Lecture 12 — Representation theory of $\mathfrak{sl}(3,\mathbb{C})$

As for $\mathfrak{sl}(2,\mathbb{C})$, we decomposed an irred. repr. V into a direct sum of eigenspaces based on the action of H: $V = \bigoplus_\alpha V_\alpha$, $\forall v \in V_\alpha$ is an eigenvector of H with eigenvalue $\alpha \in \mathbb{C}$. The major difference for $\mathfrak{sl}(3,\mathbb{C})$ is that the role of H plays a maximal commutative subalgebra (Cartan subalgebra) of $\mathfrak{sl}(3,\mathbb{C})$, and it is a 2-dimensional subspace $\mathfrak{h} \subseteq \mathfrak{g}$. A represent. is then a direct sum of weight spaces, i.e. $V = \bigoplus_\alpha V_\alpha$ and $\forall v \in V_\alpha$ is an eigenvector $\forall H \in \mathfrak{h}$: $Hv = \alpha(H)v$, $\forall H \in \mathfrak{h}$ and $\alpha \in \mathfrak{h}^*$.

The usual convention: $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is the dual vector space.

Basis of $\mathfrak{sl}(3,\mathbb{C})$: $H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, E_{ij} for $1 \le i \ne j \le 3$ (3×3 elementary matrices)

The space $\mathfrak{h} = \langle H_1, H_2 \rangle_{\mathbb{C}}$ is $\mathbb{C}$ 2-dim, Cartan subalgebra.

The dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ is the space of linear forms L_1, L_2, L_3:
$L_i\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i$, $i = 1,2,3$, where $L_1 + L_2 + L_3 = 0$. The Cartan subalgebra is defined as $\mathfrak{h}^* = \langle L_1, L_2, L_3 \rangle_{\mathbb{C}} / \langle L_1 + L_2 + L_3 \rangle$; it is of dimension 2 (e.g., L_i, L_j for $i \ne j$ are lin. independent: if not, $\exists (a,b) \ne (0,0)$ such that $aL_1 + bL_2 = 0$ $(i=1, j=2)$ in $\mathfrak{h}^*$, which is equivalent to $aL_1(H_1) + bL_2(H_1) = 0$ and $aL_1(H_2) + bL_2(H_2) = 0$ $\Rightarrow a = 0 = b$.)

Back to $\mathfrak{sl}(2,\mathbb{C})$:
$[H, X] = 2X \iff \mathrm{ad}(H)(X) = 2X$
$[H, Y] = -2Y \iff \mathrm{ad}(H)(Y) = -2Y$ $\iff$ X, Y are eigenvectors for $\mathrm{ad}(H)$

In $\mathfrak{sl}(3,\mathbb{C})$, we want to look for common eigenvectors of the adjoint repr. on $\mathfrak{sl}(3,\mathbb{C})$ of all $H \in \mathfrak{h}$ (Cartan subalgebra $\mathfrak{h}$ is commutative, hence this makes sense.) We have

$[H_1, E_{12}] = 2E_{12}$	$[H_2, E_{12}] = -E_{12}$	$(2, -1)$
$[H_1, E_{21}] = -2E_{21}$	$[H_2, E_{21}] = E_{21}$	$(-2, 1)$
$[H_1, E_{23}] = -E_{23}$	$[H_2, E_{23}] = 2E_{23}$	$(-1, 2)$
$[H_1, E_{13}] = E_{13}$	$[H_2, E_{13}] = E_{13}$	$(1, +1)$

$[H_1, E_{32}] = E_{32}$ $\qquad [H_2, E_{32}] = -2E_{32}$ $\qquad (1, -2)$

$[H_1, E_{31}] = -E_{31}$ $\qquad [H_2, E_{31}] = -E_{13}$ $\qquad (-1, -1)$

The spectral decomposition of $\mathfrak{h}$ acting on $sl(3,\mathbb{C})$ is the root space decomposion, $sl(3,\mathbb{C}) = \mathfrak{g} = \bigoplus_{\alpha} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha \right)$, α ranges over finite subset of $\mathfrak{h}^*$, $\mathfrak{g}_\alpha$ is the eigenspace corresponding to $\alpha \in \mathfrak{h}^*$:

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid [H, X] = \alpha(H) X \quad \forall H \in \mathfrak{h} \}.$$

If $\alpha = 0$, then $\mathfrak{g}_\alpha = \mathfrak{h}$.

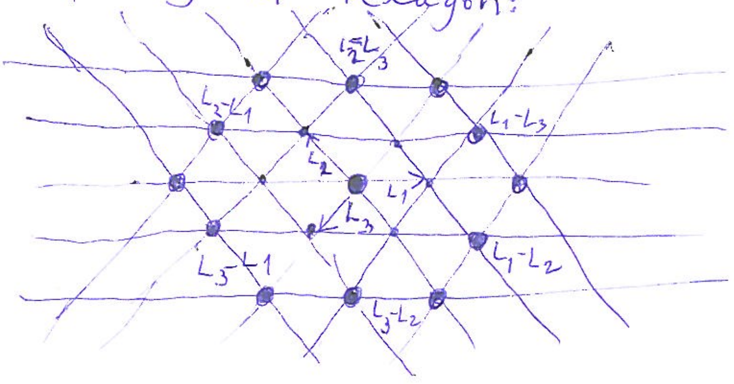
Def 1: If $\underset{\neq 0}{\alpha} \in \mathfrak{h}^*$ satisfies $\mathfrak{g}_\alpha \neq 0$, then α is a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$. If α is a root, then $\mathfrak{g}_\alpha$ is called a root space attached to α, and a vector in $\mathfrak{g}_\alpha$ is called the root vector of α.

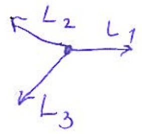
Write $H \in \mathfrak{h}$ as $H = \sum_{i=1}^{3} a_i E_{ii}$. Since

$$ad(H) E_{ij} = [H, E_{ij}] = H E_{ij} - E_{ij} H = \underbrace{(a_i - a_j)}_{(L_i - L_j)(H)} E_{ij}.$$

so we introduce $L_i - L_j$, $1 \le i \neq j \le 3$, the roots corresponding to the adjoint representation. Namely, there are six 1-dimensional root spaces $\underset{\parallel\; E_{ij}}{\mathfrak{g}_{L_i - L_j}} \leftrightarrow L_i - L_j \in \mathfrak{h}^*$, and $\mathfrak{h} = \mathfrak{g}_0$ is of dimension 2. Altogether, $6 + 2 = 8$, equal to $\dim(sl(3,\mathbb{C}))$.

The relevant picture of roots for the adjoint representation $sl(3,\mathbb{C})$: L_1, L_2, L_3, $L_1 + L_2 + L_3 = 0$ in the (real subspace) plane $\mathfrak{h}^*$ (all roots are real!), have the same length and angle between any pair is $\frac{2\pi}{3}$; then the roots $L_i - L_j$, $1 \le i \neq j \le 3$, are vertices of a regular hexagon:





V ... a finite-dim repr. of $sl(3,\mathbb{C})$; By the Jordan preservation theorem is the action of any $H \in \mathfrak{h}$ on V diagonal. Any fin.-dim. admits a decomposition $V = \bigoplus_\alpha V_\alpha$, $\forall v \in V_\alpha$ $Hv = \alpha(H)v$ $\forall H \in \mathfrak{h}$.

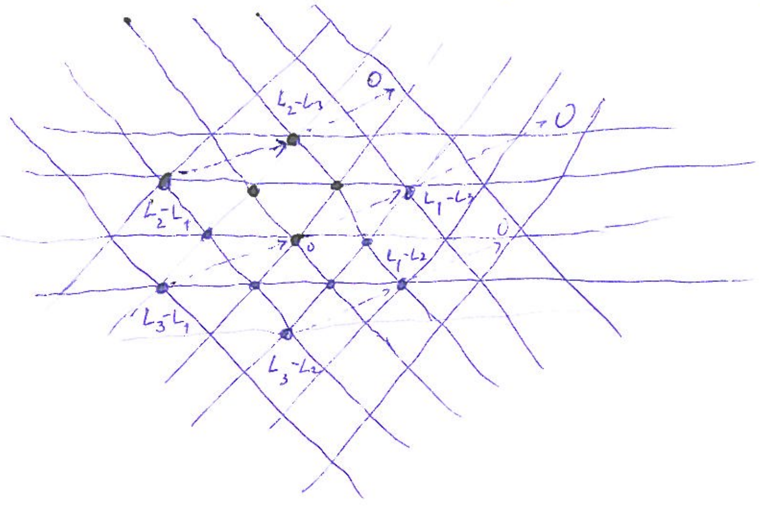
Def 2: By an eigenvector for $\mathfrak{h}$ in the representation on V we mean an eigenvector for $\forall H \in \mathfrak{h}$. Then $Hv = \alpha(H)v$, $\alpha(H)$ depending linearly on H, i.e. $\alpha \in \mathfrak{h}^*$.

(This is a feature of general semi-simple Lie algebra $\mathfrak{g}$.)

Let $X \in \mathfrak{g}_\alpha$, what is the image of $ad(X)$ acting on $Y \in \mathfrak{g}_\beta$? For $H \in \mathfrak{h}$ arbitrary, $[H,[X,Y]] = [X,[H,Y]] + [[H,X],Y] =$

$$= [X, \beta(H)Y] + [\alpha(H)X, Y] = (\alpha(H) + \beta(H))[X,Y]$$
$$= (\alpha+\beta)(H)[X,Y]$$

$\Rightarrow$ $[X,Y] = ad(X)(Y)$ is an eigenvector for $\mathfrak{h}$ again, eigenvalue $\alpha+\beta$. Hence $ad(\mathfrak{g}_\alpha): \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}$ (in particular, the root space decomposition is preserved.) For example,



$ad(\mathfrak{g}_{L_1-L_3})$

$\mathfrak{g}_{L_2-L_1} \to \mathfrak{g}_{L_2-L_3} \to 0$

$\mathfrak{g}_{L_3-L_1} \to \mathfrak{g}_0 \to \mathfrak{g}_{L_1-L_3} \to 0$

$\mathfrak{g}_{L_3-L_2} \to \mathfrak{g}_{L_1-L_2} \to 0$

$\boxed{\text{For any fin.-dim. repr. } \pi \text{ on } V:}$

$\pi(H)\pi(X)v = \pi(X)\pi(H)v + \pi([H,X])v = \pi(X)(\beta(H)v) + (\alpha(H)\pi(X))v = (\alpha(H)+\beta(H))\pi(X)v$
$= (\alpha+\beta)(H)\pi(X)v$

$\Rightarrow$ $\pi(X)v$ is an $\mathfrak{h}$-eigenvector with eigenvalue $\alpha+\beta$;

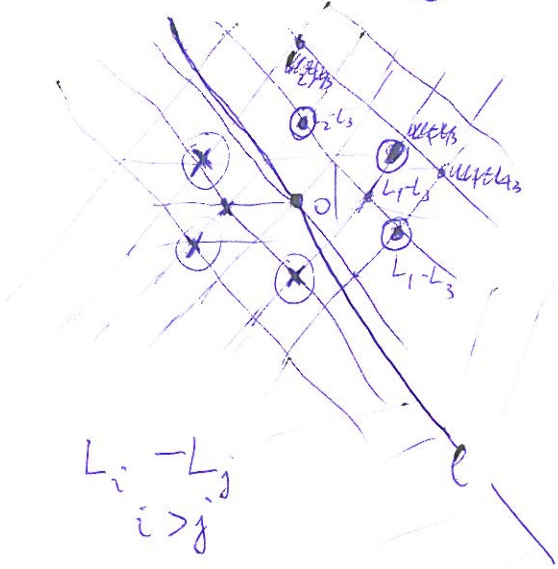
$\Rightarrow$ $\pi(\mathfrak{g}_\alpha): V_\beta \to V_{\alpha+\beta}$

Observation: the eigenvalues in an irreducible representation V of $sl(3,\mathbb{C})$ differ from each other by $\mathbb{Z}$-combinations of $L_i - L_j \in \mathfrak{h}^*$, i.e. for any $\mathfrak{h}$-eigenvalues α, β : $\alpha - \beta \in \Lambda_R$.
Here
$$\Lambda_R = \{ a(L_1-L_2) + b(L_2-L_3) + c(L_1-L_3) \mid a,b,c \in \mathbb{Z} \}.$$
For a general module V, $V' = \bigoplus_{\beta \in \Lambda_R} V_{\alpha+\beta} \subseteq V$ is a subrepresentation of V.

Recall $L_1-L_2, L_2-L_3, L_1-L_3$ are linearly dependent.

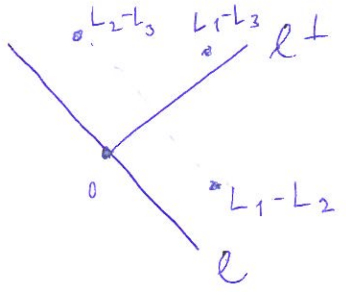
<u>Def 3</u>: The $\mathbb{Z}$-lattice $\Lambda_R \subseteq \mathfrak{h}^*$ generated by roots $L_i - L_j$ is called root lattice. The eigenvalues $\alpha \in \mathfrak{h}^*$ of $\mathfrak{h}$ in V are called weights of the representation. The eigenvectors in V_α are called weight vectors and V_α are weight spaces.

$sl(2,\mathbb{C})$ remainder: we found a vector $v \in V_\alpha$ such that $X \cdot v = 0$ and the action of Y generated the whole representation. $H \cdot v = \alpha(H) v$, The vector v had the highest weight, as an eigenvalue of H, and the remaining vectors have lower weight.
We need in $sl(3,\mathbb{C})$ to introduce an ordering among the elements in the root space; it is arbitrary, no natural ordering in the root space.
However, for any root $L_i - L_j$ there is also a root $-(L_i - L_j) = L_j - L_i$.
So consider a line $l \subseteq \mathfrak{h}^*$, passing through $0 \in \mathfrak{h}^*$



$L_i - L_j$
$i < j$

$L_i - L_j$
$i > j$



$\ell^\perp$ ... a half-line, lies in the half-plane with positive roots

Extremal point in Λ_R - its distance from ℓ is maximal

$$\left\{\begin{array}{l} sl(2,\mathbb{C}) \\ v - H\text{-eigenvect.} \\ \quad - \text{Annihilated by } X \end{array}\right\} \rightsquigarrow \left\{\begin{array}{l} sl(3,\mathbb{C}) \\ v - \text{eigenvect. for } \forall H \in \mathfrak{h} \\ \quad - \text{annihilated by } E_{12}, E_{13}, E_{23}, \\ \quad \text{i.e. } Ker(ad(\mathfrak{g}_\beta)) \text{ with } \\ \quad \beta \in \{L_i - L_j \mid 1 \le i < j \le 3\}. \\ \mathfrak{g}_{L_i - L_j} = \langle E_{ij} \rangle. \end{array}\right\}$$

<u>Lemma 4</u>: V ... a fin.-dim represent. of $sl(3,\mathbb{C})$. Then there $\exists$ $v \in V$ s.t.

1/ v is an eigenvector for $\mathfrak{h}$, i.e. $v \in V_\alpha$ for some $\alpha \in \mathfrak{h}^*$,

2/ $E_{12}(v) = 0$, $E_{13}(v) = 0$, $E_{23}(v) = 0$ (i.e. v is so called highest weight vector of V)

<u>Pf</u>: V is irred. repr. $\Rightarrow$ $\forall \alpha, \beta \in \mathfrak{h}^*$ eigenvalues of $\mathfrak{h}$ acting on V are congruent modulo Λ_R : $V = \bigoplus_{\beta \in \Lambda_R} V_{\alpha+\beta}$. As above, we have $\ell \subseteq \mathfrak{h}^*$, $\ell^\perp \subseteq \mathfrak{h}^*$, and the real line is totally ordered : the extremal point on the subset of Λ_R is the element of maximal distance between its projection on $\ell^\perp$ an $0 \in \mathfrak{h}^*$.
ℓ has irrational slope $\Rightarrow$ there are no two points of the positive root lattice at the same distance from ℓ.
The spectrum of $\pi(\mathfrak{h})$ is finite subset of

$$\{\alpha + a(L_1 - L_2) + b(L_2 - L_3) + c(L_1 - L_3) \mid a, b, c \in \mathbb{Z}\} \subseteq \underbrace{\alpha + \Lambda}_{\text{translat root latt}}$$

Because we already know $\pi(E_{ij})v \in V_{\alpha + (L_i - L_j)}$ for $1 \le i < j \le 3$, so $\pi(E_{12})(v) = 0$, $\pi(E_{13})(v) = 0$, $\pi(E_{23})(v) = 0$. $\blacksquare$

<u>Lemma 5</u>: V ... fin.-dim. irred. repr. of $sl(3,\mathbb{C})$, $v \in V$ a highest weight vector. Then V is generated by $\pi(E_{21}), \pi(E_{31})$ and $\pi(E_{32})$ acting on v.

<u>Pf</u>: As in the case of $sl(2,\mathbb{C})$: we prove that $W \subseteq V$ generated by image of v under the (subalgebra generated by) $\pi(E_{21}), \pi(E_{31}), \pi(E$

is preserved by all of $sl(3,\mathbb{C}) \Rightarrow W = V$ by irreducibility of V.

So we have to check that $\pi(E_{12}), \pi(E_{13}), \pi(E_{23})$ map W into itself.

→ v is in the kernel of E_{12}, E_{23}, E_{13}.

→ $E_{21}(v)$ is in W: $E_{12}(E_{21}(v)) = E_{21}(E_{12}(v)) + [E_{12}, E_{21}](v) =$

$$= \alpha([E_{12}, E_{21}])(v), \text{ since } \quad E_{12}v = 0, \ [E_{12}, E_{21}] \in \mathfrak{h}.$$

$$E_{23}(E_{21}(v)) = E_{21}(E_{23}(v)) + [E_{23}, E_{21}](v) = 0$$

since $E_{23}(v) = 0$ & $[E_{23}, E_{21}] = 0$.

Analogously, $E_{32}(v)$ has the same property, and their commutator $[E_{32}, E_{21}]$, too.

→ the general case can be treated by induction: $w_n \in W_n$ any word of length $\leq n$ in E_{21}, E_{32}, $W_n^v =$ the vector space spanned by $w_n(v)$ for all such words. Note $W = \bigcup_n W_n$, since E_{31} is the commutator of E_{32} and E_{21}. Then one easily proves E_{12}, E_{23} are maps from W_n to W_{n-1}. This is elementary by studying $E_{12} w_n(v)$, $E_{23} w_n(v)$. ▨

In fact, we proved little bit more:

Lemma 6: V ... a fin. dim repr. of $sl(3,\mathbb{C})$, $v \in V$ ... a highest weight vector. Then $W \subseteq V$, the subrepresent. generated by images of v by action of $\pi(E_{21}), \pi(E_{31}), \pi(E_{32})$, is irreducible.

Lemma 7: Let V be an irred. $sl(3,\mathbb{C})$-repr., $V = \bigoplus_\beta V_\beta$, and α is ~~the~~ highest weight. Then $\dim_{\mathbb{C}}(V_\alpha) = 1$.

Pf: Assume $0 \neq v_\alpha \in V_\alpha$, and we observed V is generated by images of v_α by acting E_{21}, E_{31}, E_{32}. Take $v \in V_\alpha \subseteq V$, so v is of the form $K E_{21}^{i_{12}} E_{32}^{i_{32}} E_{31}^{i_{13}}(v_\alpha)$, $i_{12}, i_{32}, i_{13} \in \mathbb{N}_0$, $K \in \mathbb{C}$.

We have (when $\mathfrak{h}$ is acting)

$$\alpha = \alpha - i_{12}(L_1 - L_2) - i_{13}(L_1 - L_3) - i_{23}(L_2 - L_3),$$

and because $L_2 - L_3 = (L_1 - L_3) - (L_1 - L_2)$, this reduces to

$$\alpha = \alpha - (i_{12} - i_{23})(L_1 - L_2) - (i_{13} + i_{23})(L_1 - L_3).$$

Since L_1-L_3 and L_1-L_2 are linearly independent, $i_{13}+i_{23}=0$. $i_{12}-i_{23}=0$ and Because i_{12}, i_{23} and i_{13} are all positive integers, $i_{13}=0, i_{23}=0$ and $i_{13}=0$, hence $V=K v_\alpha$. ■

Lemma 8: $\dim_{\mathbb{C}} V_{\alpha-(L_1-L_2)}=1$.

Pf: Prove it by yourself, analogously as Lemma 7. ■

There are $sl(2,\mathbb{C})$-subalgebras of $sl(3,\mathbb{C})$: e.g. $sl(2,\mathbb{C})$ corresponding to the root L_2-L_1 is spanned by E_{12}, E_{21}, H_1 ($[E_{12}, E_{21}]=H_1$.) $[H_1, E_{12}]=(L_1-L_2)(H_1)E_{12}$, $[H_1, E_{21}]=(L_2-L_1)(H_1)E_{21}$, hence the eigenvalues of the adjoint representation of $sl(2,\mathbb{C})=sl_{L_2-L_1}(2,\mathbb{C})$ are $2, -2$ ($(L_1-L_2)(H_1)=2$, $(L_2-L_1)(H_1)=-2$.) The string of eigenvalues $\{\alpha+k(L_2-L_1)\}_{k=0}^{m}$ for some $m\in\mathbb{N}_+$ has integer values on H_1 and is symmetric w.r.to $0\in\mathfrak{h}^*$. Consequently, there is a line $L_{L_2-L_1}\subseteq\mathfrak{h}^*$: $L_{L_2-L_1}(H_1)=0$, such that the reflection along this line preserves the eigenvalues. The equation $L_{L_2-L_1}(H_1)=0$ is equivalent to $L_{L_2-L_1}=\{\sum_{i=1}^{3}a_iL_i \mid a_1=a_2\}$.
We can analyze $sl(2,\mathbb{C})$ associated to L_2-L_3, $sl_{L_2-L_3}(2,\mathbb{C})$, resulting in the string of eigenspaces $V_{\alpha+k(L_3-L_2)}$ preserved under reflection along the line $L_{L_3-L_2}\subseteq\mathfrak{h}^*$: $L_{L_3-L_2}(H_2)=0$. This is equivalent $L_{L_3-L_2}=\{\sum_{i=1}^{3}a_iL_i \mid a_2=a_3\}$.

Def 9 Consider the lines $L_{L_i-L_j}$, orthogonal to the lines spanned by L_i-L_j. The group generated by the reflections in the lines $L_{L_i-L_j}$ is called the Weyl group.

The lines $L_{L_1-L_2}, L_{L_2-L_3}, L_{L_1-L_3}$ intersect in the origin $0 \in \mathfrak{h}^*$.

$L_{1-2} \cap L_{2-3} \cap L_{1-3}$ are the points $\sum_{i=1}^{3} a_i L_i$ with $a_1 = a_3 = a_3$, but $L_1 + L_2 + L_3 = 0$ hence the result.

Observation: For $\forall$ fin.-dim. irred. repr. of $sl(3,\mathbb{C})$, the Weyl group acts as the symmetric group S_3 on the generators $L_i, i=1,2,3$.

Consequently, there is a hexagon ($3! = 6 = \# S_3$) bounding the set of eigenvalues, obtained as the convex hull of the images of α (highest weight of an irred. repr. V) under the action of the Weyl group.

An immediate consequence of $sl(2,\mathbb{C})$-analysis, the eigenvalues of H_1, H_2 must be integers. A linear form $\sum_{i=1}^{3} a_i L_i \in \mathfrak{h}^*$ has integral values on H_1, H_2 it is necessary

$$\sum_{i=1}^{3} a_i L_i (H_1) = a_1 - a_2 \in \mathbb{Z}, \quad \sum_{i=1}^{3} a_i L_i (H_2) = a_2 - a_3 \in \mathbb{Z}.$$

Since $\sum_{i=1}^{3} L_i = 0$ in $\mathfrak{h}^*$, the eigenvalues lie in the weight lattice Λ_W: $\Lambda_W = \{ \sum_{i=1}^{3} b_i L_i \mid b_i \in \mathbb{Z} \}$.

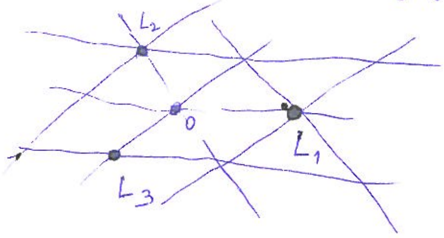
<u>Lemma 9</u>: All the eigenvalues of any irred. fin. dim. repr. of $sl(3,\mathbb{C})$ lie in the weight lattice $\Lambda_W \subseteq \mathfrak{h}^*$ generated by L_i ($i = 1,2,3$), and are congruent modulo the root lattice $\Lambda_R \subseteq \mathfrak{h}^*$ generated by $L_i - L_j$.

This is analogous to the situation for $sl(2,\mathbb{C})$: the H-eigenvalues in any fin.-dim. represent. (irreducible) lay in the lattice $\Lambda_W \simeq \mathbb{Z}$, and were congruent to one another modulo sublattice $\Lambda_R \simeq 2\mathbb{Z}$ generated by the H-eigenvalues in the adjoint represent.

Exercises 12

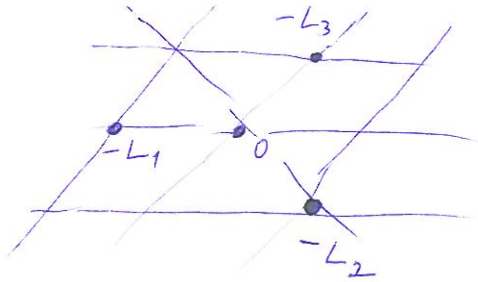
Example 1: $sl(3, \mathbb{C})$ acts on $\mathbb{C}^3$ by matrix multiplication, and so for this representation $V = \mathbb{C}^3$ the standard vectors e_1, e_2, e_3, $\langle e_1, e_2, e_3 \rangle = \mathbb{C}^3$, are the weight vectors with weights L_1, L_2, L_3:

e_1: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\Rightarrow$ e_1 has the weight L_1.



Example 2: $sl(3, \mathbb{C})$ acts on $Hom_{\mathbb{C}}(\mathbb{C}^3, \mathbb{C}) \simeq (\mathbb{C}^3)^*$ by dual action to $\mathbb{C}^3$: $(\pi^*(X)\varphi)(v) := \varphi(-\pi(X)^T v)$, $v \in \mathbb{C}^3$, $\varphi \in (\mathbb{C}^3)^*$, $X \in sl(3, \mathbb{C})$

The weights of $(\mathbb{C}^3)^*$ are negatives of the weights of V, so:



Example 3: Recall given a representation of $\mathfrak{g}$ on V and W, we get representation on $V \otimes W$. In particular, we get a representation on $V \otimes V$, and a subrepresentation on $V \otimes V / \langle v_1 \otimes v_2 + v_2 \otimes v_1 \mid v_1, v_2 \in V \rangle$ – this is called (2nd) exterior power of V. Assume v_1, v_2 are eigenvectors for the representation $\pi(V)$, then $v_1 \otimes v_2 - v_2 \otimes v_1$ is an eigenvector for the action of $(\pi \otimes \pi)(H)$ with eigenvalue $\alpha + \beta$. Given the choice of conventions for $V = \mathbb{C}^3$, the eigenvalues on $\Lambda^2 V$ are the pairwise sums of the distinct weights of V. Because $L_1 + L_2 + L_3$, these sums are

minus the third weight (i.e., for example $L_1 + L_2 = -L_3$), hence it is isomorphic to V^* ($\Lambda^2 V \xrightarrow{\sim} V^*$).

Example 4 : Analyze $\Lambda^3 V$, $S^2 V$, $\Lambda^2 V^*$, $S^2 V^*$, Adjoint representation.
Here $V = \mathbb{C}^3$, the fundamental vector represent. of $sl(3,\mathbb{C})$.

Example 5 : Λ_R ... the root lattice
Λ_W ... the weight lattice

$\left.\begin{matrix}\Lambda_R \text{ ... the root lattice}\\ \Lambda_W \text{ ... the weight lattice}\end{matrix}\right\} \Lambda_R \subseteq \Lambda_W$ at least for $sl(2,\mathbb{C})$, $sl(3,\mathbb{C})$.

Compute Λ_W / Λ_R for both $sl(2,\mathbb{C})$ and $sl(3,\mathbb{C})$.