Chapter 3

Structure of classical groups

This chapter contains a classification of finite dimensional representations of classical groups. The first main fact making possible such classification is *complete reducibility* of their regular finite dimensional representations. It claims that any such representation can be written as a direct sum of irreducible representations. Hence the classification problem is reduced to a classification of *irreducible* regular finite dimensional representations.

Reducibility of classical groups is usually proved either algebraically, or analytically. The former proof is based on properties of Casimir operators. The latter one is based on integration over the compact real form and is usually called the Weyl unitary trick.

The classification of irreducible finite dimensional representations is based on the fact that if G is a connected classical group, any irreducible regular finite dimensional representation (ρ, V) can be written as a direct sum of eigenspaces (called weight spaces) for the action of the Lie algebra $\mathfrak{h}$ (Cartan subalgebra) of a maximal torus $H \subset G$ (the Cartan subgroup). Weights are elements in the dual $\mathfrak{h}^*$ of $\mathfrak{h}$ and relative to a partial order on the set of all weights for V, there is a unique maximal element called *the highest weight* characterizing the representation (ρ, V).

We are using a close connection between irreducible representations of a connected algebraic group G and irreducible representations of its Lie algebra $\mathfrak{g}$ (see Th.1.7.7 3.3). This is a useful fact, which makes it possible to apply efficient tools of linear algebra.

The set of all highest weights of all irreducible regular finite dimensional representations is a (discrete) cone in a real subspace of $\mathfrak{h}^*$ generated by *fundamental weights*. The weights of the representations are permuted by the action of the *Weyl group* W, which is a finite group canonically attached to G.

The classification mentioned above coincides with the classification of all irreducible finite dimensional representations of G considered as a Lie group, which was already treated in the basic course on Lie groups, their Lie algberas and their representations, hence we shall present it in a reduced version, as a short overview of its main facts. On the other hand, we shall extend the classification also to linear algebraic groups which are not simple ($GL(V)$) or which are not connected ($O(V, B)$). Often is convenient to replace highest weights as parameters in the classifications by an equivalent language of Young diagrams. We shall use it systematically.

3.1 Classical groups are reductive

An attempt to understand the structure of representations of a given group is dramatically simplified, if any representation of the group can be decomposed into a direct sum of irreducible ones. Groups with this property are called reductive. One way how to prove that a group is reductive was used by Hermann Weyl and its standard name nowadays is 'the unitary trick'. Its idea is simple. Suppose that we are able to construct an invariant scalar product for any representation (ρ, V) of the given type. Then if $U \subset V$ is an invariant subspace, then it is easy to show that the orthogonal complement $U^\perp$ (with respect to an invariant scalar product) is again invariant and $V \simeq U \oplus U^\perp$. Now, to construct an invariant scalar product, it is possible to take any scalar product and to make an average of it over the given group, the result will be an invariant scalar product. Such an idea was first successfully used in the case of finite groups (where the average means just a finite sum over the group) and it can be extented to compact groups (where the integral used to average the scalar product is well defined). This it the essence of the Weyl unitary trick.

Definition 3.1. Let G be a linear algebraic group. A regular representation (ρ, V) is called **completely reducible,** if for every G-invariant subspace $W \subset V$, there is a G-invariant subspace $U \subset V$ such that $V \simeq W \oplus U$.

A linear algebraic group G is called **reductive,** if every regular representation of G is completely reducible.

Theorem 3.2. *Let G be a connected linear algebraic group which has a compact real form. Then G is reductive.*

Proof. (i) Suppose that K is a compact real form of G. The differential $d\iota$ of the embedding $\iota : K \to G$ induces an isomorphism of the Lie algebra $\mathfrak{k}$ of K onto a real Lie subalgebra of the Lie algebra $\mathfrak{g}$.

The differential $\sigma = d\tau$ of the complex conjugation τ defined by $K \subset G$ is a Lie algebra isomorphism and we can define $\mathfrak{g}_+ = \{X \in \mathfrak{g} | \sigma(X) = X\}$. Then $d\iota(\mathfrak{k})$ is a subspace of $\mathfrak{g}_+$ and both spaces have the same (real) dimension, hence they coincide. Consequently,

$$\mathfrak{g} \simeq d\iota(\mathfrak{k}) \oplus i d\iota(\mathfrak{k}).$$

(ii) Let (ρ, V) be a regular representation of G. Let us choose a Hermitian scalar product $(.,.)$ on V and define a new scalar product by the average

$$\langle u, v\rangle = \int_K (\rho(k)u, \rho(k)v) dk.$$

with respect to the invariant measure dk on K. It is easy to see that $\langle u, v\rangle > 0$ on V, because $(\rho(x)u, \rho(x)v)$ is continuous with respect to $k \in K$ and K is compact.
(iii) The new scalar product $\langle .,.\rangle$ is a Hermitean scalar product invariant with respect to the action of elements in K. Indeed, if $k' \in K$,

$$\langle \rho(k')u, \rho(k')v\rangle = \int_K (\rho(k)\rho(k')u, \rho(k)\rho(k')v) dk \int_K (\rho(u)u, \rho(u)v) du = \langle u, v\rangle.$$

Details can be found at App.D of [GW].

(iv) Consider now an invariant subspace $W \subset V$ under the action of elements $\rho(g), g \in G$ and its orthogonal complement

$$W^{\perp} = \{v \in V | \langle v, w\rangle = 0, \ w \in W\}.$$

The space W is invariant under the action of elements $\rho(K)$ and the (new) scalar product is invariant for action of K, hence also $W^{\perp}$ is invariant under the action of K. But then $W^{\perp}$ is invariant also for the action of $\mathfrak{k}$ and due to (i), also for action of $\mathfrak{g}$.
(v)Then it is possible to apply Theor.3.3 below to get that $W^{\perp}$ is invariant also for the action of G. □

It was shown in Appendix A.1 that the simple complex classical groups $\boldsymbol{SL}(V)$, $\boldsymbol{SO}(V, B)$ and $\boldsymbol{Sp}(V, \Omega)$ have a compact real form, hence are reductive. More generally, linear algebraic group with the property that their Lie algebras are semi-simple (or a direct sum of semi-simple and commutative Lie algebras) have a compact real form and are hence reductive. A classification of reductive linear algebraic groups and the related classification of connected compact Lie groups and a correspondence between reductive and compact groups can be found at Chapt. 10, Sect. 7 of the book by Processi ([P]).

3.2 Representations of a group and its Lie algebra

There is a very close connection between properties of a representation (ρ, V) of a linear algebraic group G and properties of its differential $(d\rho, V)$. One implication is easy, if $W \subset V$ is invariant under the action of $\rho(G)$, then W is invariant also under the action of $d\rho(\mathfrak{g})$. The opposite implication is much more difficult and it needs stronger assumptions. In particular, it is true, if G is generated by its unipotent elements (and hence connected).

1.7.7 **Theorem 3.3.** *Let G be a linear algebraic group with the Lie algebra $\mathfrak{g}$ and let (π, V) be a regular representation of G.*

(1) If $W \subset V$ is invariant under the action of π, then W is invariant under the action of the differential $d\pi$.

(2) Suppose that G is generated by its unipotent elements.

Then $W \subset V$ is invariant under the action of $d\pi$ iff W is invariant under the action of π. In particular, V is irreducible under the action of G iff it is irreducible under the action of $\mathfrak{g}$.

Proof. (1) Consider elements $v \in V$ and $v^* \in V^*$ and define a regular function $f_{v^*,v}$ on G by $f_{v^*,v}(g) = \langle v^*, \pi(g)v\rangle$. Let C be a rank-one linear transformation on V given by $C(u) = \langle v^*, u\rangle v$ for $u \in V$. Then

$$f_{v^*,v}(g) = tr(\pi(g)C) = (f_C \circ \pi)(g).$$

Hence for $A \in \mathfrak{g}$ we have

$$X_A f_{v^*,v}(g) = (f_{d\pi(A)C} \circ \pi)(g) = f_{v^*,d\pi(A)v}(g) \text{ for all } g \in G \qquad (3.1)$$ inv1

by definition of $d\pi$. Let $W \subset V$ be a G-invariant subspace. Set

$$W^{\perp} = \{v^* \in V^* \mid \langle v^*, w\rangle = 0 \text{ for all } w \in W\}.$$

Then $f_{v^*,w}(g) = 0$ for all $w \in W$, $v^* \in W^{\perp}$, and $g \in G$, since $\pi(g)w \in W$. Hence by (3.1), we have $f_{v^*,d\pi(A)w}(g) = 0$. Setting $g = I$, we conclude that $\langle v^*, d\pi(A)w\rangle = 0$ for all $v^* \in W^{\perp}$. This implies that $d\pi(A)w \in W$.

(2) Suppose that W is invariant under the action of $\mathfrak{g}$. Denote by $W^{\perp}$ the annihilator of W. Then $\langle w^*, (d\rho(X))^k w\rangle = 0, w \in W, w^* \in W^{\perp}, X \in \mathfrak{g}, k \in \mathbb{Z}$. Hence

$$\langle w^*, \rho(\exp X)w\rangle = \langle w^*, (\exp(d\rho(X)))w\rangle = \sum_k \frac{1}{k!}\langle w^*, d\rho(X)^k w\rangle = 0$$

and $\rho(\exp X)W \subset W$. Since G is generated by unipotent elements in G, which are images of nilpotent elements in $\mathfrak{g}$, W is invariant under the action of G.

□

3.3 Representations of a torus

3.3.1 Regular representations of $\mathbb{C}^\times$

Definition 3.4. Let G be a linear algebraic group. **A character** of G is a regular homomorphism $\chi : G \to \mathbb{C}^\times$.

The set $\mathcal{X}(G)$ of all characters of G has the natural structure of a commutative group, identity being the trivial character $\chi_0(g) = 1, g \in G$.

The classification of regular representations of the group $\mathbb{C}^\times = \boldsymbol{GL}(1, \mathbb{C})$ is described in the following theorem.

1.6.4 **Theorem 3.5.** *(1) Suppose that $(\varphi, \mathbb{C}^n)$ is a regular representation of $\mathbb{C}^\times$. For any $z \in \mathbb{C}^\times$, let $E_p, p \in Z$ be the space of all common eigenvectors for the maps $\varphi(z)$ corresponding to the eigenvalues $\chi_p(z) = z^p$, i.e.*

$$E_p = \{v \in \mathbb{C}^n | \varphi(z)\, v = z^p\, v, z \in \mathbb{C}^\times\}.$$

Then

$$\mathbb{C}^n = \oplus_{p\in Z} E_p. \tag{3.2}$$

(2) Conversely, for any decomposition of $\mathbb{C}^n$ of the form (3.2), let us define the map φ by

$$\varphi(z)v = z^p\, v, z \in \mathbb{C}^\times, v \in E_p.$$

Then $(\varphi, \mathbb{C}^n)$ is a regular representation of $\mathbb{C}^\times$ that is determined (up to isomorphism) by the set of integers given by the dimensions of the spaces E_p.

(3) In particular, any character of the group $\mathbb{C}^\times$ is given by $\chi_p(z) = z^p, z \in \mathbb{C}^\times$, for suitable $p \in \mathbb{Z}$. The group $\chi(\mathbb{C}^\times)$ of all characters of $(\mathbb{C}^\times)$ is isomorphic to $\mathbb{Z}$.

Proof. (1) The entries in the matrix $\varphi(z)$ are Laurent polynomials due to $\mathcal{O}(\mathbb{C}^\times) = \mathbb{C}[z, z^{-1}]$. Hence there is an expansion

$$\varphi(z) = \sum_{p\in\mathbb{Z}} z^p T_p,$$

where T_p are $n \times n$ complex matrices. The sum has only finite number of nontrivial terms.

The relation $\varphi(z)\varphi(w) = \varphi(zw)$ implies

$$\sum_{p,q\in\mathbb{Z}} z^p w^q T_p T_q = \sum_{r\in\mathbb{Z}} z^r w^r T_r,$$

which implies the relations $T_pT_q = 0, p \neq q; T_p^2 = T_p$. Moreover, $\varphi(1) = I_n$, hence the family $\{T_p\}$ is the resolution of identity on $\mathbb{C}^n$. If $T_pv = v$ for $v \in V$, then $\varphi(z)v = \sum_q z^q T_p T_q v = z^p v$. Hence the image of T_p is included in E_p.
(2) The decomposition (sum_p 3.2) induces the projections T_p onto E_p. Then the map φ is a regular representation of $\mathbb{C}^\times$.
(3) It is a special case of (1) and (2).

□

torus

3.3.2 Regular representations of a torus

Definition 3.6. Any linear algebraic group T isomorphic to $(\mathbb{C}^\times)^l$, where l is a positive integer, is called **algebraic torus.** The integer l is called the **rank** of T.

Lemma 3.7. *Let T be an algebraic torus of rank l. The group $\mathcal{X}(T)$ is isomorphic to $\mathbb{Z}^l$. (Hence the rank of T is uniquely determined by the algebraic group structure of T.)*

Proof. We may assume that $T = (\mathbb{C}^\times)^l$. For $\lambda = [\lambda_1, \cdots, \lambda_l] \in \mathbb{Z}^l$ and $t = [z_1, \cdots, z_l] \in T$ set

$$t^\lambda = \prod_{k=1}^{l} z_k^{\lambda_k}. \tag{3.3}$$

Then $\chi_\lambda : t \mapsto t^\lambda$ is a character of T. Since $t^{\lambda+\mu} = t^\lambda t^\mu$ for $\lambda, \mu \in \mathbb{Z}^l$ and the functions $z_1^{p_1} \cdots z_l^{p_l}$ give a basis for $\mathcal{O}(T)$, it follows that the map $\lambda \mapsto \chi_\lambda$ is an injective homomorphism from $\mathbb{Z}^l$ to $\mathcal{X}(T)$.

Conversely, let χ be a character of T. Then the functions $t \mapsto \chi_k(z) = \chi(1, \cdots, \underbrace{z}_{k}, \cdots, 1)$ for $k = 1, \cdots, l$ are characters of $\mathbb{C}^\times$. Hence by Theorem 1.6.4 3.5, $\chi_k(z) = z^{\lambda_k}$ for some $\lambda_k \in \mathbb{Z}$. Hence

$$\chi(z_1, \cdots, z_l) = \prod_{k=1}^{l} \chi_k(z_k) = \chi_\lambda(z_1, \cdots, z_l),$$

where $\lambda = [\lambda_1, \cdots, \lambda_l]$. Thus every rational character of T is of the form χ_λ for some $\lambda \in \mathbb{Z}^l$.

□

2.1.3 **Theorem 3.8.** *(1) If (ρ, V) is a regular representation of an algebraic torus T and $V(\chi) = \{v \in V | \rho(t)v = \chi(t)v\}$, then*

$$V = \oplus_{\chi\in A} V(\chi). \tag{3.4}$$

where A is a finite subset of $\mathcal{X}(T)$.

(2) If $g \in End(V)$ commutes with $\rho(t)$ for all $t \in T$, then $gV(\chi) \subset V(\chi)$.

Proof. (1) Due to $(\mathbb{C}^\times)^l \simeq (\mathbb{C}^\times)^{l-1} \times \mathbb{C}^\times$, the relation (wspaces 3.4) follows by induction.
(2) It follows from the definition of $V(\chi)$.

□

3.4 Irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$

sl2c

The complete understanding of the structure of classical complex Lie algebras and their representations is based on structure of the algebra $\mathfrak{sl}(2,\mathbb{C})$ and the behaviour of its representations. This is the reason why we repeat here classification of irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ with more details. In a study of representations of a general classical complex Lie algebra $\mathfrak{g}$, these facts are applied using many $\mathfrak{sl}(2,\mathbb{C})$ Lie subalgebras of $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, i.e. 2×2-matrices with zero trace. We observe that $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, h$=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is a basis of $\mathfrak{g}$ and satisfy the commutation relations

$$[x,y] = h, \quad [h,x] = 2x \quad [h,y] = -2y.$$

Using these relations, it is possible to prove directly from the definition that $\mathfrak{sl}(2,\mathbb{C})$ is an example of a simple Lie algebra.

Let V be a regular representation of $\mathfrak{g}$. The action of an element $x \in \mathfrak{sl}(2,\mathbb{C})$ on $v \in V$ will be denoted by $x \cdot v$. By Jordan decomposition, h acts diagonally on V. This yields a decomposition of V as a direct sum of eigenspaces $V_\lambda = \{v \in V : h \cdot v = \lambda v\}$, $\lambda \in \mathbb{C}$. Whenever $V_\lambda \neq \{0\}$, we call λ a weight of h in V and we call V_λ a weight space.

Lemma 3.9. *If $v \in V_\lambda$, then $x \cdot v \in V_{\lambda+2}$ and $y \cdot v \in V_{\lambda-2}$.*

Proof. $h\cdot(x\cdot v) = [h,x]\cdot v + x\cdot h\cdot v = 2x\cdot v + \lambda x \cdot v = (\lambda+2)x\cdot v$, and similarly for y. □

Since $dim(V) < \infty$ and $V = \bigoplus_{\lambda\in\mathbb{C}} V_\lambda$, there must exist $V_\lambda \neq \{0\}$ such that $V_{\lambda+2} = 0$ by the lemma. For such λ, any nonzero vector in V_λ will be called a **highest weight vector**, or **singular vector** of weight λ.

sl-action **Lemma 3.10.** *Let V be an irreducible $\mathfrak{g}$-representation. Let $v_0 \in V_\lambda$ be a singular vector. Set $v_{-1} = 0$, $v_i = (1/i!)y^i \cdot v_0$ for $i \in \mathbb{N}\cup\{0\}$. Then*

(1) $h\cdot v_i = (\lambda - 2i)v_i$,

(2) $y \cdot v_i = (i+1)v_{i+1}$,

(3) $x \cdot v_i = (\lambda - i + 1)v_{i-1}$.

Proof. Use the commutation relations. □

We see by (1) that nonzero v_i are in different eigenspaces of h, and therefore they are linearly independent. Let m be the smallest integer for which $v_m \neq 0$ and $v_{m+1} = 0$. This assumption on m implies $v_{m+i} = 0$ for all $i \in \mathbb{N}$. The subspace of V with basis $(v_0, v_1, \cdots, v_m)$ is a nontrivial invariant subspace, hence it coincides with V. The property (3) for $i = m+1$ implies that $(\lambda - m)v_m = 0$, hence $\lambda = m$. Every weight $\mu = m - 2i$ occurs with multiplicity one. In particular, up to nonzero constant, there is unique maximal vector. So we have

sl2rep **Theorem 3.11.** *Let V be an irreducible $\mathfrak{g}$-representation for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$.*

(1) Relative to h, V is the direct sum of weight spaces V_μ, $\mu = m, m-2, \cdots, -(m-2), -m$, where $m+1 = dim(V)$ and $dim(V_\mu) = 1$ for each μ.

(2) V has (up to nonzero scalar multiples) a unique maximal vector, whose weight (called the highest weight of V) is m.

(3) The action of $\mathfrak{g}$ on V is given by Lemma sl-action *3.10 with $\lambda = m$, if the basis is chosen in the prescribed fashion. In particular, up to isomorphism, there exists at most one irreducible $\mathfrak{g}$-representation ρ_m of each possible dimension $m+1$, $m \in \mathbb{N} \cup \{0\}$.*

Corollary 3.12. Let V be any (finite dimensional) $\mathfrak{g}$-representation. Then the eigenvalues of h on V are all integers, and each occurs along with its negative (an equal number of times). Moreover, in any decomposition of V into direct sum of irreducible subrepresentations, the number of summands is precisely $dim(V_0) + dim(V_1)$.

Proof. If $V = \{0\}$, there is nothing to prove. Otherwise use complete reducibility to write V as direct sum of irreducible subrepresentations. The latter are described by the theorem, so the first assertion of the corollary is obvious. For the second, taking possible weights in the account, each irreducible $\mathfrak{g}$-representation has unique occurrence of either the weight 0 or else the weight 1, but not both. □

3.5 Irreducible representations of $SL(2,\mathbb{C})$

We want now to classify regular irreducible representation of the group $G = \boldsymbol{SL}(2,\mathbb{C})$. It is more difficult but we can use the result proved already for representations of $\mathfrak{g}$. The strategy is to construct a sufficient number of representations of G. In more details, for every nonnegative integer m, we can construct a regular irreducible representation (ρ, V) of the group G with $dimV = m+1$. Then its differential is a regular irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ with $dimV = m+1$, isomorphic to the one described in Lemma sl-action 3.10. And it exhaust all (isomorphic classes of) regular irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$.

Let us define matrices $u(z) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$, $v(z) = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}$ and $d(a) = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$. $a \in \mathbb{C}^\times$.

The vector space of all elements $u(z), z \in \mathbb{C}$ is a commutative subgroup of $SL(2,\mathbb{C})$, we shall denote it by N^+. Similarly, let N^- be the commutative subgroup of all elements $v(z), z \in \mathbb{C}$. We have now the following classification.

Theorem 3.13. *For every integer $m \geq 0$, there is a unique (up to equivalence) regular irreducible representation (π, V) of $SL(2,\mathbb{C})$ of dimension $m+1$ whose differential is the representation ρ_m in Theorem* sl2rep *3.11. It has the following properties:*

(1) The diagonalizable operator $\pi(d(a))$ has eigenvalues $a^m, a^{m-2}, \cdots, a^{2-m}, a^{-m}$.

(2) $\pi(d(a))$ acts by the scalar a^m on the one-dimensional space V^{N^+} of N^+-fixed vectors.

(3) $\pi(d(a))$ acts by the scalar a^{-m} on the one-dimensional space V^{N^-} of N^--fixed vectors.

Moreover, any regular irreducible representation of $SL(2,\mathbb{C})$ is isomorphic to one from the list above.

Proof. Let $\mathcal{P}(\mathbb{C}^2)$ be the polynomial functions on $\mathbb{C}^2$ and let $V = \mathcal{P}^m(\mathbb{C}^2)$ be the space of polynomials that are homogeneous of degree m. Here we take $\mathbb{C}^2$ as row vectors $x = [x_1, x_2]$ with the group acting by multiplication on the right. This action gives an action of G on V by $\rho(g)\phi(x) = \phi(xg)$ for $\phi \in V$. As a basis for V, we take the monomials

$$v_k(x_1, x_2) = \binom{m}{k} x_1^{m-k} x_2^k, \quad k = 0, 1, \cdots, m.$$

We now calculate the representation $d\rho$ of $\mathfrak{g}$. Since $u(z) = \exp(zx)$ and $v(z) = \exp(zy)$, we have $\rho(u(z)) = \exp(zd\rho(x))$ and $\rho(v(z)) = \exp(zd\rho(y))$ by Theorem 2.28. Taking the z derivative, we obtain

$$d\rho(x)\phi(x_1, x_2) = \frac{d}{dz}\bigg|_{z=0} \phi(x_1, zx_1 + x_2) = x_1 \frac{\partial}{\partial x_2}\phi(x_1, x_2),$$

$$d\rho(y)\phi(x_1, x_2) = \frac{d}{dz}\bigg|_{z=0} \phi(x_1 + zx_2, x_2) = x_2 \frac{\partial}{\partial x_1}\phi(x_1, x_2).$$

Since $d\rho(h) = d\rho(x)d\rho(y) - d\rho(y)d\rho(x)$, we also have

$$d\rho(h)\phi(x_1, x_2) = \left(x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2}\right)\phi(x_1, x_2).$$

On the basis vectors v_k we thus have

$$d\rho(h)v_k = (m - 2k)v_k, \;\; d\rho(x)v_k = (m - k + 1)v_{k-1}, \;\; d\rho(y)v_k = (k + 1)v_{k+1}.$$

It follows from Theorem 3.11 that $d\rho \simeq \rho_m$ is an irreducible representation of $\mathfrak{g}$, and all irreducible representations of $\mathfrak{g}$ are obtained this way. Theorem 3.3 implies that ρ is an irreducible representation of $SL(2, \mathbb{C})$. Furthermore, ρ is uniquely determined by $d\rho$, since $\rho(u(z))$ is uniquely determined by $d\rho(u(z))$ (Theorem 2.28) and $SL(2, \mathbb{C})$ is generated by unipotent elements (see Lemma A.9). □

3.6 The structure of classical simple complex Lie algebras

Let G be a linear algebraic group and $\mathfrak{g}$ its Lie algebra. To understand well the structure of $\mathfrak{g}$, we are going to study the adjoint representation of G on $\mathfrak{g}$, its decomposeition into (simultaneous) eigenspaces (called **root spaces**) with respect to the action of the Lie algebra $\mathfrak{h}$ of the chosen Cartan subgroup H.

In the rest of this chapter, we consider just the following groups:

Type $\mathbf{A}_l$: $\boldsymbol{SL}(l + 1, \mathbb{C})$,
Type $\mathbf{B}_l$: $\boldsymbol{SO}(\mathbb{C}^{2l+1}, B)$,
Type $\mathbf{C}_l$: $\boldsymbol{Sp}(\mathbb{C}^{2l}, \Omega)$,
Type $\mathbf{D}_l$: $\boldsymbol{SO}(\mathbb{C}^{2l}, B)$,

where B, resp. Ω are defined below, see also App. (A.2).

These groups form the main part of the class of algebraic (and Lie) groups called **simple linear algebraic groups**. There is just 5 more (exceptional) cases on the complete list. We shall not discuss them in these lectures.

Understanding of their structures will be important for understanding of properties and classifications of their irreducible representations, which is the main goal of the next chapter. A study of their structures is based on the decomposition of $\mathfrak{g}$ under the adjoint representations of a maximal torus H. So we start first with the choice of such maximal torus, called the Cartan subgroup.

3.6.1 The Cartan subgroups and Cartan subalgebras

Definition 3.14. If G is a linear algebraic group, then a torus $H \subset G$ is called **maximal**, if it is not contained in any larger torus in G. Maximal tori are usually called Cartan subgroups and their Lie algebra are called Cartan subalgebras.

The explicit form of several classical group depends on an explicit choice of a bilinear form used in their definition. We would like to have the subgroup H of diagonal matrices in G as the maximal torus in G. It can be achieved in the cases of basic complex forms of classical groups by the following choices.

Denote by s_l the following $l \times l$ matrix:

$$s_l = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{3.5}$$

and define forms B and Ω as follows:

(1) $x, y \in \mathbb{C}^n$, $n = 2l$ even; $B(x,y) = x^t S y$; $\quad S = \begin{bmatrix} 0 & s_l \\ s_l & 0 \end{bmatrix}$;

(2) $x, y \in \mathbb{C}^n$, $n = 2l+1$ odd; $B(x,y) = x^t S y$; $S = \begin{bmatrix} 0 & 0 & s_l \\ 0 & 1 & 0 \\ s_l & 0 & 0 \end{bmatrix}$;

(3) $x, y \in \mathbb{C}^n$, $n = 2l$; even; $\Omega(x,y) = x^t S y$; $\quad S = \begin{bmatrix} 0 & s_l \\ -s_l & 0 \end{bmatrix}$.

The form B is nondegenerate, bilinear and symmetric, Ω is nondegenerate, bilinear and antisymmetric.

cartan **Theorem 3.15.** *Let G be one of groups $\boldsymbol{GL}(n,\mathbb{C}), \boldsymbol{SL}(n,\mathbb{C}), \boldsymbol{SO}(\mathbb{C}^n, B)$, or $\boldsymbol{Sp}(\mathbb{C}^{2l}, \Omega)$, with B and Ω as above. Define $H \subset G$ as the set of all diagonal matrices in G.*

Then H is a maximal torus in G.

Proof. Elements of the canonical basis $e_1, \ldots, e_n$ are eigenvectors for the action of an element $h \in H$ and eigenvalues are characters $\theta_i(h), \theta_i \in \mathcal{X}(H)$. Explicit forms of θ_i given in Ap. A.3 shows that θ_i are mutually distinct. If $gh = hg$ for all $h \in H$, then g preserves eigenspaces of h, so it is a diagonal matrix. It implies that H is a maximal torus. □

It looks as the form for G given above is special and priviliged but it can be proved that it is not so.

Theorem 3.16. *If G is one of groups in Theor. 3.15 and T is a maximal torus in G, then there is $g \in G$ for which $T = gHg^{-1}$.*

Details can be found in [GW], p.74.

3.7 Adjoint action

The adjoint action of H on $\mathfrak{g}$ is given by

$$\pi(h)X = h\,X\,h^{-1}, h \in H, X \in \mathfrak{g}.$$

The isomorphism between $\mathbb{Z}^l$ and the ring of characters $\mathcal{X}(H)$ sends $\lambda = [\lambda_1, \ldots, \lambda_l] \in \mathbb{Z}^l$ to the character $h^\lambda \in \mathcal{X}(H)$,

$$h^\lambda = \Pi_{k=1}^l x_k(h)^{\lambda_k}, \tag{3.6}$$

where x_k is the k-th coordinate function on H.

For calculations in $\mathfrak{h}^*$, we use elements ε_i defined on the space of all diagonal matrices by $\varepsilon_i(A) = a_i$, $A = diag[a_1, \ldots, a_n], i \leq n$. Elements $\varepsilon_1, \ldots, \varepsilon_l$ form a basis for $\mathfrak{h}^*$ if $G = \boldsymbol{GL}(l, \mathbb{C}), \boldsymbol{SO}(\mathbb{C}^{2l+1}, B), \boldsymbol{Sp}(\mathbb{C}^{2l}, \Omega)$, and $\boldsymbol{SO}(\mathbb{C}^{2l}, B)$, while in the case $\boldsymbol{SL}(l+1, \mathbb{C})$, the basis for $\mathfrak{h}^*$ is given by $\{\varepsilon_i - \frac{l}{l+1}(\varepsilon_1 + \ldots + \varepsilon_{l+1})\}, i = 1, \ldots, l$.

3.7.1 Roots and root spaces

Let G be one group from the above list of the four classical series of complex simple Lie groups and H its Cartan subgroup of G (defined as the set of all diagonal matrices in G) (see Sect. A.3 3.6.1). Now we shall quickly review well known properties of the adjoint action of the H on $\mathfrak{g}$. A classical idea is to study simultaneous eigenspaces for the adjoint action of the the (maximal) algebraic torus $H \subset G$.

For any finite-dimensional regular representation (ρ, V) and its differential $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, the operators $d\rho(h) : V \to V$ for $h \in \mathfrak{h}$ can be simultaneously diagonalised. Moreover, on a simultaneous eigenspace, the eigenvalue depends linearly on H, so it is described by a linear funcional $\lambda : \mathfrak{h} \to \mathbb{C}$. It leads to the following definitions.

Definition 3.17. For any linear functional $\lambda : \mathfrak{h} \to \mathbb{C}$ on a Cartan algebra $\mathfrak{h}$, we define the corresponding joint eigenspace by $V_\lambda = \{v \in V : \rho(h)v = \lambda(h)v \ \forall h \in \mathfrak{h}\}$. Element $\lambda \in \mathfrak{h}^*$ is called **weight** , if the **the weight space** V_λ is nontrivial. The set of all weights is a finite subset of $\mathfrak{h}^*$ and will be denoted by $wt(V)$. In the case when $\rho = Ad$ and $d\rho = ad$, the weights are called **roots** and the weight space is called **the root space**. The set of all roots will be denoted by Φ.

The key information about the structure of $\mathfrak{g}$ is containted in the following **root space decomposition.**

rootdecomp **Theorem 3.18.** *We have*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha. \tag{3.7}$$

Proof. It follows from Theor. 3.8, taking into account that the weight space corresponding to the weight zero is exactly the Cartan subalgebra $\mathfrak{h}$.

The Jacobi identity implies that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \tag{3.8}$$ gradation1

for $\alpha, \beta \in \Phi$. More generally, the same computation shows that

$$\mathfrak{g}_\alpha(V_\lambda) \subset V_{\lambda+\alpha}. \tag{3.9}$$ gradation1

□

3.8 The root system of a semisimple Lie algebra

rsprop1 **Lemma 3.19.** *Let $G \subset \boldsymbol{GL}(n, \mathbb{C})$ be a connected classical group with the maximal torus H. Suppose that $\Phi \subset \mathfrak{h}^*$ is the root system for $\mathfrak{g}$. Define a symmetric bilinear form (X, Y) on $\mathfrak{g}$ by $(X, Y) = tr_{\mathbb{C}^n}(XY)$. Then*

(1) $\dim \mathfrak{g}_\alpha = 1$ *for all* $\alpha \in \Phi$*;*

(2) $\alpha \in \Phi$ *if and only if* $-\alpha \in \Phi$; $\alpha \in \Phi, c\alpha \in \Phi, c \in \mathbb{C}$ *imply* $c = \pm 1$;

(3) The form (X, Y) *is invariant, i.e.*

$$([X, Y], Z) = -(Y, [X, Z]); X, Y, Z \in \mathfrak{g};$$

and its restriction to $\mathfrak{h}$ *is positive definite.*

(4) $(\mathfrak{h}, \mathfrak{g}_\alpha) = 0$ *and* $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ *unless* $\alpha = -\beta$;

(5) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$; $\alpha, \beta \in \Phi$;, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$.

Proof. Items (1) and (2) are visible from the explicit lists of roots in App. A.4 for all four series of classical groups.
(3) We have the relation

$$([X, Y], Z) = tr(XYZ - YXZ) = tr(YZX - YXZ) = -tr(Y[X, Z]) = -(Y, [X, Z]).$$

The scalar product restricted to diagonal matrices is clearly Euclidean one.
(4) For the second relation, suppose that there is $A \in \mathfrak{h}$ with $(\alpha + \beta)(A) \neq 0$. Then for all $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta$, we have

$$0 = ([A, X], Y) + (X, [A, Y]) = (\alpha + \beta)(A)(X, Y).$$

The first relation follows from the second taking into account that $\mathfrak{h}$ is $\mathfrak{g}_\beta$ for trivial vector β.
(5) It follows from Jacobi identity.

□

3.8.1 Simple roots

The set Φ of all roots spans $\mathfrak{h}^*$. We are going to choose a basis for $\mathfrak{h}^*$.

Definition 3.20. A subset $\Delta = \{\alpha_1, \dots, \alpha_l\} \subset \Phi$ is a set of **simple roots**, if every $\gamma \in \Phi$ can be uniquely written as a linear combination $\gamma = \sum_{i=1}^{l} n_i \alpha_i$, where n_i are integers having all the same sign.

It divides Φ as a union of two sets

$$\Phi = \Phi^+ \cup (-\Phi^+),$$

where Φ^+ contains roots with nonnegative coefficients in the above decomposition. Elements in Φ^+ are **positive roots** (with respect to the choice of Δ).

3.8.2 $\mathfrak{sl}_2$-subalgebras, the Cartan matrix

The key information for classification of irreducible regular representations of classical linear algebraic groups are Lie subalgebras of $\mathfrak{g}$ isomorphic with $\mathfrak{sl}(2, \mathbb{C})$. For every root α, there is one such subalgebra.

SL2T **Lemma 3.21.** *For every $\alpha \in \Phi^+$, there exist $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that the element $h_\alpha = [e_\alpha, f_\alpha]$ satisfies $\alpha(h_\alpha) = 2$. Then*

$$[h_\alpha, e_\alpha] = 2e_\alpha, \ [h_\alpha, f_\alpha] = -2f_\alpha,$$

hence these three elements generate the subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. We call such triples $\mathfrak{sl}_2$-triples (SL2T).

The element $h_\alpha \in \mathfrak{h}$ is fixed by the normalization condition above uniquely inside one-dimensional space $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and is called the **coroot** *to α and often denoted by $\check{\alpha}$.*

The **Cartan matrix** C is defined by $C_{ij} = \alpha_j(h_{\alpha_i})$. All diagonal elements of C are equal to 2, and all off-diagonal elements of C are non-positive and integral.

The form of the Cartan matrix is encoded in the corresponding **Dynkin diagram.** This is a graph with a node for every simple root. Nodes corresponding to α_i and α_j are connected with $C_{ij}C_{ji}$ lines. If the roots do not have the same length (with respect to the scalar product (X,Y)), the inequality sign is placed on the lines indicating which root is longer.

All such SL2 triples are described explicitely in App. A.4 for classical algebraic groups.

3.8.3 Borel subalgebra, Borel subgroup

Definition 3.22. Let $\mathfrak{g}$ be a classical simple complex Lie algebra of a linear algebraic group G equal to $\boldsymbol{SL}(n,\mathbb{C}), \boldsymbol{SO}(n,B)$, or $\boldsymbol{Sp}(l,\Omega), n = 2l$. Let $H \subset G$ be a Cartan subgroup and $\mathfrak{h}$ its Lie algebra.

Suppose that

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

is the decomposition of $\mathfrak{g}$ with $\mathfrak{n}^- = \oplus_{\alpha\in\Phi_-}\mathfrak{g}_\alpha$ and $\mathfrak{n}^+ = \oplus_{\alpha\in\Phi_+}\mathfrak{g}_\alpha$.

The Lie subalgebra $\mathfrak{b} = \mathfrak{h} \ominus \mathfrak{n}^+ \subset \mathfrak{g}$ is called the **Borel subalgebra** of $\mathfrak{g}$.

Let moreover N_n^+ be the space of all $n \times n$ upper triangular unipotent matrices and $N^+ = G \cap N_n^+$.

The group $B = H \cdot N^+ \subset G$ is called the **Borel subgroup** of G.

Then we have the following fact.

N+ **Lemma 3.23.** *Let $G \subset \boldsymbol{GL}(n,\mathbb{C})$ be a connected classical linear algebraic group and B its Borel subgroup. Let $N^+ = G \cap N_n^+$. Then:*

(1) The group N^+ is connected, unipotent, and normal in B.

(2) The Lie algebra of N^+ is $\mathfrak{n}^+$.

(3) $B = H \ltimes N^+$ is a semidirect product.

The proof of Lemma is an interesting application of action of the group G on the homogeneous space G/B, which can be realized as a flag manifold (resp. isotropic flag manifold). Details can be found in [GW], Sect. 11.3.

1

Chapter 4

Irreducible regular representations for classical groups and algebras

In this chapter, we describe the set of all regular irreducible representations for classical complex Lie algebras and for classical simple linear algebraic groups. It gives then the complete description of all regular representations, because these groups and Lie algebras are reductive and any finite dimensional representation decomposes into a direct sum of irreducible pieces. In many treatments ([FH],[H]), the attention is concentrated to the case of Lie algebras, where it is possible to use algebraic methods for the classification. As argued in previous chapter there is a very close relation between representation of a linear algebraic group G and its Lie algebra $\mathfrak{g}$. But it needs some care to specify the relation of individual cases precisely. Classification of regular irreducible representations will be given using the highest weight theory. We shall first discuss classification of irreducible representations of $\mathfrak{g}$ for four main series of classical complex Lie algebras. Then we shall show which ones are differentials of irreducible representations of the corresponding classical linear algebraic group G. Finally, we shall discuss classification of irreducible regular representations for special classical cases - the group $\boldsymbol{GL(V)}$ (which is reductive but not simple) and the group $\boldsymbol{O(V)}$, which is not connected. In this chapter, we concentrate our attention to classical cases. Hence if not specified otherwise, the group G will be one of simple classical linear algebraic groups given by the following list: $\boldsymbol{SL(n,\mathbb{C})}$, $\boldsymbol{SO(n,B)}$, or $\boldsymbol{Sp(n,\Omega)}$. The bilinear forms B and Ω giving a realisation of orthogonal, resp. symplectic groups will be chosen as in Sect. 3.6.1. It leads to a comfortable form of the Cartan and Borel subgroups (resp. subalgebras).

The Cartan subalgebra $\mathfrak{h}$ was defined as the Lie algebra of a maximal torus in G. (It could be defined also as a maximal commutative subalgebra of $\mathfrak{g}$.) The basic idea of classification below is to decompose a given irreducible representation into a direct sum of joint eigenspaces (called the weight spaces) for the action of elements of $\mathfrak{h}$ and to characterize the representation by the choice of a suitable extremal element among all weight spaces. Joint eigenvalues (called weights) for the weight spaces are elements of a lattice in the real subspace of $\mathfrak{h}^*$ (called the weight lattice), which is specific for every classical complex Lie algebra. For a given choice of a partial order on the weight lattice, it is possible to show that there is a unique highest weight among all weights of a given irreducible representation satisfying a suitable dominant codition. The set of dominant weights in the weight lattice gives us hence a parametrization of the set of (isomorphic classes of) regular irreducible representations.

4.1 Weight lattices, dominant integral weight, fundamental weights

Let $\mathfrak{g}$ be a classical simple complex Lie algebra.

Definition 4.1. The **weight lattice** $P(\mathfrak{g})$ is defined by

$$P(\mathfrak{g}) = \{\mu \in \mathfrak{h}^* | \mu(h_\alpha) \in \mathbb{Z}, \alpha \in \Phi\}.$$

The **root lattice** $Q(\mathfrak{g})$ is the additive subgroup generated by Φ. Hence $Q(\mathfrak{g}) \subset P(\mathfrak{g})$.

The weight λ is called **integral,** if $\lambda(h_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi$.

The weight λ is called **dominant,** if $\lambda(h_\alpha) \geq 0$ for all $\alpha \in \Phi$.

The semigroup $P(\mathfrak{g})_{++}$ of **integral dominant weights** is hence defined as

$$P(\mathfrak{g})_{++} = \{\mu \in \mathfrak{h}^* | \mu(h_\alpha) \in \mathbb{Z};\ \mu(h_\alpha) \geq 0\ \forall \alpha \in \Phi\}.$$

We shall define a partial order on the weight lattice as follows. Take two elements λ, μ in $\mathfrak{h}^*$. Then $\mu \prec \lambda$ iff there are elements $\beta_1, \ldots, \beta_r \in \Phi^+$ such that $\mu = \lambda - \beta_1 \ldots - \beta_r$. This order will be called the **root order.**

Let us denote by $\mathfrak{h}^*_\mathbb{R}$ the real vector subspace of $\mathfrak{h}^*$ generated by Q. The weight and root lattices are contained in it. Similarly, we denote by $\mathfrak{h}_\mathbb{R}$ the real vector subspace in $\mathfrak{h}$ generated by corrots. The bilinear from $(X, Y) = tr(XY)$ on $\mathfrak{g}$ restricts to a positive definite form on $\mathfrak{h}_\mathbb{R}$ (which consists of diagonal matrices).

Definition 4.2. Let $\alpha_1, \ldots, \alpha_l$ be the set of simple roots for classical simple complex Lie algebra $\mathfrak{g}$ and denote by $h_i = h_{\alpha_i}, i = 1, \ldots, l$ the corresponding coroots. They form a basis for $\mathfrak{h}$. Elements of the dual basis $\omega_1, \ldots, \omega_l$ of $\mathfrak{h}^*$ are called **fundamental weights**. They are characterized by the relations

$$\omega_i(h_j) = \delta_{ij}.$$

The lattice $P_{++}(\mathfrak{g})$ of integral dominant weights has a simple structure of a free semi-group, fundamental weights are its generators.

mfg **Theorem 4.3.** *Let $\mathfrak{g}$ be a classical simple complex Lie algebra. Then we have*

$$P_{++}(\mathfrak{g}) = \{\lambda \in \mathfrak{h}^* | \lambda = \sum_1^l n_i \omega_i, n_i \in \mathbb{Z}, n_i \geq 0\}.$$

In particular

(1) Let $G = \boldsymbol{SL}(n, \mathbb{C})$. Then $\mu \in P_{++}(\mathfrak{g})$ iff $\mu = k_1\varepsilon_1 + \ldots + k_n\varepsilon_n$ with $k_1 \geq k_2 \geq \ldots \geq k_n$ and $k_i - k_{i+1} \in \mathbb{Z}$.

(2) Let $G = \boldsymbol{SO}(2l+1, \mathbb{C})$. Then $\mu \in P_{++}(\mathfrak{g})$ iff $\mu = k_1\varepsilon_1 + \ldots + k_n\varepsilon_n$ with $k_1 \geq k_2 \geq \ldots \geq k_l \geq 0$ and $2k_i$ and $k_i - k_j$ are integers for all i, j.

(3) Let $G = \boldsymbol{Sp}(2l, \mathbb{C})$. Then $\mu \in P_{++}(\mathfrak{g})$ iff $\mu = k_1\varepsilon_1 + \ldots + k_n\varepsilon_n$ with $k_1 \geq k_2 \geq \ldots \geq k_l \geq 0$ and k_i are integers for all i.

(4) Let $G = \boldsymbol{SO}(2l, \mathbb{C})$. Then $\mu \in P_{++}(\mathfrak{g})$ iff $\mu = k_1\varepsilon_1 + \ldots + k_n\varepsilon_n$ with $k_1 \geq k_2 \geq \ldots \geq k_{l-1} \geq |k_l|$ and $2k_i$ and $k_i - k_j$ are integers for all i, j.

The statements in the theorem can be easily checked case-by-case by looking at explicit description of the sets $P_{++}(\mathfrak{g})$ and of fundamental weights at App. A.4.

Definition 4.4. Let $\mathfrak{g}$ be a classical simple complex Lie algebra, and (π, V) its regular representation, then its **weight** μ is an element from $\mathfrak{h}^*$ such that the weight space

$$V(\mu) = \{v \in V | \pi(X)v = \mu(X)v, X \in \mathfrak{h}\}$$

is nontrivial. Denote by $wt(V)$ the set of all weights of V.

Lemma 4.5. *Let (π, V) is a regular representation of a classical simple complex Lie algebra $\mathfrak{g}$ and $wt(V)$ the set of all weights of (π, V).*

Then $wt(V) \subset P(\mathfrak{g})$ and

$$V = \oplus_{\mu \in wt(V)} V(\mu).$$

Proof. Consider three-dimensional subalgebra $\boldsymbol{s}(\alpha)$ containing h_α, which is isomorphic to $\boldsymbol{sl}(2, \mathbb{C})$ and consider V as a representation of $\boldsymbol{s}(\alpha)$. The first claim in the lemma follows from the fact that $\boldsymbol{sl}(2, \mathbb{C})$ is reductive and from properties of irreducible finite dimensional representations of $\boldsymbol{sl}(2, \mathbb{C})$.

Coroots h_α generate $\mathfrak{h}$, the operators $\pi(h_\alpha)$ commutes and are diagonalizable, which implies the second claim of the theorem.

□

Recall that the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is given by the sum $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^+ = \oplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. The choice of positive roots implies a partial order on the weight lattice and the highest weight vector $v \in V$ can be interpreted as a joint eigenvector for the Borel subalgebra, with trivial eigenvalues for action of elements from $\mathfrak{n}^+$. All that is formalized in the following definition.

Definition 4.6. Let (ρ, V) be a regular representation of a classical Lie algebra $\mathfrak{g}$ and choose a set Δ of simple roots in Φ. A nontrivial vector $v \in V$ is called the **highest weight vector** (with respect to the choice of Δ), if:

(1) $\rho(X)v = 0$ for all $X \in \mathfrak{n}^+$;

(2) There is $\lambda \in \mathfrak{h}^*$ such that $\rho(H)v = \lambda(H)v, H \in \mathfrak{h}$.

The element $\lambda \in \mathfrak{h}^*$ is called the **highest weight** of V. The highest weight vector v is often called the **singular** vector for the representation (ρ, V).

Now we shall state and give main lines of the proof of the classification of regular irreducible representations of $\mathfrak{g}$ using highest weights. First we shall show that the highest weight of such representation is unique, then we prove that isomorphic representations have the same highest weight, and finally we shall characterize the image of the map $(\rho, V) \mapsto \lambda_V$.

4.2 Highest weight theorem for $\mathfrak{g}$

hwta **Theorem 4.7** (Highest weight theorem). *Let $\mathfrak{g}$ be a classical complex Lie algebra and (ρ, V) its regular irreducible representation. Then:*

(1) There is a unique (up to a multiple) highest weight vector $v \in V$. The corresponding highest weight will be denoted by λ_V.

(2) Two regular irreducible representations (ρ_1, V_1) and (ρ_2, V_2) are isomorphic if and only if $\lambda_{V_1} = \lambda_{V_2}$.

(3) $\lambda_V \in P_{++}(\mathfrak{g})$, *i.e.* λ_V *is integral and dominant for* $\mathfrak{g}$.

(4) For every $\mu \in P_{++}(\mathfrak{g})$, *there is a regular irreducible representation* (ρ, V) *of* $\mathfrak{g}$ *such that* $\lambda_V = \mu$.

Proof. (1) Let λ be a maximal element in $wt(V)$ with respect to the root order. By maximality, any vector in V_λ is killed by the action of elements from the root spaces $\mathfrak{g}_\alpha, \alpha \in \Phi^+$, hence by $\pi(\mathfrak{n}^+)$. Choose $v_0 \in V_\lambda$. Invariant subspace generated by the action of $\pi(\mathfrak{n}_-)$ on v_0 coincides with V (by irreducibility). Hence $V = \mathbb{C}v_0 \oplus \bigoplus_{\mu \prec \lambda} V_\mu$ and $dim V_\lambda = 1$.
(2) Isomorphic representations have the same highest weight. On the contrary, suppose that $\lambda_{V_1} = \lambda_{V_2} = \alpha$ and v_1, resp. v_2 are corresponding highest weight vectors. Then $v_1 \oplus v_2$ is a highest weight vector of weight α in $V_1 \oplus V_2$, which generates under the action of $\mathfrak{g}$ another irreducible representation $U \subset V_1 \oplus V_2$. Projections $\pi_1 : U \to V_1$ and $\pi_2 : U \to V_2$, are nontrivial, hence isomorphisms.
(3) We know already that λ_V is integral. For every $\alpha \in \Phi^+$, we can restrict the representation π to $\boldsymbol{s}(\alpha)$ generated by the corresponding triple $\{e_\alpha, f_\alpha, h_\alpha\}$ and by representation theory for $\boldsymbol{sl}(2, \mathbb{C})$, we get that λ_V should be dominant.
(4) Using the fact that the semigroup of dominant integral weights is generated by fundamental weights, we shall construct all regular irreducible representations using the Cartan product of representations corresponding to fundamental weights. Then we shall describe regular irreducible representations for fundamental weights by case-by-case analysis.

Cartan product

Let V and W be irreducible $\mathfrak{g}$-representations with highest weights λ and μ, respectively. If $v \in V$ and $w \in W$ are the highest weight vectors, then $v \otimes w$ is a highest weight vector in the tensor product representation $V \otimes W$ with the highest weight $\lambda + \mu$. This vector generates an irreducible subrepresentation of $V \otimes W$ with highest weight $\lambda + \mu$. Moreover, if $V = \bigoplus V_{\lambda'}$ and $W = \bigoplus W_{\mu'}$ are the weight decompositions, then the weight spaces in $V \otimes W$ have the form $\bigoplus_{\lambda'+\mu'=\nu} V_{\lambda'} \otimes W_{\mu'}$. In particular, for any weight ν of $V \otimes W$, we have $\nu \precsim \lambda + \mu$. The subrepresentation generated by the highest weight vector $v \otimes w$ is called Cartan product of V and W and is denoted by $V \circledcirc W$.

Recall that dominant integral weights are linear combinations of the fundamental weights with nonnegative integral coefficients. The irreducible representation V_i with the highest weight ω_i is called the ith fundamental representation. Suppose that we have constructed the fundamental representations $V_1, \cdots, V_n$. Given a dominant integral weight $\lambda = a_1\omega_1 + \cdots + a_n\omega_n$, consider the representation $V_1^{\otimes a_1} \otimes \cdots V_n^{\otimes a_n}$. From above we see that this contains a unique (up to scale) highest weight vector of weight λ. Hence by this procedure we can find all irreducible representations. However, the essential step that remains is to construct fundamental representations.

Fundamental representations:

(1) $\mathbf{A}_l, G = \boldsymbol{SL}(l+1, \mathbb{C})$ Let (π, V) be the defining representation of G on $V = \mathbb{C}^{l+1}$. It induces the representation $(\otimes^k(\pi), \otimes^k(V))$ on the tensor power of V. The subspace $\Lambda^k(V)$ is clearly invariant subspace of $\otimes^k(V)$, which defines a new representation $(\sigma_k, \Lambda^k(V))$. By definition of fundamental weights, we must have $\frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$, from which one easily concludes that $\omega_i = e_1 + \cdots + e_i$ for $i = 1, \cdots, l-1$. If $\{e_1, \dots, e_{l+1}\}$ is the canonical basis of V, then each e_i is a weight vector of weight ε_i. Consequently, the highest weight of V is ε_1, so V contains the fundamental representation V_1 as an irreducible subrepresentation. Next, consider the exterior powers $\Lambda^j V$ of the standard representations for $j = 2, \dots, l$. The

elements $e_{i_1} \wedge \ldots \wedge e_{i_j}$ for $1 \leq i_1 < \ldots < i_j \leq n$ form a basis for $\Lambda^j V$, which implies that the weights of $\Lambda^j V$ are given by all expressions of the form $\varepsilon_{i_1} + \cdots + \varepsilon_{i_j}$ for $1 \leq i_1 < \cdots < i_j \leq n$. In particular, $\omega_j = e_1 + \cdots + e_j$ is a weight (and in fact the highest weight) of $\Lambda^j V$, so this contains the fundamental representation V_j as an irreducible subrepresentation.

By Theorem akceW 4.11, the set of weights of a representation must be invariant under the action of the Weyl group. We know that the Weyl group of $\mathfrak{sl}(n, \mathbb{C})$ is the permutation group $\mathfrak{S}_n$ which permutes the e_i. But this shows that for each of the representations $\Lambda^j V$ all the weights are obtained from the highest weight by the action of the Weyl group. In particular, the fundamental representation cannot be strictly smaller, so $\Lambda^j V$ is a fundamental representation for $j = 1, \ldots, n-1$, and we have found all fundamental representations.

(2) $\mathbf{D}_l, G = \boldsymbol{SO}(\mathbb{C}^{2l}, B)$ Similarly to the previous example, one can find $\omega_j = \varepsilon_1 + \ldots + \varepsilon_j$ for $j < l-1$, $\omega_{l-1} = \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_{n-1} - \varepsilon_n)$ and $\omega_l = \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_n)$. For the standard representation $V = \mathbb{C}^{2l}$, one immediately sees that the weights are $\pm\varepsilon_i$ for $i = 1, \ldots, l$. Since the Weyl group acts by permuting the ε_i and changing the sign of an even number of ε's, we see that the orbit of the highest weight ε_1 under the Weyl group is exactly the set of all $\pm e_j$, so we see that $\mathbb{C}^{2n}$ is the first fundamental representation V_1. As in the case of $\mathfrak{sl}(n, \mathbb{C})$ above, we next conclude that for $j = 2, \cdots, n-2$ the exterior power $\Lambda^j V$ has to contain the fundamental representation V_j. It turns out that $\Lambda^j V$ is irreducible for $j = 1, \cdots, l-1$ and splits into two irreducible components for $j = l$. In particular, for $j = 1, \cdots, l-2$ the representation $\Lambda^j V$ is the jth fundamental representation. (Interested reader can find details in [GW],p.269)

The remaining two fundamental representations are the two spin representations. They cannot show up in any tensor power of the standard representations, since a tensor power contains only weights which are integral linear combinations of the e_j, while half integers cannot occur. A construction of the spin representations is described, e.g., in [GW], Chapt.6.

(3) $\mathbf{D}_l, G = \boldsymbol{SO}(\mathbb{C}^{2l+1}, B)$

One can verify that $\omega_j = \varepsilon_1 + \cdots + \varepsilon_j$ for $k = 1, \ldots, l-1$, while $\omega_l = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l)$. Similarly, as for the even orthogonal algebras, one shows that for the standard representation $V = \mathbb{C}^{2l+1}$ the exterior power $\Lambda^j V$ is the jth fundamental representation for $j = 1, \cdots, l-1$. In fact, even $\Lambda^l V$ is irreducible, but not a fundamental representation. It should be noted, however, that for the odd orthogonal algebras the weights of standard representations are not in one orbit of the Weyl group any more, since apart from $\pm e_j$ also 0 is a weight. Further details can be found in [GW,p.269]. The last fundamental representation is the spin representation (see [GW], Chapt. 6).

(4) $\mathbf{C}_l, G = \boldsymbol{Sp}(\mathbb{C}^{2l}, \Omega)$

One concludes that $\omega_j = \varepsilon_1 + \cdots + \varepsilon_j$ for all $j = 1, \cdots, l$. The weights of the standard representation $V = \mathbb{C}^{2l}$ are given by $\pm\varepsilon_j$ for $j = 1, \ldots, l$, so they lie on one orbit of the Weyl group, which acts by permutations and sign changes of the ε_j. Thus, V is irreducible and hence coincides with the fundamental representation V_1. The fundamental weight ω_j shows up as the weight of a highest weight vector in $\Lambda^j V$, but in contrast to the earlier cases, the exterior powers are not irreducible any more. The point here is that the symplectic form on V is an invariant elements of $\Lambda^2 V^*$. Contracting with this form defines a $\mathfrak{sp}(2n, \mathbb{C})$-homomorphism $\Lambda^j V \to \Lambda^{j-2} V$. Since this map is clearly nonzero, its kernel is a nontrivial subrepresentation V_j, which turns out to be the jth fundamental representation. Further details can be found in [GW],p.271.

□

4.2.1 Highest weight vectors for G.

Definition 4.8. Let G be a classical linear algebraic group and B its Borel subgroup. The **highest weight vector** $v \in V$ for a representation (ρ, V) of G is a joint eigenvector for all $\rho(b), b \in B$. Note that this condition implies that $\rho(b)v = \psi(b)v, b \in B$, where $\psi : B \to \mathbb{C}^\times$ is a character of B. We shall denote it by ψ_V.

3.11 **Lemma 4.9.** *Let G be a classical group and B its Borel subgroup Let (ρ, V) be a regular representation of G and $(d\rho, V)$ its differential.*

Then the two following claims are equivalent.

(1) The vector $v \in V$ is a highest weight vector for the representation (ρ, V) of G.

(2) Vector $v \in V$ is a highest weight vector for the representation $(d\rho, V)$ of $\mathfrak{g}$.

Moreover, λ_V is the weight of v for $(d\rho, V)$ iff the character ψ_V restricted to H is of the form e^{λ_V} (recall here (3.6)).

Proof. Write $B = H.N^+$, where H is the Cartan subgroup and N^+ is defined in Lemma 3.23. The group N^+ is unipotent, $\mathfrak{n}^+$ is nilpotent Lie algebra and exp is a $1-1$ map from $\mathfrak{n}^+$ onto N^+. Hence any character for B is trivial on N^+ and is determined by its restriction to H.

Let first consider the representation ρ of B restricted to H and suppose that a character $\psi \in \mathcal{X}(V)$ has a form e_V^λ.

Then $\rho(h)v = \psi(h)v, h \in H$ iff $d\rho(A)v = d\chi(A)v = \lambda(A)v$.

Now consider the representation ρ on N^+. We know that exp is $1-1$ map from $\mathfrak{n}^+$ onto N^+. Hence for every $u \in N^+$, there is a nilpotent element $N \in \mathfrak{n}^+$ with $u = \exp N$. Using Theorem 2.28 (2), we get $\rho(u)v = \rho(\exp N)v = \exp(d\rho(N))v$. Hence $\exp(N)v = 0$ for all $N \in \mathfrak{n}^+$ iff $\psi(u) = 1$ for all $u \in U$.

(2) Now let us restrict the character ψ to the torus H. We know that we can write ψ on H as t^λ (see (3.6)) and that $dt^\lambda(A) = (\lambda, A), A \in \mathfrak{h}$.

□

4.3 Weyl group

Definition 4.10. Let G be a classical simple complex Lie group. The scalar product $(X, Y) = tr(XY)$ is positive definite after restriction to $\mathfrak{h}_\mathbb{R}$ and it defines dually a positive definite scalar product on $\mathfrak{h}^*$.

For every positive root α, we define the reflection $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

The reflection s_α is characterized by the relations

$$s_\alpha(\alpha) = -\alpha; (\alpha, \beta) = 0 \implies \alpha(\beta) = \beta.$$

The Weyl group W is the subgroup of End$(\mathfrak{h}^*)$ generated by all reflections $s_\alpha, \alpha \in \Phi^+$.

(1) Type $\mathbf{A}_l$, $G = \mathbf{SL}(l+1, \mathbb{C})$ For the root system A_l of $\mathfrak{sl}(l+1, \mathbb{C})$, one can easily verify that the root reflection $s_{\varepsilon_i - \varepsilon_j} : \mathfrak{h}^* \to \mathfrak{h}^*$ is induced by the map which exchanges ε_i and ε_j and leaves the ε_k for $k \neq i, j$ untouched. Hence, the Weyl group W of A_l is the permutation group $\mathfrak{S}_{l+1}$ of $l+1$ elements.

(2) Type $\mathrm{B}_l, G = \mathbf{SO}(\mathbb{C}^{2l+1}, B)$

For the root system of $\mathfrak{so}(2l+1, \mathbb{C})$, the reflection $s_{\varepsilon_i - \varepsilon_j}$ again exchanges ε_i and ε_j, while the reflection s_{ε_j} changes the sign of ε_j and leaves the other ε_k untouched. Thus, we may view the Weyl group W as the subgroup of all permutations σ of the $2l$ elements $\pm\varepsilon_j$ such that $\sigma(-\varepsilon_j) = -\sigma(\varepsilon_j)\}$ for all $j = 1, \cdots, l$. In particular, W has $l!2^l$ elements.

(3) Type $\mathrm{C}_l, G = \mathbf{Sp}(\mathbb{C}^{2l}, \Omega)$

Since the reflection corresponding to $2\varepsilon_j$ coincides with the reflection for ε_j, we get the same Weyl group for the root system C_l as for B_l.

(4) Type $\mathrm{D}_l, G = \mathbf{SO}(\mathbb{C}^{2l}, B)$

For the even orthogonal root system D_l, the reflection $s_{\varepsilon_i - \varepsilon_j}$ again generate permutations of the ε_j, while the reflection $s_{\varepsilon_i + \varepsilon_j}$ maps ε_i to $-\varepsilon_j$ and ε_j to $-\varepsilon_i$, all other ε_k remain untouched. Consequently, W can be viewed as the subgroup of those permutations π of the elements $\pm\varepsilon_j$ which satisfy $\pi(-\varepsilon_j) = -\pi(\varepsilon_j)$ and have the property that the number of j such that $\pi(\varepsilon_j) = \varepsilon_k$ for some k is even. In particular, the number of elements in W equals $l!2^{l-1}$.

Note that the Weyl group W for $\boldsymbol{SL}(2, \mathbb{C})$ has just two elements and that the nontrivial element of W is represented by the right hand side of an interesting relation

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

This is an inspiration for the following theorem.

akceW **Theorem 4.11.** *Let (π, V) be a finite-dimensional representation of $\mathfrak{g}$ and*

$$V = \oplus_{\lambda \in wt(V)} V_\lambda$$

its weight decomposition. Suppose that $\{e_\alpha, f_\alpha, h_\alpha\}$ is an SL2 triple for $\alpha \in \Phi^+$, then elements $E = \pi(e_\alpha)$ and $F = \pi(f_\alpha)$ are nilpotent and we can define

$$\tau_\alpha = \exp(E)\exp(-F)\exp(E) \in \boldsymbol{GL}(V).$$

Then

(1) $\tau_\alpha \pi(Y) \tau_\alpha^{-1} = \pi(s_\alpha Y), Y \in \mathfrak{h}$;

(2) The action of W preserve the space of weights $\mathcal{X}(V)$ and the multiplicity $m_\lambda = dim V_\lambda$ is invariant under the action of W.

Proof. If $\{e, f, h\}$ have the canonical commutation relation for $\mathfrak{sl}(2, \mathbb{C})$, then images of e and f are nilpotent in any finite dimensional representation.

exponential

It follows from Theor.2.28 that $ad(X)$ is nilpotent on $\mathrm{End}(V)$ for any nilpotent element $X \in \mathrm{End}(V)$ and that we have

$$\exp(X)A\exp(-X) = \exp(adX)A, A \in \mathrm{End}(V). \tag{4.1}$$

(1) We need now to prove three claims:

(a) $\exp(E)\pi(Y)\exp(-E) = \pi(Y) - \alpha(Y)E;\ Y \in \mathfrak{h}$;

(b) $\exp(-F)\pi(Y)\exp(F) = \pi(Y) - \alpha(Y)F;\ Y \in \mathfrak{h}$;

(c) $\exp(E)F\exp(-E) = F + \pi(h_\alpha) - E$.

Having them available, we get for $Y \in \mathfrak{h}$

$$\begin{aligned} \tau_\alpha \pi(Y)\tau_\alpha^{-1} &= \exp(E)\exp(-F)[\pi(Y) - \alpha(Y)E]\exp(F)\exp(-E) && (4.2)\\ &= \exp(E)[\pi(Y) - \alpha(Y)E - \alpha(Y)\pi(h_\alpha))]\exp(-E) && (4.3)\\ &= \pi(Y) - \alpha(Y)\pi(h_\alpha) = \pi(s_\alpha Y). && (4.4) \end{aligned}$$

So to finish the proof, we have to verify the three claims above.
(a) We have

$$ad(E)\pi(Y) = -\pi(ad(Y)e_\alpha) = -\alpha(Y)E$$

for any $Y \in \mathfrak{h}$. Hence it is sufficient to use (4.1) and the relation $ad(E)^2(\pi(Y)) = 0$.
(b) The map given by

$$e_\alpha \mapsto -f_\alpha;\ f_\alpha \mapsto -e_\alpha;\ h_\alpha \mapsto -h_\alpha \qquad (4.5)$$

is an automorphism of the SL2 triple for α, so the calculation in (a) works as well for (b).
(c) The claim (a) implies $\exp(E)\pi(h_\alpha)\exp(-E) = \pi(h_\alpha) - 2E$. Moreover, $ad(E)F = \pi(h_\alpha)$ and $ad(E)^2F = -2E$. So we can use $(adE)^3F = 0$, (4.1) and automorphism (4.5) to prove the claim (c).
(2) Suppose that $\pi(Y)v = \lambda(Y)v, v \in V, Y \in \mathfrak{h}$. Then $\pi(Y)\tau_\alpha v = \tau_\alpha \pi(s_\alpha Y)v = [s_\alpha \lambda](Y)\tau_\alpha v$. □

4.4 Highest weight theorem for G

The previous section contains a classification of regular irreducible representations of simple classical complex Lie algebras (four series). Every regular irreducible representation (ρ, V) of a linear algebraic group G induces the regular irreducible representation $(d\rho, V)$ of its Lie algebra $\mathfrak{g}$ but the opposite implication is not true. There are regular irreducible representations (σ, V) of $\mathfrak{g}$, which cannot be integrated to a representation of G (i.e., which are not differentials of a regular irreducible representation of G). So we are going to discuss now the classification of regular irreducible representations of classical simple linear algebraic groups. Then, in next sections, we are also going to discuss how to classify regular irreducible representations for other (nonsimple) classical groups ($\boldsymbol{GL}(V)$) and (nonconnected) classical groups ($\boldsymbol{O}(V, B)$).

4.4.1 Weight lattices

Let G be a simple complex linear algebraic group and H its Cartan subgroup. One of important objects for further discussion is the **weight lattice** $P(G)$ of G, which is defined as the space of differentials of all characters of H. We shall see that it is an integral lattice generated by the basis $\varepsilon_1, \ldots, \varepsilon_l$ in $\mathfrak{h}^*$.

Definition 4.12. The weight lattice $P(G)$ of a classical algebraic group G is defined as

$$P(G) = \{d\chi | \chi \in \mathcal{X}(H)\}, \qquad (4.6)$$

where H is a Cartan subgroup of G.

The set $P_{++}(G)$ is defined by the formula

$$P_{++}(G) = P(G) \cap P_{++}(\mathfrak{g})$$

Lemma 4.13. *Let G be a simple complex linear algebraic group. Then*

$$P(G) = \oplus_{k=1}^{l} \mathbb{Z}\varepsilon_k. \tag{4.7}$$

Proof. Let t^λ is the character of H given by (2.20 3.6). Then $dt^\lambda(A) = \langle \lambda, A \rangle, A \in \mathfrak{h}$. Indeed, X_A acts on $\mathcal{O}(H)$ by

$$X_A = \sum_{i=1}^{l} \langle \varepsilon_i, A \rangle x_i \frac{\partial}{\partial x_i}$$

and we get

$$dt^\lambda(A) = X_A(x_1^{\lambda_1} \ldots x_l^{\lambda_l})(1) = \langle \lambda, A \rangle.$$

□

Elements of $P(G)$ are called **integral weights** for G and elements of $P_{++}(G)$ are called **integral and dominant** for G. We have already described explicitely sets $P_{++}(\mathfrak{g})$ in Theorem mfg 4.3. Now we describe also explicitely the sets $P_{++}(G)$.

Theorem 4.14. *(1) When $G = \boldsymbol{SL}(l+1, \mathbb{C})$, then $P_{++}(G) = P_{++}(\mathfrak{g})$.*

(2) When $G = \boldsymbol{SO}(2l+1, \mathbb{C})$, then $\mu \in P_{++}(G)$ iff $\mu = n_1\omega_1 + \ldots n_{l-1}\omega_{l-1} + n_l(2\omega_l)$, where n_i are nonnegative integers.

(3) When $G = \boldsymbol{Sp}(2l, \mathbb{C})$, then $P_{++}(G) = P_{++}(\mathfrak{g})$.

(4) When $G = \boldsymbol{SO}(2l, \mathbb{C})$, then $\mu \in P_{++}(G)$ iff

$$\mu = n_1\omega_1 + \ldots n_{l-2}\omega_{l-2} + n_{l-1}(2\omega_{l-1}) + n_l(2\omega_l) + n_{l+1}(\omega_{l-1} + \omega_l),$$

where n_i are nonnegative integers.

Proof. There is nothing to prove in cases (1) and (3). In odd dimensional orthogonal case, the claim is easily visible, because $P(G) = \sum_{i=1}^{l} \mathbb{Z}\varepsilon_i$, $\omega_i = \varepsilon_1 + \ldots + \varepsilon_i$, $i = 1, \ldots, l-1$ and $2\omega_l$ is equal to $\varepsilon_1 + \ldots + \varepsilon_l$.

In even dimensions, it is easy to check that the lattice generated by elements

$$\omega_1, \ldots, \omega_{l-2}, 2\omega_{l-1}, 2\omega_l, \omega_{l-1} + \omega_l$$

coincides with $P(G)$. (Note that it is the same information as the claim that vectors $\{(1,1), (1,0), (1,-1)\}$ generate the same lattice in $\mathbb{R}^2$ as vectors $\{(1,0), (0,1)\}$.)

□

hwtg **Theorem 4.15** (Highest weight theorem). *Let G be one of the four series A_l, B_l, C_l, D_l of classical linear algebraic groups and suppose that (ρ, V) is its regular irreducible representation. Then*

(1) $(d\rho, V)$ is an irreducible regular representation for $\mathfrak{g}$, hence there here is a unique (up to a multiple) highest weight vector $v \in V$ with the highest weight λ_V.

(2) Two regular irreducible representations (ρ_1, V_1) and (ρ_2, V_2) of G are isomorphic if and only if $\lambda_{V_1} = \lambda_{V_2}$.

(3) $\lambda_V \in P_{++}(G)$, i.e. λ_V is integral and dominant for G.

(4) For every $\mu \in P_{++}(G)$, there is a regular irreducible representation (ρ, V) of G such that $\lambda_V = \mu$.

Proof. (1) The representation $(d\rho, V)$ is irreducible, because (ρ, V) is. Hence Theor. 4.7 implies that there is a unique (up to multiple) highest weight vector in V for $(d\rho, V)$.

(2) The representation (ρ_i, V_i) are isomorphic iff $(d\rho, V_i)$ are.

(3) Any weight for any representation of G should be in $P(G)$ and we know already using Theor. 4.7 that the highest weight λ_V belongs to $P_{++}(\mathfrak{g})$.

(4) The proof of part (4) of the highest weight theorem for algebras shows how to construct representations for weights generating $P_{++}(G)$. All other are constructed using the Cartan product construction.

□

4.5 Frobenius reciprocity

frobenius

Definition 4.16. Let G be a linear algebraic group and H its algebraic subgroup. The ring $\mathcal{O}(H\backslash G)$ of functions invariant under the left translation by elements of H is called the **ring of regular functions** on the homogeneous space $H\backslash G$.

The right regular action of G preserves the space, which hence become a representation of G. Its decomposition under this action is described in the following theorem, called the Frobenius reciprocity.

Let $\hat{G}$ denote the set of equivalence classes of irreducible regular finite-dimensional representations of G.

frobenius

Theorem 4.17. *Under the action of G,*

$$\mathcal{O}(H\backslash G) \simeq \oplus_{\sigma \in \hat{G}} (\dim(\sigma^*)^H) V_\sigma, \tag{4.8}$$

where σ^* is the representation contragredient to σ and $(\sigma^*)^H$ is the space of its H-invariant vectors.

Proof. Let (σ, W) be an irreducible regular representation of G and let λ be an invariant vector for σ^*. Recall the definition of the matrix coefficients

$$\varphi_{\lambda,v}(g) = (\lambda(\sigma(g)v)).$$

for σ and the intertwining the map Φ_λ from σ to the right regular representation intertwining for the action of G given by $v \mapsto \varphi_{\lambda,v}$. The image of this map consists of functions invariant under the left translations (i.e., functions on the coset space $H\backslash G$).

Conversely, if $T : V \to \mathcal{O}(H\backslash G)$ is an intertwining operator for the action of G and we define $\lambda_T(v) = T(v)\mathbf{1}_G$, then λ_T is H-invariant element in V^*. Then it is easy to check that $\lambda \to \Phi_\lambda$ and $T \to \lambda_T$ are mutually inverse isomorphisms. If σ is moreover irreducible, then the multiplicity of σ in $\mathcal{O}(H\backslash G)$ is equal to $dim(V^*)^H$.

□

4.6 Representations of $\mathbf{GL}_n$.

The group $GL(n,\mathbb{C})$ is isomorphic to the direct product $SL(n,\mathbb{C})\times\mathbb{C}^\times$, hence irreducible regular representations of $GL(n,\mathbb{C})$ are isomorphic to the tensor product of irreducible representations of both factors. The set of isomorphic classes of such representations of the first factor are parametrized by highest weights

$$\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n,\lambda_1\geq\lambda_2\geq\dots\geq\lambda_n$$

modulo $\lambda=(1,\dots,1)$, while isomorphic classes of irreducible regular representations of $\mathbb{C}^\times$ are classified by an integer.

We can put both data together into one item λ from the cone

$$\mathcal{C}=\{\lambda\in\mathbb{Z}^n|\lambda_1\geq\lambda_2\geq\dots\geq\lambda_n\}$$

It leads to the following classification of regular irreducible representations of $GL(n,\mathbb{C})$

Theorem 4.18. *(Isomorphic classes) of regular irreducible representations of the group $G=GL(n,\mathbb{C})$ are classified by elements λ of the cone*

$$\mathcal{C}=\{\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n|\lambda_1\geq\lambda_2\geq\dots\geq\lambda_n\}.$$

of integral dominant weights.

Denote by 1_n element $(1,\dots,1)\in\mathcal{C}$. The representation *det* of GL_n has the highest weight $\lambda=1_n$ and it restricts to the trivial representation of SL_n. If ρ is a regular representation of GL_n with the highest weight λ, then $\rho\otimes det$ is a regular representation with the highest weight $\lambda+1_n$.

Let $\mathcal{C}^+$ denote the subcone of $\lambda\in\mathcal{C}$ with all components nonnegative. Regular representations of GL_n with highest weight λ in $\mathcal{C}^+$ are **polynomial representations**, because their matrix elements will not involve function $\det^{-1}$, they will be polynomials in coordinate functions of the matrix.

A general regular irreducible representation of $GL(V)$ is characterized by a highest weight from the conus $\mathcal{C}$. We shall characterize such general representations by a pair of weights as follows. Suppose that the weight $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ belongs to $\mathcal{C}$ but its components need not be nonnegative. Then the weight $\lambda^*=(-\lambda_n,-\lambda_{n-1},\dots,-\lambda_1)$ is again in the cone $\mathcal{C}$, hence it is the highest weight of another regular irreducible representation of G. (It can be checked as an exercise that it is the highest weight of the dual representation (ρ^*,V^*).)

Now define by

$$\lambda^+=(max\{\lambda_1,0\},max\{\lambda_2,0\},\dots,max\{\lambda_n,0\})$$

and

$$\lambda^-=-((-\lambda)^+)=(min\{\lambda_1,0\},min\{\lambda_2,0\},\dots,min\{\lambda_n,0\})$$

Then $\lambda=\lambda^++\lambda^-$ and if $\lambda\in\mathcal{C}$, then both λ^+ and λ^- belongs to $\mathcal{C}^+$. Let us denote by $\ell(\lambda)$ the values of index of the last nontrivial component of $\lambda\in\mathcal{C}^+$ and call it the **length** of λ.

So for any $\lambda\in\mathcal{C}$, we label the representation (ρ_λ,V) by the pair of weights (λ^+,λ^-). Note that the sum of lengths of λ^+ and λ^- is less or equal to n. We can also go back and for any pair (λ_1,λ_2) of two weights from $\mathcal{C}^+$ with sum of their lengths less or equal to n, we can define a regular irreducible representation of $GL(V)$ such that the associated pair of weights for it is equal to (λ_1,λ_2). So we have

Theorem 4.19. *Classes of isomorphisms of regular irreducible representations of* $G = GL(n,\mathbb{C})$ *are parametrized by the set* $\widehat{(GL_n)}$ *of all pairs* (λ_1, λ_2) *of weights from* $\mathcal{C}^+$ *with* $\ell(\lambda_1) + \ell(\lambda_2) \leq n$.

4.7 Representations of O(V, B) and SO(V, B)

repr0

We have already described and classified regular irreducible representations of the group $SO(n,\mathbb{C})$ by their highest weight $\lambda \in P_{++}(SO(n,\mathbb{C}))$. The goal of this section is to give a related description and classification of regular irreducible representations of the full group $G = O(V,B)$. The group G has two connected components, which can be described using the character det : $G \to]\{\pm 1\} \subset \mathbb{C}^\times$. The kernel of det is the simple group $G^o = SO(V,B)$. The other component is given by $\det^{-1}(-1)$. We shall treat the simpler case of odd dimension of V first, and then we shall describe two different subcases, when dimension of V is even. We shall use, as usually, the realization of orthogonal groups given by the choice of the symmetric form B on V described in A.3 3.6.1. Let H be the Cartan subgroup of diagonal matrices in G, and let $N^+ = \exp(\mathfrak{n}^+)$ be the subgroup of upper triangular matrices in G. The regular irreducible representation of G^o with the highest weight λ will be denoted by (π^λ, V^λ).

(1) dim V is odd.

In this case, $\det(-I) = -1$ and $G \simeq G^o \times \mathbb{Z}_2$. Suppose that (ρ, W) is a regular irreducible representation of G. The matrix $-I$ is in the center of G, hence it acts, by the Schur lemma, as a multiple of identity on W. But $\rho(-I)^2 = I$, hence $\rho(-I) = \pm I$.

odd **Theorem 4.20.** *Let V be complex vector space with dimV being odd. Regular irreducible representations of the group $G = O(V,B)$ are classified by pairs (λ, ε), where λ is an integral dominant weight for $(SO(V,B))$ and $\varepsilon = \pm 1$. The representation will be denoted by $(\pi^{\lambda,\varepsilon}, V^{\lambda,\varepsilon})$, when $\pi^{\lambda,\varepsilon}(-I) = \varepsilon I$. The representation $\pi^{\lambda,-}$ is isomorphic to $\pi^{\lambda,+} \otimes det$.*

(2) dim V is even.

In this case, the situation is more complicated. Let us choose an element g_0, which exchanges basis elements e_l and e_{l+1} and fixes all other basis elements. Then two connected components of G are G^o and $g_0 G^o$ and G is now a semidirect product $G \simeq \{I, g^o\} \ltimes G^o$. We have $g_0 H g_0^{-1} = H, g_0 \mathfrak{n}^+ g_0^{-1} = \mathfrak{n}^+$, and $g_0 \cdot \varepsilon_i = \varepsilon_i, i = 1, \ldots, l-1;\ g_0 \cdot \varepsilon_l = -\varepsilon_l$,

Examples.

Let $V = \mathbb{C}^{2l}$ be the defining representation of $O(2l,\mathbb{C})$. Then for any $k = 0, \ldots 2l;\ k \neq l$, representations ρ^k on the exterior powers $\Lambda^k(V)$ are irreducible both for G^o, and for G. For $k = 0, \ldots, l-1$, representations $\Lambda^k(V)$ and Λ^{2l-k} are isomorphic as representations of G^o, but not so, when considered as representations of G (note that $\rho^k(g_0) = \pm I$ for $k < l$, resp. $k > l$.) But in the middle dimension, $\Lambda^l(V)$ is irreducible only as representation of G, and it decomposes into the sum of self-dual and anti-self-dual parts as a representation of G^o.)

To construct irreducible representations of G from irreducible representations of G^o, we can use the scheme of induced representations. Starting from regular irreducible representation (π^λ, V^λ) of G^o, we define the induced representation $(\rho, I(V^\lambda))$ of G as

$$I(V^\lambda) = Ind_{G^o}^G(\pi^\lambda) = \{f \in \mathcal{O}(G, V^\lambda) | f(xg) = \pi^\lambda(x) f(g), x \in G^o, g \in G\}.$$

where $\mathcal{O}(G, V^\lambda)$ is the space of all regular maps from G to V^λ and the action ρ of G on $I(V^\lambda)$ is given by the right translation (which clearly preserves the space $I(V^\lambda)$.)

We want to show that the space $I(V^\lambda)$ decomposes under the action of G^o as a direct sum of two G^o irreducible representations. More precisely, we are going to prove the following lemma.

claim **Lemma 4.21.** *The induced representation $I(V^\lambda)$ decomposes as G^o representation as*

$$(\pi^\lambda, V^\lambda) \oplus (\pi^{g_0\cdot\lambda}, V^{g_0\cdot\lambda}).$$

Proof. The map $f \in I(V^\lambda) \mapsto (f(1), f(g_0)) \in V^\lambda \oplus V^\lambda$ is an isomorphism of vector spaces. If we define the representation π_0^λ on V^λ by $\pi_0^\lambda(x) = \pi^\lambda(g_0^{-1} x g_0)$, $x \in G^o$, then we get that under the action of G^o

$$I(V^\lambda) \simeq (\pi^\lambda, V^\lambda) \oplus (\pi_0^\lambda, V^\lambda). \tag{4.9}$$

But it is easy to see that the highest weight of the representation $(\pi_0^\lambda, V^\lambda)$ is $g_0 \cdot \lambda$, hence this representation is isomorphic to $(\pi^{g_0\cdot\lambda}, V^{g_0\cdot\lambda})$.

It is helpful to see explicitly the form of these two G^o-irreducible components and their highest weight vectors. Define the space $I_1(V^\lambda)$ as functions supported on G^o and vanishing on $G^o g_0$. Similarly, let $I_0(V^\lambda)$ be the space of functions supported on $G^o g_0$ and vanishing on G^o. Then clearly

$$I(V^\lambda) \simeq I_1(V^\lambda) \oplus I_0(V^\lambda)$$

is an alternative expression for (split 4.9).

Choose a highest weight vector $v \in (V^\lambda)^{\mathfrak{n}^+}$ and define functions f_1 and f_0 for $x \in G^o$ by

$$f_1(x) = \pi^\lambda(x)v,\ f_1(xg_0) = 0;\ f_0(x) = 0,\ f_0(xg_0) = \pi^\lambda(x)v.$$

Then $f_1 \in I_1(V^\lambda)$ and $f_0 \in I_0(V^\lambda)$ form a basis for $I(V^\lambda)^{\mathfrak{n}^+}$, because

$$\rho(h)f_1(xg_0) = \pi(xh)v = h^\lambda f_1(x) \tag{4.10}$$

$$\rho(h)f_0(xg_0) = \pi^\lambda(xg_0 h g_0^{-1})v = h^{g_0\cdot\lambda} f_0(xg_0), \tag{4.11}$$

and $\rho(g_0)f_1 = f_0$. □

Recall that a dominant integral weight $\lambda = \lambda_1\varepsilon_1 + \ldots + \lambda_l\varepsilon_l$ satisfies

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{l-1} \geq |\lambda_l|.$$

Due to $g^0 \cdot \lambda_l = -\lambda_l$, we may (and will) suppose that $\lambda_l \geq 0$. We have now to discuss whether the induced representation $I(V^\lambda)$ is irreducible as a representation of G.

The subcase (2) A.

Suppose first that $\lambda_l > 0$, hence $g_0 \cdot \lambda \neq \lambda$. Then weights of highest weight vectors f_1 and f_0 are different, f_1 is mapped by $\rho(g_0)$ to f_0. Hence $I(V^\lambda)$ is irreducible for G. We shall denote it by ρ^λ.

The subcase (2) B. If $\lambda_l = 0$, then $g_0 \cdot \lambda = \lambda$ and the highest weight vectors $F_\pm = f_1 \pm f_0$ form another basis for $I(V^\lambda)^{\mathfrak{n}^+}$ with the property $\rho(g_0)F_\pm = \pm F_\pm$. Each of the vectors $F_\pm$ generates an irreducible representation for G^o, we shall denote them by $V^{\lambda,+}$, resp. $V^{\lambda,-}$. Hence $I(V^\lambda)$ decomposes into irreducible G^o-components as

$$I(V^\lambda) \simeq V^{\lambda,+} \oplus V^{\lambda,-}.$$

It is possible to check that $V^{\lambda,-} \simeq V^{\lambda,+} \otimes det$.

The case (2), classification. We are going to show now that any regular irreducible representation of G is isomorphic to one of those induced representations or their subrepresentations.

clasO **Theorem 4.22.** *Regular irreducible representations of $G = \boldsymbol{O}(V,B)$ for $dimV$ even are isomorphic to one of representations in the following families:*
(1) $(\pi^{\lambda,\pm}, V^{\lambda,\pm})$ with λ integral and dominant for $G^o = \boldsymbol{SO}(V,B)$ and $\lambda_l = 0$;
(2) $(\rho^{\lambda}, I(V^{\lambda}))$ with λ integral and dominant for G^o and $\lambda_l > 0$.

Proof. Suppose that (σ, W) is a regular irreducible representation for G. We can restrict the action on W to G^o and we can find an irreducible representation for $(\pi^{\lambda}, V^{\lambda})$ for the action of G^o with $V^{\lambda} \subset W$. The group G^o is reductive, hence we can find G^o-equivariant projection $P : W \to V^{\lambda}$ and to define the map $S : W \to I(V^{\lambda})$ by $S(w)(g) = P(\sigma(g)w)$. The map S intertwines representations (σ, W) and $(\rho, I(V^{\lambda}))$ and S is injective, because $S(w)(1) = w$. Using our discussion of the induced representation for the image $(S(W))$ in $I(V^{\lambda})$, we get that (σ, W) is isomorphic with $(\rho, I(V^{\lambda}))$ if $\lambda_l \neq 0$, and (σ, W) is isomorphic with ρ restricted to one of the spaces $V^{\lambda,\pm}$, if $\lambda_l = 0$.

□

The case (2), an alternative classification. It is possible to simplify and unify the classification for regular irreducible representations of $G = \boldsymbol{O}(V)$ as follows. There are (in both parities) two families $(\pi^{\lambda,\pm}, V^{\lambda,\pm})$, where $\pi^{\lambda,+} \simeq \pi^{\lambda,-} \otimes det$. Suppose that the rank of G is equal to l and $\ell(\lambda) = j, 0 < j \leq l$. (Recall that $\ell(\lambda)$ is equal to the index of the last nontrivial component of λ.

We shall denote the representation $\pi^{\lambda,+}$ by π^{λ} and the representation $\pi^{\lambda,-}$ by $\pi^{\bar{\lambda}}$, where $\bar{\lambda}_i = \lambda_i$ for $i = 0, \ldots, j$ and $\bar{\lambda}_i = 1$ for $i = j+1, \ldots, 2l-j$. It means that $\bar{\lambda}$ is equal to λ complemented by components equal to 1 up to the length symmetric to the length of λ with respect to the 'origin' $i = l$.

Before going further in discussion, we introduce a helpful notation. Note that the length $\ell(\lambda)$ of a weight λ can be writen also as $\ell(\lambda) = |\{k|\lambda_k \geq 1\}|$, where the absolute value denotes the number of elements in the set. More generally, define the new weight λ' associated to λ by

$$\lambda'_i = |\{k|\lambda_k \geq i\}|.$$

So, e.g., $\ell(\lambda) = \lambda'_1$. If an element λ is represented by the Young diagram D (see the next section), then λ' is represented by the transposed diagram D^t obtained from D by the reflection with respect to the diagonal. Note also that $\lambda' \in \mathcal{C}^+$ for $\lambda \in \mathcal{C}^+$.

Returning back to representations $\pi^{\lambda,\pm}$, it is easy to check that using the introduced notation, the weight $\bar{\lambda}$ satisfies an additional condition $\bar{\lambda}'_1 + \bar{\lambda}'_2 \leq 2l$, because $\bar{\lambda}'_2 = \lambda'_2$.

As a summary, we get the following alternative classification.

Theorem 4.23. *Regular irreducible representations of the group $G = \boldsymbol{O}(V,B)$, of rank l are classified by a weight $\mu \in \mathcal{C}^+$ with $\mu'_1 + \mu'_2 \leq 2l$.*

Proof. If $dimV$ is odd, then the parameter $(\lambda, \pm)$ is replaced by $\mu = \lambda$ for $\varepsilon = +$ and $\mu = \bar{\lambda}$ for $\varepsilon = -$.

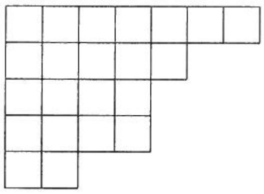
The same is true for $dimV$ even and $\lambda_l = 0$.

If $dimV$ is even and $\lambda_l > 0$, then we define $\mu = \lambda$. It is easy to check that the condition $\mu'_1 + \mu'_2 \leq 2l$ holds.

On contrary, if μ is an element from $\mathcal{C}^+$ with $\mu'_1 + \mu'_2 \leq 2l$, then μ corresponds to the representation $\pi^{(\lambda,+)}$, if $\ell(\mu) \leq l$, to the representation $\pi^{(\lambda,-)}$, with only nontrivial components of λ given by $\lambda_i = \mu_i, i = 0, \ldots, 2l - \ell(\mu)$, if $\ell(\mu) > l$, and to the representation (ρ^{μ}), if $dimV$ is even and $\mu_l > 0$. □

4.8 The diagram notation.

We now introduce an alternative notation for weights, which is often used in description of representations for classical linear algebraic groups. A typical dominant weight λ is an element of the cone $\mathcal{C}^+$, i.e., a sequence of nonincreasing nonnegative integers. Such sequences can be represented graphically using Young diagrams (often also called Ferrers diagrams). Suppose that $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a sequence of nonnegative integers satisfying the relation $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n$. We shall identify such sequence with the array of square boxes, arranged in left-justified rows, with the top row of length λ_1, the second row of length λ_2, and so forth. For example, the diagram corresponding to the sequence $(7, 5, 4, 4, 2, 0, 0, \cdots)$ is



Diagrams will be denoted by capital letters, such as D, E. We say that the Young diagram D with p nontrivial rows has **length** (or **depth**) equal to p and denote it by $\ell(D)$. The total number of boxes in the diagram is $|D| = \sum_i \lambda_i$. If λ is a weight, then associated character will be denoted by ψ_D.

For a diagram D, we denote by D^t the diagram corresponding to the sequence given by number of boxes in individual columns of the diagram. Geometrically, D^t is the diagram D transposed along the diagonal axis. For example, if $D = (7, 5, 4, 4, 2)$, then $D^t = (5, 5, 4, 4, 2, 1, 1)$. If $D = (\lambda_1, \ldots, \lambda_k)$, then it can be checked that $D^t = (\lambda'_1, \ldots, \lambda'_r)$, where $\lambda'_i = |\{k|\lambda_k \geq i\}|$, where the absolute value denotes the number of elements in the set. So, for examples, the condition for weights μ in alternative classification of regular irreducible representations of the orthogonal group Theorem clas0 4.22 is saying that sum of lenghts of the first two columns in the corresponding Young diagram D is less or equal to $2l$.

4.9 Notation for regular irreducible representations of classical groups

The case of $G = GL_n$ Let p, q be nonnegative integers and n a positive integer with $p + q \leq n$ and suppose that D_1 with $\ell(D_1) = p$ and D_2 with $\ell(D_2) = q$ are two Young diagrams. Then the regular irreducible representation on V corresponding to the pair (D_1, D_2) will be denoted by $\rho_V^{D_1, D_2}$.

The case of $G = O_n$ Let D be a Young diagram satysfying the conditions that the sum of lenghts of the first two columns is less or equal to n. Then we denote the regular irreducible representation of $G = O_n$ on V corresponding to the diagram D by σ_V^D.

The case of $G = Sp_{2n}$ Let D be a Young diagram satisfying the conditions that $\ell(D) \leq 2n$. Then we denote the regular irreducible representation of $G = Sp_n$ on V corresponding to the diagram D by τ_V^D.