LECTURE 10

In order to understand the relationship between Lie groups vs Lie algebra representations, we recall (and at the same time extend) the following results:

<u>Theorem 1</u>: G, H ... Lie groups, $\mathfrak{g}, \mathfrak{h}$ ... Lie algebras of G, H. Suppose $\Phi : G \to H$ is a Lie group homomorphism. Then there is a unique linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that $\forall X \in \mathfrak{g}$ $\Phi(\exp_G(X)) = \exp_H(\varphi(X))$. This map fulfills

1/ $\varphi(\mathrm{Ad}(g)X) = \mathrm{Ad}(\Phi(g))\varphi(X)$ $\quad \forall X \in \mathfrak{g}, g \in G,$

2/ $\varphi([X,Y]_{\mathfrak{g}}) = [\varphi(X), \varphi(Y)]_{\mathfrak{h}}$ $\quad \forall X, Y \in \mathfrak{g},$

3/ $\varphi(X) = \frac{d}{dt}\Big|_{t=0} \Phi(\exp_G(tX))$ $\quad \forall X \in \mathfrak{g}.$

<u>Pf</u>: Recall that for Φ a Lie group homom, $\Phi(\exp_G(tX))$ is a 1-param. subgroup of H, $\forall X \in \mathfrak{g}$. Therefore there is a unique $Z \in \mathfrak{h}$ such that $\Phi(\exp_G(tX)) = \exp_H(tZ)$, $t \in \mathbb{R}$. We then define $\varphi(X) = Z$, i.e. φ is the tangent map of Φ at $e \in G$, and all required conditions of φ are easy to check.

We have for any $g \in G$

$$\exp_H(t\varphi(\mathrm{Ad}(g)X)) = \exp_H(\varphi(t\mathrm{Ad}(g)X)) = \Phi(\exp_G(t\mathrm{Ad}(g)X)),$$

so that

$$\exp_H(t\varphi(\mathrm{Ad}(g)X)) = \mathrm{Ad}(\Phi(g))\Phi(\exp_G(tX)) = \mathrm{Ad}(\Phi(g))\exp_H(t\varphi(X)),$$

and the application of $\frac{d}{dt}\Big|_{t=0}$ to both sides gives the claim 1/.

The point 2/ was already proved, while 3/ follows from

$$\Phi(\exp_G(tX)) = \exp_H(\varphi(tX)) = \exp_H(t\varphi(X))$$

by application of $\frac{d}{dt}\Big|_{t=0}$ to both sides. ∎

The application of Theorem 1 to representation theory is based on the choice ~~of~~ $H = GL(V)$, $\mathfrak{h} = \mathfrak{gl}(V)$ for a finite-dim vector space V (over $\mathbb{R}, \mathbb{C}$.)

<u>Corollary 2</u>: $G, \mathfrak{g}$ as above, Π a finite-dim (over $\mathbb{R}, \mathbb{C}$) represent. of G, acting on V. Then there is a unique repr. π of $\mathfrak{g}$ acting on V such that $\Pi(\exp_G(X)) = e^{\pi(X)}$ $\forall X \in \mathfrak{g}$.

exponential in $GL(V)$

The representation π can be computed as

$$\pi(X) = \frac{d}{dt}\Big|_{t=0} \Pi(e^{tX})$$

and satisfies

$$\pi(Ad(g)X) = Ad(\Pi(g))\pi(X) \quad \forall X \in \mathfrak{g}, \forall g \in G.$$

<u>Lemma 3</u>: 1. $G, \mathfrak{g}$ as above, and G assumed to be connected. Let Π be a representation of G, π associated representation of $\mathfrak{g}$. Then Π is irreducible iff π is irreducible.

2. $G, \mathfrak{g}$ as in 1., Π_1, Π_2 represent. of G, π_1, π_2 associated representations of $\mathfrak{g}$. Then π_1, π_2 are isomorphic iff Π_1, Π_2 are isomorphic.

<u>Pf</u>: 1. Assume Π is irreducible $\overset{?}{\Rightarrow}$ π is irreducible.

Let $W \subseteq V$ invariant under $\pi(X)$ $\forall X \in \mathfrak{g}$, and show that W is either $\{0\}$ or V. Assuming $g \in G$, G connected implies $g = \exp(X_1) \cdot \ldots \cdot \exp(X_m)$ for some $m \in \mathbb{N}$, $X_1, \ldots, X_m \in \mathfrak{g}$. Since W is invariant w.r.t. $\pi(X_j)$, it is invariant w.r.t. $\exp(\pi(X_j)) = e^{\pi(X_j)}$. Hence, it is invariant under

$$\Pi(g) = \Pi(\exp(X_1) \cdot \ldots \cdot \exp(X_m)) = \Pi(\exp(X_1)) \ldots \Pi(\exp(X_m)) = e^{\pi(X_1)} \cdot \ldots \cdot e^{\pi(X_m)}.$$

Since Π is irreducible and $W \subseteq V$ is invariant for all $\Pi(g)$, W is either $\{0\}$ or V $\Rightarrow$ π is irreducible.

The second implication is analogous (based on $\pi(X) = \frac{d}{dt}\Big|_{t=0} \Pi(\exp_G(tX))$), the point 2. is also straightforward.

There are several ways to obtain more complicated representations from the old ones:

<u>Def 4</u>: G .. a Lie group, $\Pi_1, \dots, \Pi_m$ representations of G on $V_1, \dots, V_m$. Then the direct sum of $\Pi_1, \dots, \Pi_m$ is a representation $\Pi_1 \oplus \dots \oplus \Pi_m$ of G acting on $V_1 \oplus \dots \oplus V_m$ by

$$\left[(\Pi_1 \oplus \dots \oplus \Pi_m)(g)\right](v_1, \dots, v_m) = \left(\Pi_1(g)v_1, \dots, \Pi_m(g)v_m\right)$$

for all $g \in G$, $v_1, \dots, v_m \in V_1, \dots, V_m$.

Analogous formula works for $\mathfrak{g}$, $\pi_1, \dots, \pi_m$ and the representation $\pi_1 \oplus \dots \oplus \pi_m$. It is elementary to check the homomorphism property.

$\otimes$ .. tensor product of vector spaces; $e_1, \dots, e_n$ .. a basis of U, $f_1, \dots, f_m$ .. a basis of V

$\dim U = n$, $\dim V = m$ $\Rightarrow$ $e_j \otimes f_k$, $1 \le j \le n$, $1 \le k \le m$ a basis of $U \otimes V$

$\dim U \otimes V = m \cdot n$

<u>Def 5</u>: U, V .. fin-dim vectorspaces (over $\mathbb{R}, \mathbb{C}$), then the tensor product of U, V is a vector space W together with $\alpha : U \times V \to W$ bilinear map such that $\forall$ vector space X and a bilinear map $\psi : U \times V \to X$ $\exists!$ linear map $\tilde{\psi} : W \to X$ which makes the diagram

$$\begin{tikzcd} U \times V \arrow[r, "\alpha"] \arrow[dr, "\psi"'] & W \arrow[d, "\tilde{\psi}"] \\ & X \end{tikzcd}$$

Def 6: $G, H \ldots$ Lie groups, G: Π_1, U representation of G,
H: Π_2, V representation of H.

The tensor product of Π_1 and Π_2 is a representation $\Pi_1 \otimes \Pi_2$ of $G \times H$ acting on $U \otimes V$, defined by $(\Pi_1 \otimes \Pi_2)(g, h) := \Pi_1(g) \otimes \Pi_2(h)$, $\forall g \in G, h \in H$.

It is elementary to check the homomorphism property.

Lemma 7: $G, H \ldots$ Lie groups, Π_1 resp Π_2 repr. of G resp. H, and $\Pi_1 \otimes \Pi_2$ repr. of $G \times H$. If $\pi_1 \otimes \pi_2$ denotes the associated representation of $\mathfrak{g} \oplus \mathfrak{h}$, then

$$(\pi_1 \otimes \pi_2)(X, Y) = \pi_1(X) \otimes Id + Id \otimes \pi_2(Y)$$

$\forall X, Y$
$\quad\in\mathfrak{g}\ \in\mathfrak{h}$

Pf: For $t \to u(t)$ a smooth curve in U
$\quad$ — " — $\to v(t)$ $\quad$ — " — $\quad V$ $\Big\}$ $\Rightarrow$

$$\frac{d}{dt}\left(u(t) \otimes v(t)\right) = \frac{du}{dt}(t) \otimes v(t) + u(t) \otimes \frac{dv}{dt}(t).$$

This implies

$$(\pi_1 \otimes \pi_2)(X, Y)(u \otimes v) = \frac{d}{dt}\Big|_{t=0} (\Pi_1 \otimes \Pi_2)\left(\exp(tX), \exp(tY)\right)(u \otimes v) =$$

$$= \frac{d}{dt}\Big|_{t=0} \left(\Pi_1(\exp(tX))u \otimes \Pi_2(\exp(tY))v\right) =$$

$$= \left(\frac{d}{dt}\Big|_{t=0} \Pi_1(\exp(tX))u\right) \otimes v + u \otimes \left(\frac{d}{dt}\Big|_{t=0} \Pi_2(\exp(tY))v\right).$$

Because for $u \in V, v \in V$ the elements $u \otimes v$ span $U \otimes V$, we are done. ∎

Again, it is easy to see that this formula gives a representation of $\mathfrak{g} \oplus \mathfrak{h}$ on $U \otimes V$. E.g. the formula $(\pi_1 \otimes \pi_2)(X, Y) := \pi_1(X) \otimes \pi_2(Y)$ is NOT a representation of $\mathfrak{g} \oplus \mathfrak{h}$. (Why?)

Def 8: The diagonal embedding $G \hookrightarrow G\times G$, $g \mapsto (g,g)$ implies

G ... a Lie group, Π_1, Π_2 representations of G on V_1, V_2. Then the tensor product representation of G, acting on $V_1 \otimes V_2$, is defined by
$$(\Pi_1 \otimes \Pi_2)(g) := \Pi_1(g) \otimes \Pi_2(g) \quad \forall g \in G.$$

Similarly for the diagonal embedding of Lie algebras $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$, $X \mapsto (X, X)$. If π_1, π_2 representations of $\mathfrak{g}$, we define tensor product repr. of $\mathfrak{g}$ on $V_1 \otimes V_2$ by
$$(\pi_1 \otimes \pi_2)(X) := \pi_1(X) \otimes Id + Id \otimes \pi_2(X).$$

One proves easily that $\Pi_1 \otimes \Pi_2$, $\pi_1 \otimes \pi_2$ are representations. (for both of G and $G\times G$, so this is a bit ambiguous.)

Once Π_1, Π_2 (and so π_1, π_2) are irreducible, $\Pi_1 \otimes \Pi_2$ is typically not, so have to decompose on direct sum of irred. repr.

Let π be a representation of $\mathfrak{g}$ on fin.-dim V, $V^* = Hom(V, k)$ for $k = \mathbb{R}, \mathbb{C}$. For $A \in End(V)$, $A^* \in End(V^*)$ defined by $(A^*\varphi)(v) = \varphi(Av)$, $\varphi \in V^*$, $v \in V$. In the basis of V and dual basis of V^*, A^* is the transpose matrix of A. Then $(AB)^* = B^* A^*$, i.e. $(AB)^T = B^T A^T$.

Def 9: G, Π, V. Then the dual representation Π^* to Π is the repr. of G acting on V^* and given by $\Pi^*(g) = (\Pi(g^{-1}))^* \overset{\shortparallel}{} \Pi(g^{-1})^T$

In the case of Lie algebra $\mathfrak{g}$, π, V, then π^* is the repr. of $\mathfrak{g}$ acting on V^*, given by $\pi^*(X) = -\pi(X)^* \overset{\shortparallel}{} -\pi(X)^T$. For matrix Lie groups/algebras, $Z^* = Z^T$ (the transposition.)

The dual representation is also called contragradient, and one ~~proves~~ easily the ~~homomorphism~~ property, and the fact that the dual representation is irreducible iff the former is irreducible. Moreover, $(\Pi^*)^*$ is isomorphic to Π.

Exercises 10

Example 1: In the case of matrix Lie group/algebra, i.e. $G \subseteq GL(n, {\mathbb{R} \atop \mathbb{C}})$ and $\mathfrak{g} \subseteq M_n({\mathbb{R} \atop \mathbb{C}})$, the action on underlying vector space $\mathbb{R}^n$, $\mathbb{C}^n$ gives standard ~~(vector)~~ representation of G, $\mathfrak{g}$.

$SO(3) \subseteq GL(\mathbb{R}^3)$ gives the standard represent. on $\mathbb{R}^3$

$SU(2) \subseteq GL(\mathbb{C}^2)$ —— " —— $\mathbb{C}^2$

Trivial represent.: $V = {\mathbb{R} \atop \mathbb{C}}$, $\Pi : G \to GL(1, \mathbb{C})$

$g \mapsto 1 \quad \forall g \in G$

(it is irreducible)

$\pi : \mathfrak{g} \to gl(1, \mathbb{C})$

$X \mapsto 0$

(again, adjoint representation)

Example 2: $\mathrm{Ad}: G \to GL(\mathfrak{g})$, $\mathrm{ad} : \mathfrak{g} \to gl(\mathfrak{g})$
adjoint representation.

Example 3: V_m = vector space of polynomials in two variables, which are homogeneous of degree m.

$f(z_1, z_2) = a_0 z_1^m + a_1 z_1^{m-1} z_2 + \dots + a_m z_2^m$, $a_i \in \mathbb{C} \ \forall i$

$\dim V_m = m+1$

For $U \in SU(2)$, $\Pi_m(U)$ is given by $\left(\Pi_m(U) f\right)(z) = f(U^{-1} z)$, $z \in \mathbb{C}^2$,

so that

$$\left[\left(\Pi_m(U)\right) f\right](z_1, z_2) = \sum_{k=0}^{m} a_k \left(U_{11}^{-1} z_1 + U_{12}^{-1} z_2\right)^{m-k} \left(U_{21}^{-1} z_1 + U_{22}^{-1} z_2\right)^k.$$

Π_m on V_m is a representation: $\cap V_m$

$$\Pi_m(U_1)\left(\Pi_m(U_2) f\right)(z) = \left(\Pi_m(U_2) f\right)(U_1^{-1} z) = f(U_2^{-1} U_1^{-1} z) = \Pi_m(U_1 U_2) f(z)$$

The associated repr of $su(2)$ is $\left(\pi_m(X)f\right)(z) = \frac{d}{dt}\Big|_{t=0} f(e^{-tX}z)$, $X \in su(2)$.

$t \to z(t) = (z_1(t), z_2(t))$ a curve in $\mathbb{C}^2$, so by the chain rule
$\quad = e^{-tX}z$

$$\left(\pi_m(X)f\right) = \frac{\partial f}{\partial z_1}\frac{dz_1}{dt}\Big|_{t=0} + \frac{\partial f}{\partial z_2}\frac{dz_2}{dt}\Big|_{t=0}$$

$z = (z_1, z_2)$

Because $\frac{dz}{dt}\Big|_{t=0} = -X\cdot z$, we obtain

$$\pi_m(X)f = -\frac{\partial f}{\partial z_1}\left(X_{11}z_1 + X_{12}z_2\right) - \frac{\partial f}{\partial z_2}\left(X_{21}z_1 + X_{22}z_2\right).$$

We may consider the (unique) $\mathbb{C}$-linear extension of π to $sl(2,\mathbb{C}) = su(2)_{\mathbb{C}}$: $\quad H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,

$\pi_m(H) = -z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}$

$\pi_m(X) = -z_2\frac{\partial}{\partial z_1}$, $\quad \pi_m(Y) = -z_1\frac{\partial}{\partial z_2}$.

In the basis $\langle z_1^m, z_1^{m-1}z_2, \ldots, z_1^{m-k}z_2^k, \ldots, z_2^m\rangle$ of V_m:

$\pi_m(H)\left(z_1^{m-k}z_2^k\right) = (-m+2k)\, z_1^{m-k}z_2^k$

$\pi_m(X)\left(z_1^{m-k}z_2^k\right) = (m-k)\, z_1^{m-k-1}z_2^{k+1}$

$\pi_m(Y)\left(z_1^{m-k}z_2^k\right) = -k\, z_1^{m-k+1}z_2^{k-1}$

$$\begin{tikzcd}
0 & z_1^m \arrow[l, "Y"'] \arrow[r, bend left, "X"] & z_1^{m-1}z_2 \arrow[l, bend left, "Y"] \arrow[r, bend left, "X"] & \cdots \arrow[l, bend left, "Y"] \arrow[r, bend left, "X"] & z_2^m \arrow[l, bend left, "Y"] \arrow[r, "X"] & 0
\end{tikzcd}$$

and each $z_1^{m-k}z_2^k$ is an eigenvector of eigenvalue $(-m+2k)$ for H.

<u>Lemma</u>: $\forall m \geq 0$, π_m is irreducible.

Pf: Show that any non-zero invariant subspace is V_m.

Let $W \subseteq V_m$ be an invariant (non-zero) subspace, $w \neq 0$.
$\quad\quad \in W$

Then $w = a_0 z_1^m + a_1 z_1^{m-1}z_2 + \cdots + a_m z_2^m$, at least on $a_i \neq 0$.

Let k_0 be the smallest value of k, for which $a_k \neq 0$, apply $\pi_m(X)^{m-k_0}$ to w.

The action $\pi_m(X)$ raises power of z_2 by 1, $\pi_m(X)^{m-k_0}$ kills all terms in w up to $a_{k_0} z_1^{m-k_0} z_2^{k_0}$. Since $\pi_m(X)$ is zero on $z_1^{m-k}z_2^k$ iff $k = m$, $\pi_m(X)^{m-k_0} w$ is a non-zero multiple of z_2^m. Since W is assumed invariant, W contains this multiple of z_2^m, hence z_2^m itself. Then for $0 \le k \le m$, $\pi_m(Y)^k z_2^m$ is a nonzero multiple of $z_1^k z_2^{m-k}$, therefore W contains $z_1^k z_2^{m-k}$ for all $0 \le k \le m$. Since these elements are a basis of V_m, $W = V_m$ as desired. $\blacksquare$

The matrix realization of $\pi_m(H), \pi_m(X), \pi_m(Y)$ in the basis $\langle z_1^m, z_1^{m-1} z_2, \ldots, z_1 z_2^{m-1}, z_2^m \rangle$ of V_m:

$$\pi_m(H) = \begin{pmatrix} -m & & & & 0 \\ & -m+2 & & & \\ & & \ddots & & \\ & & & m-2 & \\ 0 & & & & m \end{pmatrix}$$

$$\pi_m(X) = \begin{pmatrix} 0 & -m & 0 & \cdots & 0 \\ 0 & 0 & -(m-1) & \cdots & 0 \\ & & 0 & \ddots & \\ & & & \ddots & \\ 0 & & & & -1 \end{pmatrix}$$

$$\pi_m(Y) = \begin{pmatrix} 0 & 0 & & & 0 \\ -1 & 0 & & & \\ 0 & -2 & & & \\ \vdots & 0 & \ddots & & \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & -m & 0 \end{pmatrix}$$

Example: In the previous example we studied $sl(2,\mathbb{C})$-action on polynomial on $\mathbb{C}^2$. Now we condised the representation called fundamental (vector) representation, which is given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Bigg\} \longrightarrow \overbrace{V = \mathbb{C}[x,y]}^{\text{symmetric algebra in } x, y}$$

$$\pi_V : \quad \begin{aligned} X &\longmapsto x\frac{\partial}{\partial y} = \pi_V(X) \\ H &\longmapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} = \pi_V(H) \\ Y &\longmapsto y\frac{\partial}{\partial x} = \pi_V(Y) \end{aligned}$$

and it is elementary to check this is a representation.

Because the action preserves homogeneity (degree) of symmetric tensors, the restriction to degree k-tensors gives a representation π_V^k dual to the one studied in the previous example (V_k).

The fundamental vector representation is identified with

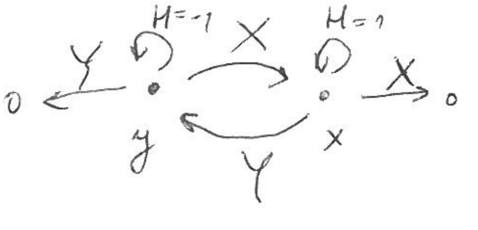
$$x \longleftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad y \longleftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$H \cdot x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x$$

$$\Leftrightarrow \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)x = x$$

$$H \cdot y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -y$$

$$\Leftrightarrow \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)y = -y$$



$X \cdot x = \ldots \qquad X \cdot y = \ldots$

$Y \cdot x = \ldots \qquad Y \cdot y = \ldots$

Now consider the second tensor power of the representation π_V^1: the underlying vector space is of dimension 4, $\langle x\otimes x,\ x\otimes y,\ y\otimes x,\ y\otimes y\rangle$, the representation of $sl(2,\mathbb{C})$ given by

$$\pi^1_{V\otimes V}(Z)(v_1\otimes v_2) = \pi_V^1(Z)v_1 \otimes v_2 + v_1 \otimes \pi_V^1(Z)v_2.$$

We get

$$\pi^{1}_{V\otimes V}(X)(x\otimes x) = 0,$$

$$\pi^{1}_{V\otimes V}(Y)(x\otimes x) = \pi^{1}_{V}(Y)x\otimes x + x\otimes\pi^{1}_{V}(Y)x = y\otimes x + x\otimes y$$

$$\pi^{1}_{V\otimes V}(Y)(y\otimes x + x\otimes y) = \overbrace{\pi^{1}_{V}(Y)\cdot y}^{=0}\otimes x + y\otimes \overbrace{\pi^{1}_{V}(Y)x}^{y} + \underbrace{\pi^{1}_{V}(Y)x}_{y}\otimes y + x\otimes \underbrace{\pi^{1}_{V}(Y)y}_{=0} = 2\,y\otimes y,$$

$$\pi^{1}_{V\otimes V}(Y)(y\otimes y) = 0.$$

By the action of $\pi^{1}_{V\otimes V}(X), \pi^{1}_{V\otimes V}(H)$ we see that the space of dimension 3, $\langle x\otimes x,\ x\otimes y + y\otimes x,\ y\otimes y\rangle \subseteq \mathbb{C}^2\otimes\mathbb{C}^2$, is stable and irreducible by the action of $sl(2,\mathbb{C})$.

On the other hand, the vector $x\otimes y - y\otimes x$ fulfills

$$\pi^{1}_{V\otimes V}(X)(x\otimes y - y\otimes x) = 0$$

$$\pi^{1}_{V\otimes V}(Y)(x\otimes y - y\otimes x) = 0$$

$$\pi^{1}_{V\otimes V}(H)(x\otimes y - y\otimes x) = \overbrace{\pi^{1}_{V}(H)x}^{=1\cdot x}\otimes y + x\otimes\overbrace{\pi^{1}_{V}(H)y}^{=-1\cdot y} - \underbrace{\pi^{1}_{V}(H)y}_{-1\cdot y}\otimes x - y\otimes\underbrace{\pi^{1}_{V}(H)x}_{1\cdot x} = 0.$$

Because $x\otimes y - y\otimes x \notin \langle x\otimes x,\ x\otimes y + y\otimes x,\ y\otimes y\rangle$, we see

$$\mathbb{C}^2\otimes\mathbb{C}^2 \xrightarrow{\sim} \underbrace{Sym^2\mathbb{C}^2}_{\cong\ \mathbb{C}^3} \oplus \underbrace{\Lambda^2\mathbb{C}^2}_{\cong\ \mathbb{C}} \quad \leftarrow \text{direct sum of two irred. of } sl(2,\mathbb{C}).$$

$$2\cdot 2 = 3+1$$