1 (Polarization and restitution II.)

<u>Theorem 1</u> (Multilinear FFT for matrices) Assuming char(K)=0, the multilinear invariants on $End(V)^m$ are linearly generate by the functions Tr_σ, $\sigma \in S_m$.

<u>Pf</u>: This result follows from the multilinear FFT for GL(V), which we proved the last week.

Again, we have a more precise statement:

<u>Lemma 2</u>: Theorem 1 is equivalent to "Multilinear FFT for GL(V)".

<u>Pf</u>: The multilinear functions on $End(V)^m$ are identified with $(End(V)^{\otimes m})^*$. We use the canonical isomorphism $\beta : V \otimes V^* \xrightarrow{\sim} End(V)$

$$(v \otimes \varphi) \mapsto \beta(v \otimes \varphi)(u) = \varphi(u) v$$

($\beta(v\otimes\varphi)$ is a rank 1 endomorphism of V, $Im(\beta) = \langle v \rangle$, $Ker(\beta) = Ker(\varphi)$.) It is elementary to verify

1/ $Tr(\beta(v\otimes\varphi)) = \varphi(v)$, and 2/ $\beta(v\otimes\varphi) \circ \beta(w \otimes \psi) = \beta(v \otimes \varphi(w)\psi)$.

Now β induces GL(V)-equivariant isomorphism

$$\tilde{\beta} : V^{\otimes m} \otimes V^{*\otimes m} \xrightarrow{\sim} End(V)^{\otimes m},$$

and its adjoint (dual map) $\tilde{\beta}^*$ identifies the multilinear invariants of $End(V)^m$ with those of $V^m \otimes V^{*m}$ ($\tilde{\beta}^* : End(V)^{m}_{*} \to V^{\otimes m *}_{*} \otimes V^{*\otimes m *}$

$$\cong$$

$$V^{\otimes m} \otimes V^{*\otimes m},$$

and $End(V) \xrightarrow{\varphi} End(V)^*$ is induced by the bilinear form $(A, B) \mapsto Tr(AB)$, which is GL(V)-equivariant.)

We have $\varphi = \varphi^*$.)

The claim follows once we show $\tilde{\beta}^*(Tr_\sigma) = f_\sigma$. Let $\sigma \in S_m$, and $\sigma = (i_1, \dots, i_k)(j_1, \dots, j_r) \dots (l_1, \dots, l_s)$ be the decomposition into disjoint cycles. Then using 1/ and 2/, we get

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$$\mathrm{Tr}_\sigma\,\tilde\beta(v_1\otimes\cdots\otimes v_m\otimes\varphi_1\otimes\cdots\otimes\varphi_m)=\mathrm{Tr}_\sigma\big(\beta(v_1\otimes\varphi_1)\beta(v_2\otimes\varphi_2)\dots\big)=$$
$$=\mathrm{Tr}\big(\beta(v_{i_1}\otimes\varphi_{i_1})\beta(v_{i_2}\otimes\varphi_{i_2})\dots\big)\,\mathrm{Tr}\big(\beta(v_{j_1}\otimes\varphi_{j_1})\beta(v_{j_2}\otimes\varphi_{j_2})\dots\big)\dots$$
$$=\mathrm{Tr}\big(\beta(v_{i_1}\otimes\varphi_{i_1}(v_{i_2})\varphi_{i_2}(v_{i_3})\dots\varphi_{i_{k-1}}(v_{i_k})\varphi_{i_k})\big)\,\mathrm{Tr}(\beta\dots)\dots$$
$$=\varphi_{i_1}(v_{i_2})\varphi_{i_2}(v_{i_3})\dots\varphi_{i_k}(v_{i_1})\,\varphi_{j_1}(v_{j_2})\varphi_{j_2}(v_{j_3})\dots$$

and because the cycle structure implies $\sigma(i_\nu)=i_{\nu+1}$ for $\nu<k$ and $\sigma(i_k)=i_1$, and similarly for the other cycles, the last product equals

$$\prod_{i=1}^m\varphi_i(v_{\sigma(i)})=f_\sigma(v_1\otimes\cdots\otimes v_m\otimes\varphi_1\otimes\cdots\otimes\varphi_m),$$

which proves the claim. ∎

Polarization: $f\in K[V]$, homogeneous of degree d. For $v=\sum_{i=1}^d t_iv_i$, $t_i\in K$, $v_i\in V$, we have
$$f(t_1v_1+\dots+t_dv_d)=\sum_{\substack{s_1,\dots,s_d\\ \sum_{i=1}^d s_i=d}} t_1^{s_1}\dots t_d^{s_d}\,f_{s_1,\dots,s_d}(v_1,\dots,v_d),$$
where $f_{s_1,\dots,s_d}\in K[V^d]$ are well-defined and multihomogeneous of (multi)degree $(s_1,\dots,s_d)$.

Def 3: The multilinear polyn. $f_{1,\dots,1}\in K[V^d]$ is called the (full) polarization of f, denoted $\mathcal{P}f$.

Lemma 4: The linear operator
$$\mathcal{P}: K[V]_d\to K[V^d]_{(1,1,\dots,1)},\quad f\mapsto\mathcal{P}f$$
has following properties:

1/ $\mathcal{P}$ is $GL(V)$-equivariant,

2/ $\mathcal{P}f$ is symmetric,

3/ $\mathcal{P}f(v,v,\dots,v)=d!\,f(v)$.

Pf: 1/, 2/ are verified easily, 3/: substitute for the polarization identity $v_i=v$, $i=1,\dots,d$. The LHS is
$$f((\textstyle\sum_i t_i)v)=(\sum t_i)^d f(v)=(d!\,t_1\cdots t_d+\dots)f(v),$$ and the proof follows. ∎

Restitution: (inverse operator to polarization)

Def 5: For a multilinear map $F \in K[V^d]$ (i.e., $V \times \ldots \times V \to K$), linear in each component

the homogeneous polynom. RF, $RF(v) := F(v, v, \ldots, v)$ of hom. $= d$, is the (full) restitution of F.

$R : K[V^d]_{(1,1,\ldots,1)} \to K[V]_d$ is a linear $GL(V)$-equivariant operator,

$RPf = d!f$ by 3/ of Lemma 4. Consequently, we get

Proposition 6: Assume char $K = 0$, V fin. dim repr of G. Then $\forall$ homog. invariant $f \in K[V]^G$ of degree d is the full restitution of a multilinear invariant $F \in K[V^d]^G$.

Here f is the full restitution of $\frac{1}{d!}Pf$, which is a multihomogeneous invariant (Lemma 4).

Exercise 7: Show that PRF is the symmetrization of F, defined by

$$\underset{PR}{\overset{\shortparallel}{\mathrm{Sym}}}(F)(v_1, \ldots, v_d) := \sum_{\sigma \in S_d} F(v_{\sigma(1)}, \ldots, v_{\sigma(d)}).$$

Generalization to several representations: (a generalization of previous results)

$f \in K[V_1 \oplus \ldots \oplus V_r]$ multihomog. pol. of (multi)degree $(d_1, \ldots, d_r) = d$.

$P_i \equiv$ full polarization for V_i, $i = 1, \ldots, r$. Then

$$Pf := P_r P_{r-1} \ldots P_1 f \in K[V_1^{d_1} \oplus V_2^{d_2} \oplus \ldots \oplus V_r^{d_r}]$$

is multilinear polynomial (polarization of f again.)

The restitution RF of $F \in K[V_1^{d_1} \oplus \ldots \oplus V_r^{d_r}]$ by

$$RF(v_1, \ldots, v_d) := F(\underbrace{v_1, \ldots, v_1}_{d_1}, \underbrace{v_2, \ldots, v_2}_{d_2}, \ldots, \underbrace{v_r, \ldots, v_r}_{d_r}).$$

Lemma 8: The linear operators

$$P : K[V_1 \oplus \ldots \oplus V_r]_{(d_1, \ldots, d_r)} \to K[V_1^{d_1} \oplus \ldots \oplus V_r^{d_r}]_{\text{multilinear}}$$

(4)

$$R : K[V_1^{d_1} \oplus \dots \oplus V_r^{d_r}]_{\text{multilinear}} \longrightarrow K[V_1 \oplus \dots \oplus V_r]_{(d_1, \dots, d_r)}$$

are $GL(V_1) \times \dots \times GL(V_r)$-equivariant and satisfy

1/ Pf is multisymmetric (i.e. symmetric w.r. to the variables in V_i $\forall i$),

2/ $RPf = d_1!\, d_2! \dots d_r!\, f$,

3/ PRF is the multisymmetrization of F:

$$PRF(v_1^{(1)}, \dots, v_1^{(d_1)}, v_2^{(1)}, \dots, v_2^{(d_2)}, \dots) = \sum_{\substack{(\sigma_1, \dots, \sigma_r) \\ \cap \\ S_{d_1} \times \dots \times S_{d_r}}} F(v_1^{\sigma_1(1)}, \dots, v_1^{\sigma_1(d_1)}, \dots).$$

As before, we get

<u>Proposition 9</u>: $\mathrm{char}(K) = 0$, $V_1, \dots, V_r$ representations of the group G. Then $\forall$ multihomogeneous invariant $f \in K[V_1 \oplus \dots \oplus V_r]^G$ of degree $d = (d_1, \dots, d_r)$ is the restriction of a multilinear invariant $F \in K[V_1^{d_1} \oplus \dots \oplus V_r^{d_r}]^G$.

<u>Proof of the FFTs (completion of the required proof):</u>

"Multilinear FFT for $GL(V)$": $\forall$ mult. inv. F of $V^p \oplus V^{*p}$ is a lin. comb. of invariants $f_\sigma = (1 \mid \sigma(1)) \dots (p \mid \sigma(p))$, $\sigma \in S_p$.

Any (partial) restitution of f_σ is a monomial in the $(i \mid j)$. By Proposition 6, this shows $\forall$ invariant of $V^p \oplus V^{*p}$ is a polynomial in the $(i|j)$; in other words:

If $\mathrm{char}(K) = 0$, "First Fund. theorem for $GL(V)$" for vectors and covectors is a consequence of its multilinear version "Multilin. FFT for $GL(V)$".

Analogously: $\mathrm{char}(K) = 0$, "FFT for matrices" is a consequence of its multilinear version "Multilinear version of FFT for matrices."

Example 10: (Invariants of forms and vectors) $F_d \equiv$ homog. forms of degree d on V, $F_d := K[V]_d$.

Question: Invariants of $F_d \oplus V$, under $GL(V)$?

Obvious invariant: $\epsilon \in K[F_d \oplus V]^{GL(V)}$, $\epsilon(f, v) := f(v)$.

Claim: ϵ is a generator of the ring of invariants, $K[F_d \oplus V]^{GL(V)} = K[\epsilon]$.

Pf: Let $h: F_d \oplus V \to K$ be a bi-homogeneous invariant of degree (r, s), $\tilde{h}$ be the full polarization of h w.r. to the first variable. Then $\tilde{h}: F_d^r \oplus V \to K$ is a multihomog. invariant of degree $(1, 1, \dots, 1, s)$. Composing $\tilde{h}$ with the d-th power map $\begin{matrix} V^* \to F_d \\ \varphi \mapsto \varphi^d \end{matrix}$)

obtain a multi-homogeneous invariant $H: V^{*r} \oplus V \to K$ of degree $(\underbrace{d, d, \dots, d}_{r}, s)$. Then FFT for $GL(V)$ $\Rightarrow$

$rd = s$, and H is a scalar multiple of the invariant $(1|1)^d (2|1)^d \dots (r|1)^d$.

On the other hand, starting with $h = \epsilon$ we find $\tilde{h} = \epsilon$ and $H(\varphi, v) = \varphi(v)^d$, hence $H = (1|1)^d$. Since F_d is spanned by (symmetric) d-th powers of linear forms, $\tilde{h}$ is completely determined by H and therefore h is a scalar multiple of ϵ^r.

Exercise 11: Show that the invariants of a form f of degree d, and two vectors v, w are generated by invariants ϵ_i, $i = 0, 1, 2, \dots, d$:

$$K[F_d \oplus V^2]^{GL(V)} = K[\epsilon_0, \epsilon_1, \dots, \epsilon_d], \quad \epsilon_i(f, v, w) := f_i(v, w),$$

where f_i are the partial polarization of f in variables $i = 0, 1, \dots, d$ of F_d.

The notion of partial polarization is defined as follows: let $f \in K[V]_d$ (a polyn. of degree d), and write

(6)

$$f(sv+tw) = \sum_{i=0}^{d} s^i t^{d-i} f_i(v,w) \quad , \quad s,t \in K \ , \quad v,w \in V.$$

Then the polynomials f_i are bihomogeneous of degree $(i, d-i)$ and the operators $f \mapsto f_i$ are $GL(V)$ equivariant. Moreover, $f_i(v,v) = \binom{d}{i} f(v)$. In particular, for $G < GL(V)$ is f G-invariant iff f_i is G-invariant $\forall i = 0,1,\dots,d$. The f_i are called partial polarizations of f.