Warming up: describe irreducible $sl(3,\mathbb{C})$-representation of highest weight $(1,1)$!?

We know that $V=\mathbb{C}^3$ (the fundamental vector representation of $sl(3,\mathbb{C})$ has highest weight $(1,0)$, $V^*=(\mathbb{C}^3)^*$ $(0,1)$, and consider the representation generated by highest weight vectors acting on by operators $\pi_{1,1}(E_{21})$, $\pi_{1,1}(E_{32})$ (note that $E_{31}=-[E_{21},E_{32}]$) in the tensor product representation.

$V=\mathbb{C}^3$: highest weight vector $e_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\left.\begin{matrix} H_1 e_1 = e_1 \\ H_2 e_1 = 0\end{matrix}\right\} (1,0)$

e_1, e_2, e_3

$$\pi_{1,0}(E_{21})\, e_1 = e_2 \qquad \pi_{10}(E_{21})\, e_2 = 0,\quad e_3 = 0 \qquad \pi_{10}(E_{32})\, e_1 = 0,\quad e_2 = e_3,\quad e_3 = 0$$

$$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$V^*=(\mathbb{C}^3)^*$: the action on V^* is given by $\pi_{01}(Z)=-Z^T$ ← transpose of Z

e_1^*, e_2^*, e_3^* for any $Z\in sl(3,\mathbb{C})$

$$E_{12}=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix} \qquad E_{23}=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}$$

$$\pi_{01}(E_{12})\, e_3^* = -E_{21}\cdot e_3^* \qquad \left.\begin{aligned} (E_{21}e_3^*)(e_1) &= e_3^*(\overbrace{E_{12}e_1}^{0}) = 0 \\ (E_{21}e_3^*)(e_2) &= e_3^*(\underbrace{E_{12}e_2}_{e_1}) = 0 \\ (E_{21}e_3^*)(e_3) &= e_3^*(\underbrace{E_{12}e_3}_{=0}) = 0 \end{aligned}\right\} \Rightarrow \pi_{01}(E_{12})e_3^* = 0$$

$$\pi_{01}(E_{23})\, e_3^* = -E_{32}\cdot e_3^* \qquad \left.\begin{aligned} (E_{32}e_3^*)(e_1) &= e_3^*(\overbrace{E_{23}e_1}^{0}) = 0 \\ (E_{32}e_3^*)(e_2) &= e_3^*(\underbrace{E_{23}e_2}_{=0}) = 0 \\ (E_{32}e_3^*)(e_3) &= e_3^*(E_{32}e_3) = e_3^*(e_2) = 0 \end{aligned}\right\} \Rightarrow \underbrace{\pi_{01}(E_{23})e_3^*}_{=0}$$

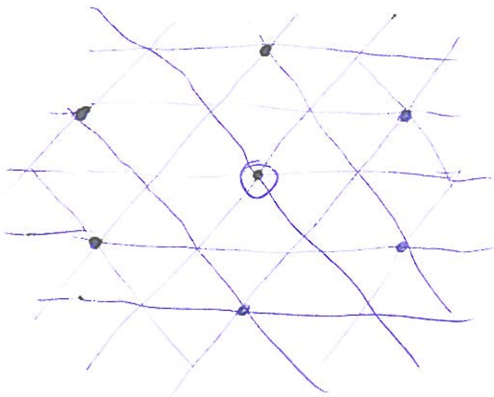
$\Rightarrow e_3^*$ is the highest weight vector of V^*:

$$\begin{cases} \pi_{01}(H_1)\, e_3^* = -H_1 e_3^* = 0 \\ \pi_{01}(H_2)\, e_3^* = -H_2 e_3^* \Rightarrow (\pi_{01}(H_2)e_3^*)(e_3) = (-H_2 e_3^*)(e_3) = -e_3^*(H_2 e_3) \\ \qquad\qquad = -e_3^*(-e_3) = 1 \end{cases}$$

↳ weight of e_3^* is $(0,1)$, analogously weights of e_2^* $(1,-1)$, e_1^* $(-1,0)$

- write basis of $V \otimes V^*$, $\{e_i \otimes e_j^*\}_{i,j=1}^{3}$ dim $= 9$
- tensor product $sl(3,\mathbb{C})$ action $\pi_{V\otimes V^*} = \pi_{10} \otimes Id + Id \otimes \pi_{01}$
- act by $\pi_{V\otimes V^*}(E_{21})$, $\pi_{V\otimes V^*}(E_{32})$, ...
- realize the decomposition as $V \otimes V^* \simeq V_{1,1} \oplus V_{0,0}$

$V_{1,1}$:



dim $= 8$

$V_{0,0}$ dim $= 1$

e.g. find the vector $v' \in V \otimes V^*$ such that $\pi_{V\otimes V^*}(E_{12})$, $\pi_{V\otimes V^*}(E_{23})$, $\pi_{V\otimes V^*}(E_{21})$, ... annihilates v', and v' is of weight $(0,0)$.

The class of semi-simple Lie groups/algebras - can classify the irred. repr. ~~for~~ in a scheme analogous to $sl(3,\mathbb{C})$. There are several equivalent ways to define semi-simple Lie algebra as a reductive Lie algebra with trivial center; a complex Lie algebra $\mathfrak{g}$ is reductive if there is a compact matrix Lie group K such that $\mathfrak{g} \simeq \mathbb{k}_{\mathbb{C}}$, Lie$(K) = \mathbb{k}$.

$\underset{su(n)}{sl(n,\mathbb{C})}$ $n \geq 2$, $\underset{so(n)}{so(n,\mathbb{C})}$, $n \geq 3$, $\underset{sp(n)}{sp(n,\mathbb{C})}_{n\geq 1}$ - semisimple

$\underset{U(n)}{gl(n,\mathbb{C})}$, $\underset{so(2)}{so(2,\mathbb{C})}$ - reductive (not semi-simple)

Because K is compact, there is a ($\mathbb{R}$-valued) inner product on $\mathbb{k}$ invariant under adjoint action of K. It extends to a complex inner product on $\mathfrak{g} = \mathbb{k}_{\mathbb{C}}$ for which the adjoint action of K is unitary.

Lemma 1: Let $\mathfrak{g} := \mathfrak{k}_{\mathbb{C}}$ be a reductive Lie algebra. Then there exists an inner product on $\mathfrak{g}$ that is real-valued on $\mathfrak{k}$ an the adjoint action of $\mathfrak{k}$ is unitary: $\langle ad(X)Y, Z\rangle = -\langle Y, ad(X)Z\rangle$ $\forall X \in \mathfrak{k}$, $Y, Z \in \mathfrak{g}$. The definition of $X \to X^*$ on $\mathfrak{g}$ by $(X_1 + iX_2)^* = -X_1 + iX_2$, $X_1, X_2 \in \mathfrak{k}$, then the inner product $\langle \,,\, \rangle$ on $\mathfrak{g}$ satisfies $\langle ad(X)Y, Z\rangle = \langle Y, ad(X^*)Z\rangle$ $\forall X, Y, Z \in \mathfrak{g}$. The scalar product is non-degenerate.

Decompositions of semi-simple Lie algebras:

Def 2: $\mathfrak{g}$ ... complex semi-simple Lie algebra, then a Cartan subalgebra of $\mathfrak{g}$ is a (complex) subspace $\mathfrak{h} \subseteq \mathfrak{g}$ such that

1/ $\forall H_1, H_2 \in \mathfrak{h}$, $[H_1, H_2] = 0$
2/ If for $X \in \mathfrak{g}$ $[H, X] = 0$ $\forall H \in \mathfrak{h}$, then $X \in \mathfrak{h}$. } maximal commut. sub., ..
3/ $\forall H \in \mathfrak{h}$, $ad(H)$ is diagonalizable. } simult. diagonalizable

In the case $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ a complex semi-simple Lie algebra, $\mathfrak{t} \subseteq \mathfrak{k}$ maximal comm. subalgebra. Then $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}} = \mathfrak{t} + i\mathfrak{t} \subseteq \mathfrak{g}$ is its Cartan subalgebra.

Def 3: For $\mathfrak{g}$ a complex semi-simple Lie algebra, the rank of $\mathfrak{g}$ is dim of any Cartan subalgebra (This is well defined, because any two Cartan subalgebras $\mathfrak{h}_1, \mathfrak{h}_2 \subseteq \mathfrak{g}$ are conjugated by an inner automorphism of $\mathfrak{g}$.)

Def 4: A non-zero element $\alpha \in \mathfrak{h}$ is a root (relative to the pair $\mathfrak{g}, \mathfrak{h}$) if $\exists X \in \mathfrak{g}$, $X \neq 0$ such that $[H, X] = \langle \alpha, H\rangle X$ for all $H \in \mathfrak{h}$. The set of roots is denoted by R.

Here we used the inner product in Lemma 1 to identify $\mathfrak{h}$ with $\mathfrak{h}^*$. $\forall$ root α belongs to $i\mathfrak{t} \subset \mathfrak{h}$, because $\forall H \in \mathfrak{t}$ is skew-adjoint operator, i.e. $ad(H)$ has purely imaginary eigenvalues $\Rightarrow \langle \alpha, H\rangle$ is purely imaginary for α a root and $H \in \mathfrak{t}$ (the scalar product is real on $\mathfrak{t} \subset \mathfrak{k}$.)

<u>Def 5</u>: For α a root, its root space $\mathfrak{g}_\alpha$ is the space of $X \in \mathfrak{g}$ for which $[H, X] = \langle \alpha, H \rangle X$ for all $H \in \mathfrak{h}$. A non-zero element in $\mathfrak{g}_\alpha$ is root vector for α.

We have $\mathfrak{g}_0 = \mathfrak{h}$, if $\alpha \neq 0$ is not a root, then $\mathfrak{g}_\alpha = \{0\}$. By Jordan theorem, $\mathrm{ad}(H)$ is semi-simple so that

<u>Lemma 6</u>: $\mathfrak{g}$ is a direct sum decomposition of vector spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha .$$

<u>Lemma 7</u>: $\forall \alpha, \beta \in \mathfrak{h}$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

<u>Pf</u>: $\mathrm{ad}(H)$ is a derivation ($\Leftarrow$ Jacobi identity)

$$[H, [X, Y]] = [[H, X], Y] + [X, [H, Y]]$$

$$\Rightarrow [H, [X, Y]] = \langle \alpha+\beta, H \rangle [X, Y], \quad \forall H \in \mathfrak{h}. \quad \square$$

<u>Lemma 8</u>: If $\alpha \in \mathfrak{h}$ is a root, so is $-\alpha$. In particular, if $X \in \mathfrak{g}_\alpha$ then $X^* \in \mathfrak{g}_{-\alpha}$ (see lemma 1 for $?^*$ operation)

In addition, the roots span $\mathfrak{h}$.

<u>Pf</u>: Based on the fact that if $X = X_1 + i X_2$, $X_1, X_2 \in \mathbb{k}$, $[H, X] = \langle \alpha, H \rangle X$, then $\bar{X} = X_1 - i X_2$ fulfills $[H, \bar{X}] = -\langle \alpha, H \rangle \bar{X}$. $\square$

A key tool in the study of semi-simple Lie algebra $\mathfrak{g}$ is the existence of certain subalgebras of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

<u>Lemma 9</u>: $\forall \alpha \in \mathfrak{h}$ a root, we can find lin. independ. elements $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$ and $H_\alpha \in \mathfrak{h}$ such that $H_\alpha \sim \alpha$ and $[H_\alpha, X_\alpha] = 2X_\alpha$, $[H_\alpha, Y_\alpha] = -2Y_\alpha$, $[X_\alpha, Y_\alpha] = H_\alpha$;

~~which~~ Moreover Y_α can be chosen to be equal to X_α^*.

Because $\left.\begin{array}{l}[H_\alpha, X_\alpha] = 2X_\alpha \\ [H_\alpha, X_\alpha] = \langle \alpha, H_\alpha \rangle X_\alpha\end{array}\right\}$ $H_\alpha = \dfrac{2\alpha}{\langle \alpha, \alpha \rangle}$, so H_α is called coroot associated to root α.

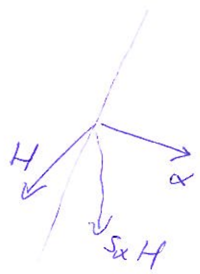
<u>Remark 10</u> For $X_\alpha, Y_\alpha, H_\alpha$ as in lemma 9, with $Y_\alpha = X_\alpha^*$, the elements
$E_1^\alpha := \frac{i}{2} H_\alpha$, $E_2^\alpha := \frac{i}{2}(X_\alpha + Y_\alpha)$, $E_3^\alpha := \frac{1}{2}(Y_\alpha - X_\alpha)$
are lin. independ. elements of $\mathfrak{k}$ and satisfy
$[E_1^\alpha, E_2^\alpha] = E_3^\alpha$, $[E_2^\alpha, E_3^\alpha] = E_1^\alpha$, $[E_3^\alpha, E_1^\alpha] = E_2^\alpha$, i.e.
$\langle E_1^\alpha, E_2^\alpha, E_3^\alpha \rangle$ is isomorphic to $su(2)$.

<u>Lemma 11</u>: For each root α, the only multiples of α that are roots are α and $-\alpha$. $\forall \alpha$ is the root space of dimension 1.
~~If both α and $c\alpha$ are roots, and $|c|>1$, then $c = \pm 2$.~~

The set of roots R has an important symmetry of a reflection group called Weyl group of $\mathfrak{g}$.

<u>Def 12</u>: $\forall \alpha \in R$, define a linear map $s_\alpha : \mathfrak{h} \to \mathfrak{h}$ by $H \mapsto H - \dfrac{2\langle \alpha, H \rangle}{\langle \alpha, \alpha \rangle}\alpha$.
The Weyl group of R, denoted W, is a subgroup of $GL(\mathfrak{h})$ generated by all s_α, $\alpha \in R$. ($H \in i\mathfrak{t} \rightsquigarrow s_\alpha H \in i\mathfrak{t}$)

s_α ... reflection along the hyperplane OG to α
in fact, $W \leq O(i\mathfrak{t})$



<u>Lemma 13</u>: The action of W on $i\mathfrak{t}$ preserves R. This means that if α is a root, then $w \cdot \alpha$ is a root for all $w \in W$.

<u>Pf</u>: For each $\alpha \in R$, the invertible operator $S_\alpha := e^{ad(X_\alpha)} e^{-ad(Y_\alpha)} e^{ad(X_\alpha)}$ maps any root vector $X_\beta \in \mathfrak{g}_\beta$ to the root space $s_\alpha \cdot \beta$:
$ad(H)(S_\alpha^{-1} X) = \langle (s_\alpha^{-1} \cdot \beta), H \rangle S_\alpha^{-1} X \quad \forall H \in \mathfrak{h}$.
(we use $S_\alpha \, ad(H) S_\alpha^{-1} = ad(s_\alpha \cdot H)$). ∎

Lemma 14: $\forall \alpha, \beta$ roots in R, $\langle \beta, H_\alpha \rangle = 2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

The proof is a consequence of the structure of $\mathfrak{sl}(2,\mathbb{C})$-subalgebras discussed above. This can be interpreted as the fact that the OG-projection of α onto β must be an integer or ½ integer multiple of β. Regarding $R \subseteq i\mathfrak{t}$, the properties may be summarized/formalize by the notion of root system.

Theorem 15: The root system is the set of roots R (a finite set of non-zero elements of a real inner product space E, $R \subseteq E$) with additional properties:

1/ the roots in R span E,

2/ if $\alpha \in R$, then $-\alpha \in R$ and the only multiples of α in R are $+\alpha, -\alpha$,

3/ if $\alpha, \beta \in R$, so is $s_\alpha \cdot \beta$, where s_α is the reflection defined by $\alpha \in R$,

4/ $\forall \alpha, \beta \in R$, $2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ (or half-integer)

The root systems of the classical Lie algebras:

Special linear Lie algebras $\mathfrak{sl}(n+1, \mathbb{C})$:

$$\mathfrak{k} = \mathfrak{su}(n+1), \quad \mathfrak{t} = \left\{ \begin{pmatrix} i a_1 & & 0 \\ & \ddots & \\ 0 & & i a_{n+1} \end{pmatrix} \;\middle|\; a_j \in \mathbb{R}, \sum a_i = 0 \right\}$$

$$\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n+1} \end{pmatrix} \;\middle|\; \lambda_j \in \mathbb{C}, \sum \lambda_i = 0 \right\}$$

This is a Cartan subalgebra of $\mathfrak{sl}(n+1, \mathbb{C})$.

E_{jk} ... elementary matrix $(n+1)\times(n+1)$, then

$$H E_{jk} = \lambda_j E_{jk} \quad \text{and} \quad E_{jk} H = \lambda_k E_{jk} \implies$$

$[H, E_{jk}] = (\lambda_j - \lambda_k) E_{jk}$. If $j \neq k$, then E_{jk} is in $sl(n+1,\mathbb{C})$ and E_{jk} is an eigenvector for $\forall$ $ad(H)$, $H \in \mathfrak{h}$, with eigenvalue $\lambda_j - \lambda_k$. There is a direct sum decomposition $sl(n+1,\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{\substack{j,k=1 \\ j\neq k}}^{n+1} E_{jk}$. If we first start with roots as elements of $\mathfrak{h}^*$, they are linear funcnals α_{jk} which associate to each $H \in \mathfrak{h}$ the quantity $\lambda_j - \lambda_k$.

We identify $\mathfrak{h} \subseteq \mathbb{C}^{n+1}$ as the vectors whose components sum to $0 \in \mathbb{C}$. The inner product on $\mathfrak{h}$ is given by restriction of $\langle\ ,\ \rangle : X, Y \mapsto \mathrm{Tr}(\underset{\text{transpose + conjugate}}{X^*} \cdot Y)$, which is standart inner product on $\mathbb{C}^{n+1}$. Using this inner product, the roots in $\mathfrak{h}^*$ transfer to $\mathfrak{h}$, we obtain the vectors $e_j - e_k$, $j \neq k$, $\{e_1, \dots, e_{n+1}\}$ the canonical basis of $\mathbb{C}^{n+1}$.

The roots of $sl(n+1,\mathbb{C})$ form a root system denoted A_n, n is the rank of $sl(n+1,\mathbb{C})$. $\forall$ root has length $\sqrt{2}$, and $\langle \alpha_{jk}, \alpha_{j'k'} \rangle$ equals to $0, \pm 1, \pm 2$, depending whether $\{j,k\}$ and $\{j',k'\}$ have zero, one or two elements in common (recall $j\neq k$, $j'\neq k'$). This implies

$$2\,\frac{\langle \alpha_{jk}, \alpha_{j'k'} \rangle}{\langle \alpha_{jk}, \alpha_{jk} \rangle} \in \{0, \pm 1, \pm 2\}.$$

If $\alpha, \beta \in R$, and $\alpha \neq \beta$ $\overbrace{\text{and } \alpha \neq -\beta}^{\pm 2}$, the angle between α and β is either $\overset{1}{\pi/3}$, $\overset{0}{\pi/2}$, $\overset{-1}{2\pi/3}$, depending on $\langle \alpha, \beta \rangle$ has value $1, 0, -1$.

It is easy to see that for any j, k, the reflection $s_{\alpha_{jk}}$ acts on $\mathbb{C}^{n+1}$ by interchanging j-th and k-th entry of each vector. It follows that the Weyl group of A_n root system is the permutation group on $(n+1)$ elements.

Example: Do the same for orthogonal Lie algebras $so(2n, \mathbb{C})$.

Example: ——————— " ——————— $so(2n+1, \mathbb{C})$.

Example: ——————— " ——————— symplectic Lie algebra $sp(n, \mathbb{C})$.