LECTURE 6

G... Lie group, $\mathfrak{g}$ .. Lie algebra of G, $X, Y \in \mathfrak{g}$ commute if $(\operatorname{ad} X)(Y) = [X, Y] = 0$

Lemma 1: $X, Y \in \mathfrak{g}$ commuting elements. Then $\exp(X), \exp(Y) \in G$ commute, and $\exp(X+Y) = \exp(X) \cdot \exp(Y)$.

Proof: As we know, $x \cdot y \cdot x^{-1} = \exp(\operatorname{Ad}(x) Y)$ and $\operatorname{Ad}(x) Y = e^{\operatorname{ad} X} Y$. Since we assume $\operatorname{ad}(X) Y = [X, Y] = 0$, $\operatorname{ad}(X)^n Y = 0 \quad \forall n \geq 1$. Then $\operatorname{Ad}(x) Y = e^{\operatorname{ad} X} Y = Y$, $x \cdot y \cdot x^{-1} = y$, i.e. x and y commute.

Second claim: because $[sX, tY] = st[X, Y] = 0 \; \forall s, t \in \mathbb{R}$, $\exp(sX)$ and $\exp(tY)$ commute $\forall s, t$. The map

$$\alpha : t \to \exp(tX) \exp(tY), \quad t \in \mathbb{R},$$

satisfies $\alpha(0) = e$ and

$$\alpha(s+t) = \exp((s+t)X) \cdot \exp((s+t)Y) = \exp(sX) \cdot \exp(sY) \cdot \exp(tX) \cdot \exp(tY) = \alpha(s)\alpha(t),$$

so α is 1-parameter subgroup of G and so $\alpha = \alpha_Z$, $Z = \alpha'(0)$, for some $Z \in \mathfrak{g}$. We have

$$\alpha'(0) = \left(\frac{d}{dt}\Big|_{t=0} \exp(tX) \right) \exp(0) + \frac{d}{dt}\Big|_{t=0} \exp(0) \exp(tY)$$

$$= X + Y \quad \Rightarrow \alpha(t) = \alpha_Z(t) = \exp(tZ) = \exp(t(X+Y))$$

for $t \in \mathbb{R}$.

The result follows by $t = 1$. $\blacksquare$

The chain rule for differentiation of smooth maps leads to

Lemma 2: M... a smooth manifold, $(0,0) \in U \overset{\text{open}}{\subseteq} \mathbb{R}^2$, and $\varphi : U \to M$ a smooth map at $(0,0)$. Then

$$\frac{d}{dt}\Big|_{t=0} \varphi(t, t) = \frac{d}{dt}\Big|_{t=0} \varphi(t, 0) + \frac{d}{dt}\Big|_{t=0} \varphi(0, t).$$

<u>Definition 3</u>: The subgroup $G_e \subseteq G$ generated by the elements $\exp(X)$, $X \in \mathfrak{g}$, is called the component of identity of G.

$$G_e = \{ \exp(X_1) \cdot \ldots \cdot \exp(X_k) \mid k \geq 1, \ X_1, \ldots, X_k \in \mathfrak{g} \}$$

Open subgroup of a Lie group G is a subgroup H of G such that H is an open subset of G as a topological subspace.

<u>Lemma 4</u>: G_e is an open subgroup of G.

<u>Proof</u>: $a \in G_e$, then $\exists\, k \geq 1$ and $X_1, \ldots, X_k \in \mathfrak{g}$ such that $a = \exp(X_1) \cdot \ldots \cdot \exp(X_k)$. The map $\exp : \mathfrak{g} \to G$ is a local diffeomorphism at $0 \in \mathfrak{g}$, i.e. $\exists\, 0 \in \Omega \subseteq \mathfrak{g}$ open and $\exp$ is diffeomorph. of Ω on $\exp(\Omega) \underset{\text{open}}{\subseteq} G$. L_a is diffeomorphism $\forall a \in G \Rightarrow L_a(\exp(\Omega)) \underset{\text{open}}{\subseteq} G$ is open neigh. of $a \in G$, and

$$L_a(\exp(\Omega)) = \{ \exp(X_1) \cdot \ldots \exp(X_k) \exp(X) \mid X \in \Omega \} \subseteq G_e$$

$\Rightarrow$ a is an interior point of G_e $\Rightarrow$ $G_e \overset{\text{open}}{\subseteq} G$

("a" was arbitrary). ▨

<u>Lemma 5</u>: Let $H \subseteq G$ be an open subgroup. Then H is closed as well.

<u>Proof</u>: $\forall\, x, y \in G$ either $xH = yH$ or $xH \cap yH = \emptyset$.

Hence there is a subset $S \subseteq G$ such that $G = \dot{\bigcup}_{s \in S} sH$ (disjoint union). Then $G \setminus H$ is a disjoint union of the sets sH, $s \in S$, $s \notin H$. Because H is open, sH is open an $\dot{\bigcup}_{\substack{s \in S \\ s \notin H}} sH$ a union of open, hence open. Then its complement $(= H)$ is closed. ▨

Another consequence is that G_e equals the connected component of G containing e. Then G is connected iff $G_e = G$.

Lemma 6: G a Lie group, $x \in G$. Then a/ and b/ are equivalent.
a/ x commutes with G_e, and b/ $Ad(x) = Id$.

Proof. Assume a/. Then $\exp(tY) \in G_e$ for $\forall Y \in \mathfrak{g}, t \in \mathbb{R}$, hence $\exp(t\,Ad(x)Y) = x \cdot \exp(tY) \cdot x^{-1} = \exp(tY)$.

Apply $\frac{d}{dt}\big|_{t=0}$, so $Ad(x)Y = Y \quad \forall Y \in \mathfrak{g} \Rightarrow$ b/ is true.

Assume b/. If $Y \in \mathfrak{g}$, then $x \cdot \exp(Y) \cdot x^{-1} = \exp(Ad(x)Y) = \exp(Y)$
$\Rightarrow x$ commutes with $\exp(\mathfrak{g}) \Rightarrow x$ commutes with G_e. ▨

Recall that so far we discussed the Lie subgroups (of a given Lie group), which are both subgroups and smooth submanifolds (hence, closed.) There is more general

Definition 7: A Lie subgroup of a Lie group G is a subgroup $H \subseteq G$, equipped with the structure of a Lie group, such that the inclusion map $\iota: H \to G$ is a Lie group homomorphism.

This definition applies to 1-parameter subgroups (these allow examples of Lie subgroups which are not submanifolds)

Lemma 8: G a Lie group, $X \in \mathfrak{g}$. The image of the one-parameter subgroup α_X is a Lie subgroup of G, $\operatorname{Im} \alpha_X \subseteq G$. $\quad (\alpha_X : \mathbb{R} \to G)$

Proof: $X = 0 \in \mathfrak{g}$ is trivial, so assume $X \neq 0$. The map $\alpha_X : \mathbb{R} \to G$ is a Lie group homom, its image H is a subgroup of G.

Assume α_X is injective: then $H = \operatorname{Im} \alpha_X$ has unique structure of smooth submanifold for which the bijection $\alpha_X : \mathbb{R} \to H$ is a diffeomorphism. This turns H into a Lie group, and $i : H \to G$ is a Lie group homomorphism.

Assume now α_X is not injective: as $\alpha_X'(0) = X \neq 0$, the map $t \mapsto \alpha_X(t) = \exp(tX)$ is injective on an open interval I, $0 \in I$.

Hence $\operatorname{Ker}(\alpha_X)$ is a discrete subgroup of $\mathbb{R}$, $\operatorname{Ker}(\alpha_X) = \mathbb{Z}\gamma$ for some $\gamma \in \mathbb{R}$. This implies that there is a unique group homom. $\bar{\alpha} : \mathbb{R}/\mathbb{Z}\gamma \to G$ such that $\alpha_X = \bar{\alpha} \circ pr$

($pr : \mathbb{R} \to \mathbb{R}/\mathbb{Z}\gamma$ is canonical projection.) Since pr is a local diffeom., $\bar{\alpha}$ is smooth, hence a Lie group homomorphism. Because $H = \operatorname{Im} \bar{\alpha}$ is compact, and $\bar{\alpha}$ is an injective immersion, so $\bar{\alpha}$ is an embedding of $\mathbb{R}/\mathbb{Z}\gamma$ onto a smooth embedded submanifold of G. In conclusion,

$H = \operatorname{Im} \bar{\alpha}$ is a smooth submanifold of G, hence a Lie subgroup. ∎

Exercises - The matrix exponential

For matrix Lie groups, $\exp : T_e G \to G$ is given by matrix exponential map "e". For $X \in Mat(n \times n)$, define $\exp(X) := e^X$ by formal power series $e^X := \sum_{m=0}^{\infty} \frac{X^m}{m!}$,

$X^0 = Id$, $X^m = \underbrace{X \cdot X \cdot \ldots \cdot X}_{m \times}$ ("$\cdot$" is the matrix multiplication.)

<u>Lemma 1</u>: The series $\sum_{m=0}^{\infty} \frac{X^m}{m!}$ converges for $\forall X \in Mat(n \times n, \mathbb{C})$,

and $\begin{array}{ccc} Mat(n\times n, \mathbb{C}) & \to & Mat(n \times n, \mathbb{C}) \\ X & \mapsto & e^X \end{array}$ is continuous.

Before proof, we introduce

<u>Def 2</u>: $\forall X \in Mat(n \times n, \mathbb{C}) \simeq \mathbb{C}^{n^2}$, we define the Hilbert-Schmidt norm $\|X\|$ of X by

$$\|X\| := \left(\sum_{j,k=1}^{n} |X_{jk}|^2 \right)^{1/2}.$$

There is a basis-independent way to compute $\|X\|$:

$$\|X\| = \left(Tr(X^* X)\right)^{1/2}, \quad X^* = \overline{X}^T.$$

This norm satisfies

- $\|X + Y\| \le \|X\| + \|Y\|$ $\Leftarrow$ (Δ-inequality for $\mathbb{C}^{n^2}$)
- $\|X \cdot Y\| \le \|X\| \, \|Y\|$ $\Leftarrow$ (Cauchy-Schwarz inequality)

$\{X_m\}_{m \in \mathbb{N}}$ a sequence in $Mat(n \times n)$, then ($X \in Mat(n\times n)$)

$$\left\{ \begin{array}{l} X_m \xrightarrow{m\to\infty} X \text{ iff} \\ (X_m)_{jk} \xrightarrow{m\to\infty} X_{jk} \\ \quad \forall j,k = 1,\ldots,n \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} X_m \xrightarrow{m\to\infty} X \text{ iff} \\ \|X_m - X\| \xrightarrow{m\to\infty} 0 \end{array} \right\}$$

Proof of Lemma 1: $\|X^m\| \le \|X\|^m$ $\forall m \in \mathbb{N}$, so that

$$\sum_{m=0}^{\infty} \left\| \frac{X^m}{m!} \right\| \le \|Id\| + \sum_{m=1}^{\infty} \frac{\|X\|^m}{m!} < \infty$$

$\Rightarrow \sum_m \frac{X^m}{m!}$ converges absolutely. Continuity: X^m is continuous fcn of X, by Weirstrass M-test ($\{f_n\}_{n\in\mathbb{N}}$, sequence of fcns on a set A s.t. $\forall x \in A$ $\exists M_n \in \mathbb{R} : |f_n(x)| \leq M_n$ & $\sum_n M_n < \infty \Rightarrow$ $\sum_n f_n(x)$ is abs. convergent uniformly on A), the series uniformly converges on the set $\{\|X\| \leq R\}$. Therefore, e^X is continuous on each such set, so continuous on all $Mat(n\times n, \mathbb{C})$.

<u>Lemma 3</u>: $X, Y \in Mat(n\times n, \mathbb{C})$. Then

$X^* = \overline{X}^T$

1. $e^0 = Id$
2. $(e^X)^* = e^{(X^*)}$
3. e^X is invertible, $(e^X)^{-1} = e^{-X}$
4. $e^{(\alpha+\beta)X} = e^{\alpha X} \cdot e^{\beta X}$ $\quad \forall \alpha, \beta \in \mathbb{C}$
5. If $X\cdot Y = Y \cdot X$, then $e^{X+Y} = e^X \cdot e^Y = e^Y \cdot e^X$
6. If C is invertible matrix, $e^{C\cdot X \cdot C^{-1}} = C e^X C^{-1}$

<u>Pf</u>: Almost all points are elementary. As for 5., we have

$$e^X \cdot e^Y = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{X^k}{k!} \frac{Y^{m-k}}{(m-k)!} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} X^k Y^{m-k},$$

and because $XY = YX$ (i.e, X and Y commute)

$$(X+Y)^m = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} X^k Y^{m-k},$$

so that

$$e^X \cdot e^Y = \sum_{m=0}^{\infty} \frac{1}{m!} (X+Y)^m = e^{X+Y}.$$

Point 6. follows from $(C\cdot X \cdot C^{-1})^m = C \cdot X^m \cdot C^{-1}$ $\forall m \in \mathbb{N}$.

$\blacksquare$

Lemma 4: $X \in Mat(n\times n, \mathbb{C})$. Then $t \to e^{tX}$ is a smooth curve in $Mat(n\times n, \mathbb{C})$ and $\frac{d}{dt} e^{tX} = X\cdot e^{tX} = e^{tX}\cdot X$. In particular,
$$\frac{d}{dt}\Big|_{t=0} e^{tX} = X.$$

Pf: $\forall j,k = 1,\dots,n$, $(e^{tX})_{jk}$ is a (uniformly) convergent power series in $t \in \mathbb{R}$, can differentiate the power series for e^{tX} termwise inside the radius of convergence. ▨

Question: What about the inverse to the matrix exponential, i.e. the matrix logarithm?

In the case of one complex variable, the function log:
$$\log(z) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(z-1)^m}{m}$$
is defined and analytic in a ~~disc~~ circle of radius 1 at $z=1$. For all z, $|z-1| < 1$, holds $e^{\log z} = z$; for all u, $|u| < \log 2$, we have $|e^u - 1| < 1$ and $\log e^u = u$.

(e.g. if $|u| < \log 2$, $|e^u - 1| = |u + \frac{u^2}{2!} + \dots| \le |u| + \frac{|u|^2}{2!} + \dots = e^{|u|} - 1 < 1$)

Definition 5: $A \in Mat(n\times n)$, define $\log A$ by
$$\log A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A - Id)^m}{m},$$
whenever the series converges.

Since $\|(A-Id)^m\| \leq \|A-I\|^m$ and the complex valued logarithmic series has radius convergence 1, the matrix-valued series converge if $\|A-Id\|<1$. Even if $\|A-Id\|>1$, the series might converge (e.g., if $A-Id$ is nilpotent matrix.)

Theorem 6: The function log given by ($A \in Mat(n\times n)$)

$$\log A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A-Id)^m}{m}$$

is defined and continuous on $Mat(n\times n, \mathbb{C})$ with $\|A-Id\|<1$. For all A with $\|A-Id\|<1$, $e^{\log A}=A$. For all X with $\|X\|<\log 2$, $\|e^X - Id\|<1$ and $\log e^X = X$.

The proof is straightforward, and we shall omit it.

Remark: Is it true that $\log e^X = X$ whenever log series is convergent? No! For $X = 2\pi i\, Id$, then $e^X = e^{2\pi i} Id = Id$. Then $e^X - Id = 0$, so $\log e^X = 0$, so $\log e^X \neq X$. This means that $\|X\|<\log 2$ is a crucial assumption.

It is easy to prove the following estimate: $\exists\, K \in \mathbb{R}_+$ such that for all $B \in Mat(n\times n)$, $\|B\|<\frac{1}{2}$, holds

$$\|\log(Id+B) - B\| \leq K\|B\|^2.$$

Example $\forall\, X, Y \in Mat(n\times n, \mathbb{C})$, we have $e^{X+Y} = \lim_{m\to\infty}\left(e^{\frac{X}{m}}\cdot e^{\frac{Y}{m}}\right)^m$

by realizing that

$\rightarrow$ $e^{\frac{X}{m}}\cdot e^{\frac{Y}{m}} = Id + \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)$,

$\rightarrow$ $\log(\text{—}\|\text{—}) = \frac{X}{m} + \frac{Y}{m} + O\left(\frac{1}{m^2}\right)$,

$\rightarrow$ $\left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)^m = \exp\left(X+Y+O\left(\frac{1}{m}\right)\right)$.

<u>Example:</u> The exponential map is not surjective. We shall demonstrate this fact for $G = SL(2,\mathbb{C})$ (it works analogously for $SL(2,\mathbb{R})$, for example.) This is a matrix Lie group, hence $\exp = e$ (the matrix exponential.) Let $M \in SL(2,\mathbb{C})$, assume $M = \exp(A)$ for some $A \in Mat(2\times 2, \mathbb{C})$, $Tr(A) = 0$. There is an invertible matrix P such that the Jordan normal form J_A of A is given by $J_A = PAP^{-1}$. Then $e^{J_A} = e^{PAP^{-1}} = Pe^{A}P^{-1} = PMP^{-1}$.

There are two possible Jordan normal forms for 2×2 trace-free matrices:

Either $\tilde{J} := \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$, $\lambda \in \mathbb{C}^*$ or $\tilde{\tilde{J}} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

We have

$$e^{\tilde{J}} = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix}, \quad \text{while} \quad e^{\tilde{\tilde{J}}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

both of them in the Jordan form already.

($\Rightarrow$ the Jordan normal form of M is the exponential image of the Jordan form of A!)

To find the image of exponential map, we have to find the matrices whose Jordan form is one of the two types above. However, the matrix

$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in SL(2,\mathbb{C})$ is in its unique normal Jordan form, different from the two types of normal Jordan forms in the image of exp.

<u>Example:</u> $G = \mathbb{R}^*$; then $T_1 G \simeq \mathbb{R}$ and for any $x \in T_1 G$ the map $\varphi : \mathbb{R} \to G$, $t \mapsto e^{tx}$, is the 1-parameter subgroup of G with $\dot{\varphi}(0) = x$, $\varphi(0) = 1 \in G$. It follows $\exp(x) = e^x$.

<u>Example:</u> $G = S^1 \, (\subseteq \mathbb{C})$, $T_1 S^1 \simeq i\mathbb{R}$. The 1-parameter subgroup corresponding to $ix \in T_1 S^1 \simeq i\mathbb{R}$ is $\varphi : \mathbb{R} \to S^1$, $t \mapsto e^{itx}$, so the exponential map is $\exp(ix) = e^{ix}$.

<u>Example:</u> $G = \mathbb{R}$, then $T_0 G \simeq \mathbb{R}$. The 1-parameter subgroup for $x \in \mathbb{R}$ is $\varphi : \mathbb{R} \to \mathbb{R}$, $t \mapsto tx$, so the exponential map is $\exp(x) = x$.