & 0 \\ 0 & & & a_0 & a_1 & \cdots & a_d \\ b_0 & b_1 & \cdots & \cdots & b_e & 0 & \cdots \\ 0 & b_0 & b_1 & \cdots & & b_e & 0 \cdots \\ & & \ddots & & & & \\ & & & b_0 & b_1 & \cdots & b_e \end{vmatrix}$$

and is called the resultant of f and g.

<u>Pf</u>: Claim: $f,g \in \mathbb{C}[x]$ posseces a non-constant common factor iff there $\exists$ non-zero $u,v \in \mathbb{C}[x]$, $\deg(u) < \deg(f)$ and $\deg(v) < \deg(g)$ s.t. $vf + ug = 0$.

(4) If f, g have a non-constant common factor, then $f = uh$, $g = vh$, $u, v, h \in \mathbb{C}[x]$, and these u, v have required properties. The opposite direction: $f = p_1^{r_1} \dots p_s^{r_s}$ the unique factorization of $f \in \mathbb{C}[x]$ into irreducibles. Then $\forall\, p_i^{r_i} \mid ug$, so p_i divides u or g. Since $\deg u < \deg f$, some p_i must divide g, and this proves the claim.

$\dim \mathbb{C}[x]_d \equiv$ v.sp. of pol. deg $\leq r$, $\dim = r$, basis $1, x, \dots, x^r$. The preceding claim is equivalent to

$$\rho_{f,g} : \mathbb{C}[x]_{n-1} \oplus \mathbb{C}[x]_{m-1} \to \mathbb{C}[x]_{m+n-1}$$
$$(u, v) \mapsto vf + ug,$$

saying f, g have a non-constant common factor iff $\rho_{f,g}$ has non-zero kernel. Since dim of the source and target vector spaces are equal $(= m+n)$, this is equivalent to $\det \rho_{f,g} = 0$. The standard basis argument shows that $\det \rho_{f,g}$ equals $R(f,g)$, which proves the claim. ▨

For $f(x) \in \mathbb{C}[x]$, $\lambda_1, \dots, \lambda_d$ its roots, $D(f) := a_0^{2d-1} \prod\limits_{1 \leq i \neq j \leq d} (\lambda_i - \lambda_j)$ is called discriminant of f.

Lemma 4: The discriminant of $f(x) \in \mathbb{C}[x]$ is equal to $R(f, f')$, $\leftarrow \frac{\partial f}{\partial x}(x)$

$$D(f) = R(f, f').$$

Pf: $g(x) := f(x+\epsilon) = a_0' x^d + a_1' x^{d-1} + \dots + a_{d-1}' x + a_d'$, $a_0' = a_0$, $a_i' = a_i'(\epsilon)$.

By Def/Lemma 3, $a_0^{2d} \prod\limits_{i,j} (\lambda_i + \epsilon - \lambda_j) =$

$$= \begin{vmatrix} a_0 & a_1 & \dots & a_d & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_d & \dots \\ & & \ddots & & \ddots & \\ & & a_0 & a_1 & \dots & a_d \\ a_0 & a_1' & \dots & a_d' & & \\ & a_0 & a_1' & \dots & a_d' & \\ & & \ddots & & \ddots & \\ & & a_0 & a_1' & \dots & a_d' \end{vmatrix} = \epsilon^d \cdot \begin{vmatrix} a_0 & a_1 & \dots & a_d & \dots & \\ 0 & a_0 & a_1 & \dots & a_d & \dots \\ & & \ddots & & \ddots & \\ & & a_0 & a_1 & \dots & a_d \\ 0 & \frac{a_1'-a_1}{\epsilon} & \dots & \frac{a_d'-a_d}{\epsilon} & \dots & \\ & 0 & \frac{a_1'-a_1}{\epsilon} & \dots & \frac{a_d'-a_d}{\epsilon} & \dots \\ & & \ddots & & \ddots & \\ & & 0 & \frac{a_1'-a_1}{\epsilon} & \dots & \frac{a_d'-a_d}{\epsilon} \end{vmatrix}$$

(5) Cancellation of $a_0\epsilon^d$ and the limit $\epsilon \to 0$ show that

$$a_0^{2d-1} \prod_{\substack{i,j \\ i\neq j}} (\lambda_i - \lambda_j) = \begin{vmatrix} a_0 & a_1 & \cdots & a_d & & \\ & a_0 & a_1 & \cdots & a_d & \\ & & \ddots & & & \ddots \\ & & a_0 & a_1 & \cdots & a_d \\ da_0 & (d-1)a_1 & \cdots & a_{d-1} & & \\ & da_0 & (d-1)a_1 & \cdots & & \\ & & \ddots & & & \ddots \\ & & da_0 & (d-1)a_1 & \cdots & \end{vmatrix} = R(f, f') \qquad \blacksquare.$$

$[x:y] \in \mathbb{P}1$, homogeneous pol. $f(x,y) := \sum_{i=0}^{d} a_i \binom{d}{i} x^{d-i} y^i =$

$$= a_0 x^d + d a_1 x^{d-1} y + \binom{d}{2} a_2 x^{d-2} y^2 + \ldots$$

($a_0 \neq 0$: for $a_0 = 0$ is $\infty = [1:0]$ a zero of $f(x,y)$.)

A multiple root of f : common zero of $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$, i.e.

$R\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = 0$. We write the coefficients of f as $\xi_0, \ldots, \xi_d$:

$$f(x,y) = \sum_{i=0}^{d} \xi_i \binom{d}{i} x^{d-i} y^i.$$

__Def 5__: $\underset{\xi_0,\ldots,\xi_d}{\overset{\|}{D(\xi)}} := R\left(\frac{1}{d}\frac{\partial f}{\partial x}, \frac{1}{d}\frac{\partial f}{\partial y}\right) = \begin{vmatrix} \xi_0 & (d-1)\xi_1 & \cdots & \xi_{d-1} & & \\ & \xi_0 & (d-1)\xi_1 & \cdots & \xi_{d-1} & \\ & & \ddots & \ddots & & \ddots \\ & & \xi_0 & (d-1)\xi_1 & \cdots & \xi_{d-1} \\ \xi_1 & (d-1)\xi_2 & \cdots & \xi_d & & \\ & \xi_1 & (d-1)\xi_2 & \cdots & \xi_d & \\ & & \ddots & \ddots & & \ddots \\ & & \xi_1 & (d-1)\xi_2 & \cdots & \xi_d \end{vmatrix}$

is called the discriminant of the (degree d -) form $f = f(x,y)$.

__Example 6__: For the quadratic form $f(x,y) = \xi_0 x^2 + 2\xi_1 xy + \xi_2 y^2$ we get $D(\xi) = \begin{vmatrix} \xi_0 & \xi_1 \\ \xi_1 & \xi_2 \end{vmatrix} = \xi_0\xi_2 - \xi_1^2$.

__Example 7__: — ‖ — cubic form $f(x,y) = \xi_0 x^3 + 3\xi_1 x^2 y + 3\xi_2 x y^2 + \xi_3 y^3$

is

$$D(\xi) = \begin{vmatrix} \xi_0 & 2\xi_1 & \xi_2 & 0 \\ 0 & \xi_0 & 2\xi_1 & \xi_2 \\ \xi_1 & 2\xi_2 & \xi_3 & 0 \\ 0 & \xi_1 & 2\xi_2 & \xi_3 \end{vmatrix} = \xi_0^2\xi_3^2 - 3\xi_1^2\xi_2^2 - 3\xi_0\xi_1\xi_2\xi_3 + 4\xi_1^3\xi_3 + 4\xi_0\xi_2^3

⑥ We consider the action of $g \in GL_2$, $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, on forms $f(x,y)$:

$(x,y) \mapsto (\alpha x + \beta y, \gamma x + \delta y)$, we obtain $f(x,y) \to f(\alpha x + \beta y, \gamma x + \delta y)$, "$f(g\binom{x}{y})$",

$$f(g\cdot x) = \sum_{i=0}^{d} \xi_i(g) \binom{d}{i} x^{d-i} y^i, \quad \xi_i(g) = \sum_j \underbrace{\tilde{g}_i^{\,j}(\alpha,\beta,\gamma,\delta)}_{\text{homogeneous of degree } d \text{ in } \alpha,\beta,\gamma,\delta} \xi_j$$

$$\xi(g) = (\xi_0(g), \dots, \xi_d(g))$$

<u>Proposition 8</u>: $\forall g \in GL_2$, the discriminant satisfies

$$D(\xi(g)) = \det(g)^d \cdot D(\xi).$$

<u>Pf</u>: $f(x,1) = \sum_{i=0}^{d} \xi_i \binom{d}{i} x^{d-i} = 0$ as $f \in \mathbb{C}(\xi_0, \dots, \xi_d)[x]$, denote its roots $\lambda_1, \dots, \lambda_d$. Thus $f(x,1) = \xi_0 \prod_{i=1}^{d} (x - \lambda_i y)\big|_{y=1}$, and its discriminant is $D(\xi) = \xi_0^{2d-2} \prod_{1 \le i,j \le d} (\lambda_i - \lambda_j)$. The transformation by $g \in GL_2$ gives

$$f(\alpha x + \beta y, \gamma x + \delta y) = \xi_0 \prod_{i=1}^{d} (\alpha x + \beta y - \lambda_i(\gamma x + \delta y)) =$$
$$= \left[\xi_0 \prod_{i=1}^{d} (\alpha - \lambda_i \gamma)\right] \prod_{j=1}^{d} \left(x - \frac{\delta \lambda_j - \beta}{-\gamma \lambda_j + \alpha} y\right)$$

and $g \in GL_2$ "$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$" transforms $\lambda_i - \lambda_j \to \frac{\delta \lambda_i - \beta}{-\gamma \lambda_i + \alpha} - \frac{\delta \lambda_j - \beta}{-\gamma \lambda_j + \alpha} = \frac{(\alpha\delta - \beta\gamma)(\lambda_i - \lambda_j)}{(\gamma\lambda_i - \alpha)(\gamma\lambda_j - \alpha)}$

and hence

$$D(\xi(g)) = \left[\xi_0 \prod_{i=1}^{d} (\alpha - \lambda_i \gamma)\right]^{2d-2} \prod_{1 \le i,j \le d} \frac{(\alpha\delta - \beta\gamma)(\lambda_i - \lambda_j)}{(\gamma\lambda_i - \alpha)(\gamma\lambda_j - \alpha)} = (\det g)^{d(d-1)} D(\xi).$$

In general, we have for any homog. pol. in $\xi_0, \dots, \xi_d$ the definition:

<u>Def 9</u>: $F = F(\xi_0, \dots, \xi_d)$ hom. polyn. satisfying $F(\xi(g)) = F(\xi)$, for all $g \in SL_2$. F is termed a classical binary invariant.

(7) The set of binary forms of degree d is vector space V_d of dim $= d+1$, GL_2 acting by $\xi \mapsto \xi(g)$. V_i, $i \in \mathbb{N}$, are irred. repr. of GL_2.

Classical binary invariant (of degree $e \in \mathbb{N}$) is an element of $\mathbb{C}[\xi_0, \ldots, \xi_d]_e^{SL(2,\mathbb{C})}$. Proposition 8 implies that the discriminant is a classical binary invariant of degree $2d-2$.