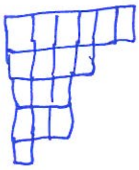
① Irreducible characters of GL_n and S_m

<u>Schur polynomials</u>: (some material is a review of the exposition for finite group invariant theory)

$\mathcal{P} := \{\lambda = (\lambda_1, \lambda_2, \dots) \mid \lambda_i \in \mathbb{N},\ \lambda_1 \geq \lambda_2 \geq \cdots,\ \lambda_i = 0 \text{ for } i \gg 0\}$

partitions, geometrically represented by Young diagrams



$\equiv (6,4,3,3,1)$

The highest weights $\sum_i \lambda_i \epsilon_i$ of irreducible polynomial representations of GL_n can be identified with the partitions $(\lambda_1, \lambda_2, \dots)$, $\lambda_i = 0$ for $i > n$.

<u>Def 1</u>: For $\lambda \in \mathcal{P}$ we define the height (length) of λ by

$$ht(\lambda) := \max\{i \mid \lambda_i \neq 0\} = \text{length of the first column of Young diagr. } \lambda$$

and the degree of λ by

$$|\lambda| := \sum_i \lambda_i = \#\text{ boxes in Young diagram } \lambda$$

<u>Example 2</u>: $\lambda = (6,4,3,3,1)$, $ht(\lambda) = 5$, $|\lambda| = 17$.

We have $\mathcal{P} = \bigcup_{m \geq 0} \mathcal{P}_m$, where $\mathcal{P}_m := \{\lambda \in \mathcal{P} \mid |\lambda| = m\}$

partition of m

Fix $n \in \mathbb{N}$, let $\lambda \in \mathcal{P}$ be of height $\leq n$, $\lambda = (\lambda_1, \dots, \lambda_n)$. Define

$$v_\lambda(x_1, \dots, x_n) := \det \begin{pmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_2+n-2} & \cdots & x_1^{\lambda_n} \\ x_2^{\lambda_1+n-1} & x_2^{\lambda_2+n-2} & \cdots & x_2^{\lambda_n} \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, x_{\sigma(1)}^{\lambda_1+n-1} x_{\sigma(2)}^{\lambda_2+n-2} \cdots x_{\sigma(n)}^{\lambda_n}.$$

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$v_\lambda \in \mathbb{Z}[x_1,\dots,x_n]$ is an alternating polynomial, of degree $|\lambda| + \binom{n}{2}$.

Moreover, $v_{(0,\dots,0)}(x_1,\dots,x_n) = \prod_{i<j}(x_i - x_j)$, the Vandermonde determinant (denoted by $\Delta(x_1,\dots,x_n) \equiv v_{(0,\dots,0)}(x_1,\dots,x_n)$.)

The polynomial $s_\lambda(x_1,\dots,x_n) := \dfrac{v_\lambda(x_1,\dots,x_n)}{\Delta(x_1,\dots,x_n)} \in \mathbb{Z}[x_1,\dots,x_n]$

is Schur polynomial associated to λ. s_λ is a symmetric pol. of homogeneity (degree) $|\lambda|$.

Exercise 3: Show $s_{(1,1,\dots,1)} = x_1 \cdot x_2 \cdots x_n$, $s_{(k,\dots,k)} = (x_1 \cdots x_n)^k$.

Show that for $ht(\lambda) = k$, $s_\lambda(x_1^{-1},\dots,x_n^{-1})(x_1 \cdots x_n)^k = s_{\lambda^c}(x_1,\dots,x_n)$ with $\lambda^c := (n-\lambda_k, n-\lambda_{k-1},\dots,n-\lambda_1)$ the complementary partition (transpose of Young diagram for λ.)

$\mathbb{Z}[x_1,\dots,x_n]_{sym}$ $\equiv$ the subring of $\mathbb{Z}[x_1,\dots,x_n]$ of symm. pol.,

$\mathbb{Z}[x_1,\dots,x_n]_{alt.}$ $\equiv$ the (abelian) subgroup of alternating pol.

Lemma 4: 1/ $\{v_\lambda \mid ht(\lambda) \le n\}$ is a $\mathbb{Z}$-bases of $\mathbb{Z}[x_1,\dots,x_n]_{alt.}$

2/ $\{s_\lambda \mid ht(\lambda) \le n\}$ is a $\mathbb{Z}$-bases of $\mathbb{Z}[x_1,\dots,x_n]_{sym.}$

Pf: 1/ Let $f \in \mathbb{Z}[x_1,\dots,x_n]_{alt.}$ and $a \cdot x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$ be the leading term of f w.r.t. the lexicographic ordering of the exponent $(r_1,\dots,r_n)$. Then $r_1 > r_2 > \dots > r_n \ge 0$ for f alternating. Define $\lambda := (r_1 - n + 1, r_2 - n + 2, \dots, r_n) \in \mathcal{P}$. Then v_λ has leading term $x_1^{r_1} \cdots x_n^{r_n}$, and it cancels out in $f_1 := f - a\, v_\lambda$. The claim follows by (finite) induction.

2/ A consequence of 1/: $\forall$ altern. fcion is divisible by the Vandermonde determinant Δ and so $\mathbb{Z}[x_1,\dots,x_n]_{alt.} = \Delta \cdot \mathbb{Z}[x_1,\dots,x_n]_{sym.}$ ▨

<u>Exercise 5</u>: Pieri's formula – for λ a partition of height $\leq n$, show that $s_\lambda \prod_{i=1}^{n} \frac{1}{1-x_i} = \sum_\mu s_\mu$, ← formal power series comparison formula

where the sum on RHS is over all partitions μ such that $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \dots \geq \mu_n \geq \lambda_n$. This corresponds to Young diagrams for μ are obtained from Young diagrams for λ by adding some boxes to the rows and at most one box to every column. [Hint: multiply by Δ.]

The first fundamental theorem for $\left.\begin{matrix} O_n/SO_n \\ Sp_n \end{matrix}\right\}$ groups

char$(K)=0$ $\quad V=K^n$, $(v|w) := \sum_{i=1}^{n} x_i y_i$, $\begin{matrix} v=(x_1,\dots,x_n) \\ w=(y_1,\dots,y_n) \end{matrix} \in V$ bilinear symm. form on V

$O_n := O_n(K) := \{ g \in GL_n(K) \mid (gv|gw) = (v|w) \ \forall v,w \in V \}$
(orthogonal)

$SO_n := SO_n(K) := \{ g \in O_n(K) \mid \det g = 1 \} = O_n(K) \cap SL_n(K)$.
(special orthogonal)

$SO_n < O_n$ of index 2; matrices – $A \in O_n \Leftrightarrow A^T A = E$

<u>Exercise 6</u>: $SO_2(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^*$, $\quad O_2(\mathbb{C}) \xrightarrow{\sim} \mathbb{Z}/_2 \ltimes \mathbb{C}^*$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a+ib$

$\langle v|w \rangle := \sum_{j=1}^{m} \left(x_{2j-1} y_{2j} - x_{2j} y_{2j-1} \right)$ bilinear skew-symmetric form on V

$Sp_n := Sp_n(K) := \{ g \in GL_n(K) \mid \langle gv|gw \rangle = \langle v|w \rangle \text{ for } v,w \in V \}$

The matrix

$$J := \begin{pmatrix} (I) & & & 0 \\ & (I) & & \\ & & (I) & \\ 0 & & & \ddots \\ & & & & (I) \end{pmatrix} \quad \text{with } I := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$n \times n$ matrix

fulfills $\langle v|w \rangle = v J w^T \Rightarrow A \in Sp_n \Leftrightarrow A^T J A = J$.

<u>Lemma 7</u>: $Sp_{2m} < SL_{2m}$.

Pf: S ... skew-symmetric $2m\times 2m$ matrix $S\in M_{2m}(K)$ is a homogeneous pol. of degree m in the entries of S, determined by conditions $\det S = (\mathrm{Pf}\, S)^2$, $\mathrm{Pf}(J)=1$.

We claim $\mathrm{Pf}(g^T S g) = \det g \, \mathrm{Pf}(S) \quad \forall g\in GL_{2m}(K)$.

Why this is true? Define $f(g) := \mathrm{Pf}(g^T S g)(\det g \, \mathrm{Pf}(S))$, defined for $g\in GL_{2m}(K)$ and S an invertible skew-symm. $2m\times 2m$ matrix. The first condition above $\Rightarrow$

$f(g)^2 = \det(g^T S g)\cdot((\det g)^2 \det S)^{-1} = 1$, and since

$f(e)=1$ the claim follows.

For $g\in Sp_{2m}$, $g^T J g = J$ and so $1 = \mathrm{Pf}\, J = \mathrm{Pf}(g^T J g)$ $= \det g \ \mathrm{Pf}(J) = \det(g)$, and the lemma follows. ∎

Exercise 8: Let $v_1,\dots,v_{2m}\in K^{2m}$. Then

$$\mathrm{Pf}(\langle v_i | v_j\rangle)_{ij} = [v_1,\dots,v_{2m}] = \det(v_1,\dots,v_{2m})$$

stand. scalar product on K^{2m} determinant $[i_1,\dots,i_n]: V^p\to K$

LHS = pol. of homog. $2m$ ($\equiv$ pol. hom. m in $\langle v_i | v_j\rangle$)

RHS = pol. —"— $2m$ (determinant)

Hint: If A is the matrix with rows $v_1,\dots,v_{2m}$, then

$$AJA^T = (\langle v_i | v_j\rangle)_{ij}.$$

Consider p-copies of natural/fundamental vector representation $V=K^n$. For $v=(v_1,\dots,v_p)\in V^p$ and $g\in O_n$, SO_n or Sp_{2m} set $g\cdot v := (gv_1,\dots,gv_p)$. The application of symmetric bilinear form to the i-th + j-th factors of V^p we obtain $\forall\ 1\le i,j\le p$ an O_n-invariant fction

$$(i|j): v\equiv (v_1,\dots,v_p)\longmapsto (v_i|v_j)$$

$$(i|j)(v_1,\dots,v_p) := (v_i, v_j).$$

FFT Theorem claims that such functions generate the ring of invariants:

Theorem 10 (FFT for O_n, SO_n):

1/ The invariant algebra $K[V^p]^{O_n}$ is generated by the invariants $(i|j)$, $1 \leq i \leq j \leq p$.

2/ The invariant algebra $K[V^p]^{SO_n}$ is — " —

— " — $(i|j)$, — " —, together with the determinants $[i_1, \dots, i_n]$, $1 \leq i_1 < i_2 < \dots < i_n \leq p$.

Analogously, for every pair $1 \leq i, j \leq p$ there is an invariant function on V^p:

$$\langle i | j \rangle : V^p \longrightarrow K$$
$$(v_1, \dots, v_p) \longmapsto \langle i | j \rangle (v_1, \dots, v_p) := \langle v_i | v_j \rangle.$$

Theorem 11 (FFT for Sp_{2m}):

The algebra of invariants for Sp_{2m}, $K[V^p]^{Sp_{2m}}$, is generated by the invariants $\langle i | j \rangle$, $1 \leq i < j \leq p$.

Proof of FFT for O_n, SO_n: Firstly, 1/ follows from 2/. In fact, since a determinant $[i_1, \dots, i_n]$ is mapped to $-[i_1, \dots, i_n]$ under any $g \in O_n \setminus SO_n$, we see from 2/ that O_n-invariants are generated by the $(i|j)$ and $\forall$ products $[i_1, \dots, i_n][j_1, \dots, j_n]$ of two determinants. But

$$[i_1, \dots, i_n][j_1, \dots, j_n] = \det\left((i_k | j_\ell)_{k,\ell=1}^{n} \right)$$

and the claim follows. ▨

We shall omit the proof of 2/, because it is based on the use of several results not discussed in our lectures. Namely

⑥ A/ If K is not alg. closed (e.g., $K=\mathbb{R}$), we have $\overline{K}[V^p]^{SO_n(K)} = \overline{K}\otimes_K \left(K[V^p]^{SO_n(K)}\right)$, with $\overline{K}$ the alg. closure of K, and $\overline{K}[V^p]^{SO_n(K)} = \overline{K}[V^p]^{SO_n(\overline{K})}$ because $SO_n(K)$ is Zariski-dense in $SO_n(\overline{K})$. ($\Rightarrow$ we can use geometric arguments)

B/ These are Weyl's theorems. Because GL_p acting on V^p commutes with diagonal action of $\underset{G}{\underset{\vee\!\!\!\mid}{GL}}(V)$, then the ring of invariants $K[V^p]^G$ is stable under GL_p. Denoting

$\langle S\rangle_{GL_p} \equiv GL_p$-module generated by a subset S of a represent. of GL_p

Th 1 (Weyl): $\forall p \geq n := \dim V$, $K[V^p]^G = \langle K[V^n]^G\rangle_{GL_p}$.
In particular, if $S\subset K[V^n]^G$ is a system of generators then $\langle S\rangle_{GL_p}$ generates the invariant ring $K[V^p]^G$.

"One can see all invariants already on $n=\dim V$ copies of V"

Th 2 (Weyl): $G < SL(V)$. For $p \geq n=\dim V$, the invariant ring $K[V^p]^G$ is generated by $\langle K[V^{n-1}]^G\rangle_{GL_p}$ together with all determinants $[i_1, \dots, i_n]$.

These results give reduction of Theorem to $p=n-1$ and the case of alg. closed field.