(1) <u>Double centralizer theorem</u> for $\mathrm{char}(K)=0 \overset{\text{(theorem of Maschke)}}{\Longrightarrow}$ the group algebra $K[S_m]$ is semisimple, i.e. $\forall$ representation of S_m is completely reducible. Consequently, the homomorphic image of $K[S_m]$ in $\mathrm{End}(V^{\otimes m})$, denoted by $\langle S_m \rangle$, is semisimple subalgebra of $\mathrm{End}(V^{\otimes m})$. We have the general result (independent of $\mathrm{char}(K)$):

<u>Proposition 1</u>: $A \subseteq \mathrm{End}(W)$ a semisimple subalgebra,
$A' := \{ b \in \mathrm{End}(W) \mid ab = ba \ \forall a \in A \}$ its centralizer.

Then 1/ A' is semi-simple, and $(A')' = A$,

2/ W has a unique decomposition $W = W_1 \oplus \dots \oplus W_r$ into simple, non-isomorphic $A \otimes A'$-modules W_i. In addition, this is the isotypic decomposition as an A-mod and as an A'-mod.

3/ $\forall$ simple summand W_i is of the form $U_i \otimes_{D_i} U_i'$ with U_i a simple A-mod, U_i' a simple A'-mod, and D_i is the division algebra $\mathrm{End}_A(U_i)^{op} = \mathrm{End}_{A'}(U_i')^{op}$.

Note: G-mod W, a simple ~~sub~~module U ~~...~~, the isotypic component of W of type U is the sum of all G-submodules of W isomorphic to U. The isotypic components form a direct sum which is all of W iff W is semisimple, and then it is called the isotypic decomposition.

<u>Pf</u>: $W = W_1 \oplus \dots \oplus W_r$ isotypic decomp. of W as an A-module, $W_i \xrightarrow{\sim} U_i^{s_i}$ with simple A-mod U_i pairwise nonisomorphic $(i \neq j)$.
As a semisimple algebra, $A = \prod_{i=1}^{r} A_i$ with $A_i \xrightarrow{\sim} M_{n_i}(D_i)$ for some division algebra $D_i \supset K$. Moreover, $U_i \xrightarrow{\sim} D_i^{n_i}$ as an A-module, with A-mod structure on $D_i^{n_i}$ given by $A \overset{pr}{\twoheadrightarrow} A_i \xrightarrow{\sim} M_{n_i}(D_i)$. We have

$$A' := \mathrm{End}_A(W) = \prod_i \mathrm{End}_{A_i}(W_i), \quad \text{and}$$

$$A_i' := \mathrm{End}_A(W_i) = \mathrm{End}_{A_i}(W_i) \xrightarrow{\sim} M_{s_i}(D_i')$$

with $D_i' := \mathrm{End}_{A_i}(U_i) = D_i^{op}$. In particular, A' is semisimple and

$$\dim A_i \cdot \dim A_i' = n_i^2 \dim D_i \, s_i^2 \dim D_i' = (n_i s_i)^2 (\dim D_i)^2$$
$$= (\dim W_i)^2 = \dim(\mathrm{End}(W_i))$$

This implies that the homomorphism $A_i \otimes A_i' \xrightarrow{\sim} \mathrm{End}(W_i)$ is an isomorphism. Because $A_i \otimes A_i'$ is simple, W_i is simple $A \otimes A'$-module. Namely, regarding $U_i = D_i^{n_i}$ as a right D_i-mod, and $U_i' := (D_i^{op})^{s_i}$ as a left D_i-module. Then these structures commute with the A- resp. A'-module structure, hence $U_i \otimes_{D_i} U_i'$ is a $A \otimes A'$-module such that $U_i \otimes_{D_i} U_i' \xrightarrow{\sim} U_i^{s_i} \xrightarrow{\sim} W_i$.

It remains to show $(A')' = A$. We can apply the same reasoning to A' as for A above, and the dimensional formulas imply
$\dim A_i' \cdot \dim A_i'' = \dim \mathrm{End}(W_i) = \dim A_i \cdot \dim A_i' \quad \forall i$.
It follows $\dim A'' = \dim A$, and since $A'' \supset A$ we are done. ▨

<u>Exercise 2</u>: V... irred. fin. dim repr. of G, $\mathrm{End}_G(V) = K$; W... a fin. dim. G-mod. Then the lin. map $j: \mathrm{Hom}_G(V, W) \otimes V \to W$, $\alpha \otimes v \mapsto \alpha(v)$, is injective and G-equivariant. Its image is the sum of all simple submodules of W isomorphic to V.

Decomposition of $V^{\otimes m}$

We prove the theorem on centralizing properties of $\langle S_m \rangle$ and $\langle GL(V) \rangle$ acting on $V^{\otimes m}$.

<u>(Decomposition) Theorem 3</u>: $\mathrm{char}(K) = 0$.

1/ $\langle S_m \rangle$ and $\langle GL(V) \rangle$ are both semisimple and are centralizers of each other.

2/ There is a canonical decomposition of $V^{\otimes m}$ as an $S_m \times GL(V)$ module into simple non-isomorphic modules V_λ:
$$V^{\otimes m} = \bigoplus_\lambda V_\lambda .$$

3/ $\forall$ simple factor V_λ is of the form $M_\lambda \otimes L_\lambda$, where M_λ is a simple S_m-module and L_λ a simple $GL(V)$-module. The modules M_λ (resp. L_λ) are all non-isomorphic.

Pf: We already proved $\langle S_m \rangle' = \langle GL(V) \rangle$, and so both 1/, 2/ follow from Proposition 1. For the last statement 3/, it remains to show that $\forall$ simple S_m-module M_λ, $\mathrm{End}_{S_m}(M_\lambda) = K$. This is clear if K is alg. closed, for K arbitrary of charact. zero will be proved later on. $\blacksquare$

$\forall$ irred. repr. of S_m occurs in $V^{\otimes m}$ provided $\dim V \geq m$, in fact the regular representation of S_m occurs as a subrepresentation. Let us fix such a representation M_λ. Then

$$L_\lambda = L_\lambda(V) = \mathrm{Hom}_{S_m}(M_\lambda, V^{\otimes m})$$

as a consequence of $\mathrm{End}_{S_m}(M_\lambda) = K$ (by Schur lemma.) This implies that $L_\lambda(V)$ is a functor of V: a linear map $\varphi: V \to W$ determines a linear map $L_\lambda(\varphi): L_\lambda(V) \to L_\lambda(W)$ with $L_\lambda(\varphi \circ \psi) = L_\lambda(\varphi) \circ L_\lambda(\psi)$ and $L_\lambda(Id_V) = Id_{L_\lambda(V)}$.

For $\left.\begin{array}{l} M_0 = \text{triv. } S_m\text{-represent.} \\ M_{sgn} = \text{sign } S_m\text{-represent} \end{array}\right\}$ functors $\begin{array}{l} L_0(V) = S^m(V), \\ L_{sgn}(V) = \Lambda^m(V). \end{array}$

The functor L_λ is called Schur functor.

Remark 4: The endomorphism rings of the simple modules L_λ, M_λ are the base field K $\Rightarrow$ representations are defined over $\mathbb{Q}$: $M_\lambda = M_\lambda^{\mathbb{Q}} \otimes_{\mathbb{Q}} K$, $L_\lambda = L_\lambda^{\mathbb{Q}} \otimes_{\mathbb{Q}} K$ with $\begin{array}{l} M_\lambda^{\mathbb{Q}} \\ L_\lambda^{\mathbb{Q}} \end{array}$ simple $\begin{array}{l} \mathbb{Q}[S_m] \\ GL(\mathbb{Q}) \end{array}$- modules.

Exercise 5: $\rho: G \to GL(V)$ a completely reducible representation. $\forall$ K'/K field extension the represent. of G on $V \otimes_K K'$ is completely reducible, too. As a hint prove that because the subalgebra $A := \langle \rho(G) \rangle \subseteq \mathrm{End}(V)$ is semisimple, the subalgebra $A \otimes_K K'$ of $\mathrm{End}(V \otimes_K K')$ is semisimple as well.

④

Polarization and restitution

We prove multilinear versions of FFT, in char $=0$ and using polarization and restitution.

Multihomogeneous invariants: $V = V_1 \oplus \dots \oplus V_r$, $\dim V_i/_K < \infty$,

$f \in K[V_1 \oplus \dots \oplus V_r]$ is multihomogeneous of degree $h = (h_1, \dots, h_r)$ if f is homog. of degree h_i in V_i: $\forall\, v_1, \dots, v_r \in V$, $t_1, \dots, t_r \in K$

$$f(t_1 v_1, t_2 v_2, \dots, t_r v_r) = t_1^{h_1} \cdots t_r^{h_r} f(v_1, \dots, v_r).$$

$\forall f$ is uniquely written as $f = \sum_h f_h$, $f_h \equiv$ multihomogeneous component corresponding to $h = (h_1, \dots, h_r)$

Therefore,
$$K[V_1 \oplus \dots \oplus V_r] = \bigoplus_{h \in \mathbb{N}^r} K[V_1 \oplus \dots \oplus V_r]_h$$
degree h multihom. functions

This gives grading on $K[V_1 \oplus \dots \oplus V_r]$:

$$K[V_1 \oplus \dots \oplus V_r]_h \cdot K[V_1 \oplus \dots \oplus V_r]_k = K[V_1 \oplus \dots \oplus V_r]_{h+k}$$

$K[V_1 \oplus \dots \oplus V_r]_h$ is stable $\forall h$ w.r. to the action of linear algebr. group $GL(V_1) \times \dots \times GL(V_r)$; if V_i are representations of G via $G \to GL(V_i)$, we get

$$K[V_1 \oplus \dots \oplus V_r]^G = \bigoplus_{h \in \mathbb{N}^r} K[V_1 \oplus \dots \oplus V_r]_h^G .$$

(f is an invariant $\Leftrightarrow$ $\forall$ multihomogeneous component is invariant)

Multilinear invariants of vectors and covectors: $V^p \oplus V^{*q} \circlearrowleft GL(V)$

$f : V^p \oplus V^{*q} \to K$ a multilinear invariants; if $f \neq 0$, we must have $p = q$: If we apply $\lambda \in K^* \subseteq GL(V)$ to $(v, \varphi) = (v_1, \dots, v_p, \varphi_1, \dots, \varphi_q)$ we obtain $(\lambda v_1, \dots, \lambda v_p, \lambda^{-1}\varphi_1, \dots, \lambda^{-1}\varphi_q)$, hence $f(\lambda \cdot (v, \varphi)) = \lambda^{p-q} f((v, \varphi))$.

FFT for $GL(V)$ claims that the invariants are generated by the contractions $(i \mid j)$ defined by $(i \mid j)(v, \varphi) = \varphi_j(v_i)$. Therefore, a multil. invariant of $V^p \oplus V^{*q}$ is a lin. comb. of products

$$(1 \mid i_1)(2 \mid i_2) \dots (p \mid i_p),$$ with $(i_1, i_2, \dots, i_p)$ a permutation of $(1, 2, \dots, p)$.

5

Theorem 6 (Multilinear FFT for $GL(V)$): Assume $\mathrm{char}(K)=0$, then multilinear inv. of $V^p \oplus V^{*q}$ exists only for $p=q$. They are linearly generated by $f_\sigma := (1|\sigma(1))\dots(p|\sigma(p))$, $\sigma\in S_p$.

(holds in arbitrary characteristic by the work of de Concini & Procesi.)

Pf: We shall prove a more general statement:

Claim: The theorem above is equivalent to the previous Theorem stating that $\mathrm{End}_{GL(V)} V^{\otimes m} = \langle S_m \rangle$.

Let M be the multilinear fctions on $V^m \oplus V^{*m}$. Then

$$M = (\underbrace{V\otimes\dots\otimes V}_{m} \otimes \underbrace{V^*\otimes\dots\otimes V^*}_{m})^* = (W\otimes W^*)^*, \quad W := V^{\otimes m}.$$

There is canonical isomorphism $\alpha : \mathrm{End}(W) \xrightarrow{\sim} (W\otimes W^*)^*$

$$A \longmapsto \alpha(A)(w\otimes\psi) = \psi(Av),$$

which is clearly $GL(W)$-equivariant. Hence we get a $GL(V)$ equivariant isomorphism $\mathrm{End}(V^{\otimes m}) \xrightarrow{\sim} (V^{\otimes m}\otimes V^{*\otimes m})^* = M$ inducing an isomorphism $\mathrm{End}_{GL(V)}(V^{\otimes m}) \to M^{GL(V)}$

$$M^{GL(V)} = \{\text{invariant multilinear functions}\}$$

The image of $\sigma\in\mathrm{End}(V^{\otimes m})$ under α is

$$\alpha(\sigma)(v_1\otimes\dots\otimes v_m\otimes\varphi_1\otimes\dots\otimes\varphi_m) = (\varphi_1\otimes\dots\otimes\varphi_m)(\sigma.(v_1\otimes\dots\otimes v_m))$$

$$= (\varphi_1\otimes\dots\otimes\varphi_m)(v_{\sigma^{-1}(1)}\otimes\dots\otimes v_{\sigma^{-1}(m)})$$

$$= (\varphi_1 | v_{\sigma^{-1}(1)})(\varphi_2 | v_{\sigma^{-1}(2)})\dots(\varphi_m | v_{\sigma^{-1}(m)})$$

$$= f_{\sigma^{-1}}(v_1\otimes\dots\otimes v_m\otimes\varphi_1\otimes\dots\otimes\varphi_m),$$

and so $\alpha\langle S_m\rangle = \langle f_\sigma \mid \sigma\in S_m\rangle$ and the claim follows. ∎

(6) <u>Multilinear invariants of matrices</u> m copies of $End(V)$, so $End(V)^m$

$\sigma \in S_m$: $\sigma = (i_1, .., i_k)(j_1, ..., j_r) \cdots (l_1, .., l_s)$ product of disjoint cycles

Define a function $Tr_\sigma : End(V)^m \to K$

$$(A_1, .., A_m) \mapsto Tr_\sigma(A_1, .., A_m) :=$$

$$Tr(A_{i_1} \cdot \ldots \cdot A_{i_k}) Tr(A_{j_1} \cdot \ldots \cdot A_{j_r}) \cdot \ldots \cdot Tr(A_{l_1} \cdot \ldots \cdot A_{l_s})$$

wher Tr_σ is a multilinear invariant. It is independent of the presentation of σ as a product of disjoint cycles (exercise, check it.) Now special case of FFT for matrices is

<u>Theorem 7 (Multilinear FFT for matrices)</u> char$(K)=0$, the multilinear invariants on $End(V)^m$ are linearly generated by the functions Tr_σ, $\sigma \in S_m$.

<u>Pf</u>: