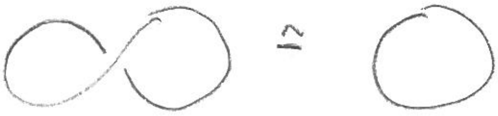
# KNOT INVARIANTS

**def** • *Knot* is an injective smooth map $K: S^1 \to \mathbb{R}^3$ w. nowhere vanishing derivative.
We identify knots with the same images $\overset{K}{\overset{\shortparallel}{K(S^1)}}$.
(i.e. we consider equivalence classes of maps $K$)

• *Isotopy* of two knots $K_0, K_1: S^1 \to \mathbb{R}^3$ is a smooth map $\sigma: [0,1] \times S^1 \to \mathbb{R}^3$ s.t.

$\forall t \quad \sigma(t,-): S^1 \to \mathbb{R}^3$ is a knot

$\sigma(0,-) = K_0, \quad \sigma(1,-) = K_1$


*goal* of the knot theory: find an easy way of telling whether two given knots are isotopic.
(in particular: can a knot be unknotted?)

assign an object $\overset{I(K)}{\shortparallel}$ (number, polynomial, seq. of v.s., ...) to each knot $K$ such that $I(K) = I(K') \Leftarrow K \simeq K'$ (isotopy invariance of $I$) and hope for "$\Rightarrow$" (or get as close to "$\Rightarrow$" as possible)

Overview of the lecture series:

1) Jones polynomial
2) more systematic way of producing knot invariants using ribbon categories
3) applications / related topics: - Yang-Baxter in statistical physics
(?) - invariants of 3-manif.
4) producing ribbon categories out of quantum groups
5) other knot invariants (HOMFLY ...)
6) Khovanov homology as categorification of Jones pol.

Lit.: Rosso, Turaev, Kassel: Quantum groups and knot invariants

...

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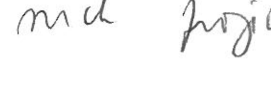
def • Link with n components is a smooth inj. map $S^1 \amalg \ldots \amalg S^1 \longrightarrow \mathbb{R}^3$ w. nowh. vanishing derivative.
(generalizes knots)
• isotopy of links ...
• link projection (aka link diagram) is the image of a projection $\mathbb{R}^3 \xrightarrow{\pi} \mathbb{R}^2$, which has only "+" singularities, and for each such projection "above/below" data:

example:



We denote the projection of $\pi L$)

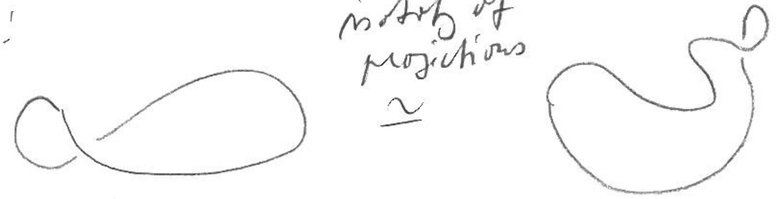
the usual picture of links are in fact link projections.

• isotopy of link projections $\pi_0 L_0$ and $\pi_1 L_1$: $\iota : \mathbb{R}^2 \times I \longrightarrow \mathbb{R}^2$ smooth satisfying $\iota(-,0) = 1_{\mathbb{R}^2}$ and $\forall t \in I$ $\iota(-,t)$ is a homeo

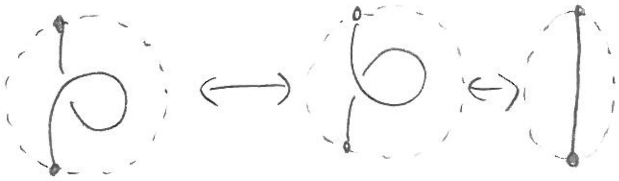
and $\left(\underbrace{\iota(-,0)}_{1_{\mathbb{R}^2}}\right) \circ \underbrace{\pi_1 L_1}_{\mathbb{R}^2 \leftarrow \mathbb{R}^3 \leftarrow S^1} = \iota(-,1) \circ \pi_0 L_0$

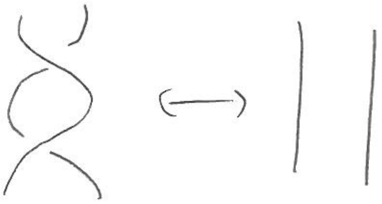
example:

isotopy of projections $\simeq$



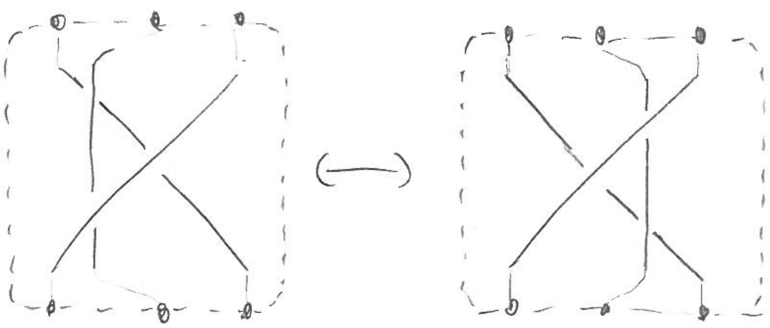
$\boxed{\text{Thm}}$ (Reidemeister) Let $\pi_1 L_1$ and $\pi_2 L_2$ be link projections of links $L_1, L_2$ resp. Then the links $L_1, L_2$ are isotopic iff one can transform $\pi_1 L_1$ into $\pi_2 L_2$ by a sequence of isotopies of link projections and the following "local moves":

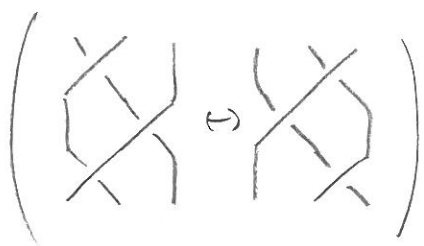
RM1  changes the projection only in a small set $\bigcirc \subset \mathbb{R}^2$ (fixed on boundary)

RM2 

this requires orientation (here implicit)

all the crossings are "positive"

RM3 



Application: isotopy invariants of links can be defined on link projection and then we check invariance under (link proj. isotopies – usually obv.) and RM1–3.

# Kauffman bracket and Jones polynomial



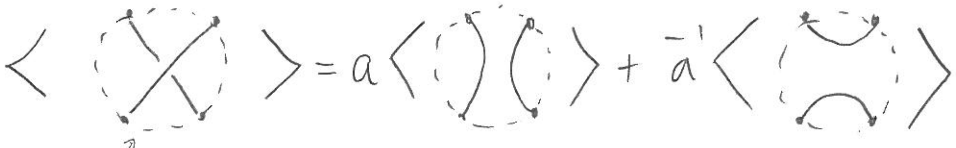
$\boxed{\text{def.}}$ $\langle\ \rangle$ : link projections $\longrightarrow \mathbb{C}[a, a^{-1}]$ (or $\mathbb{C}$ and $a \in \mathbb{C}$ a param. s.t. $a^2 + a^{-2} \neq 0$)

is defined as follows:

(1) $\langle \bigcirc \rangle = 1$

(2) $\langle L \sqcup \bigcirc \rangle = -(a^2 + a^{-2}) \langle L \rangle$ if $\bigcirc$ bounds a disc which doesn't intersect $L$

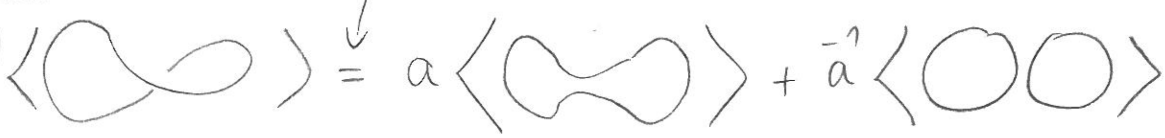
any link projection (abuse of notation)

(3) 

L locally somewhere

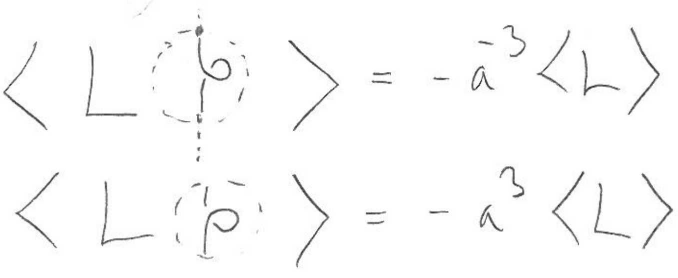
$\boxed{\text{Example}}$

(3) (look at (3) from left or right)

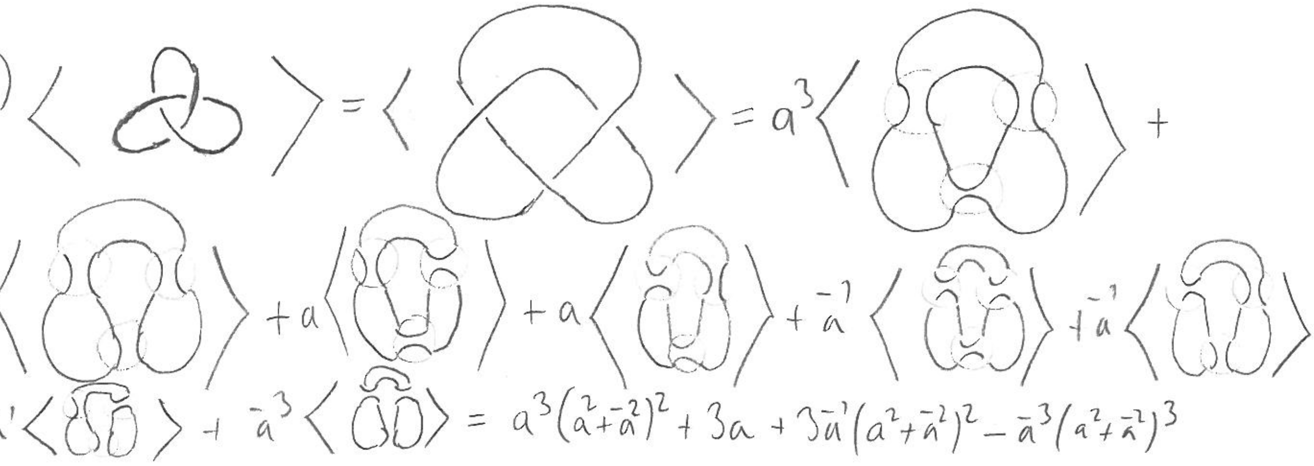
(a) 

$= a - a^{-1}(a^2 + a^{-2}) = -a^{-3}$

(b) more generally: L locally somewhere L  then



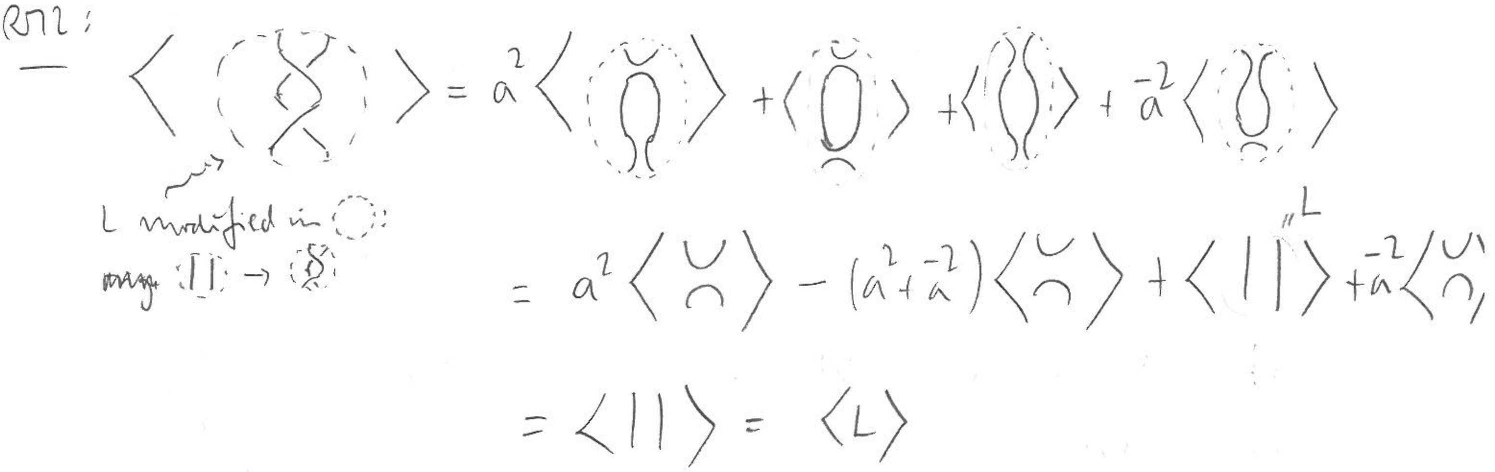
Hence the Kauf. b. is NOT isotopy invariant

OMIT → (c) 

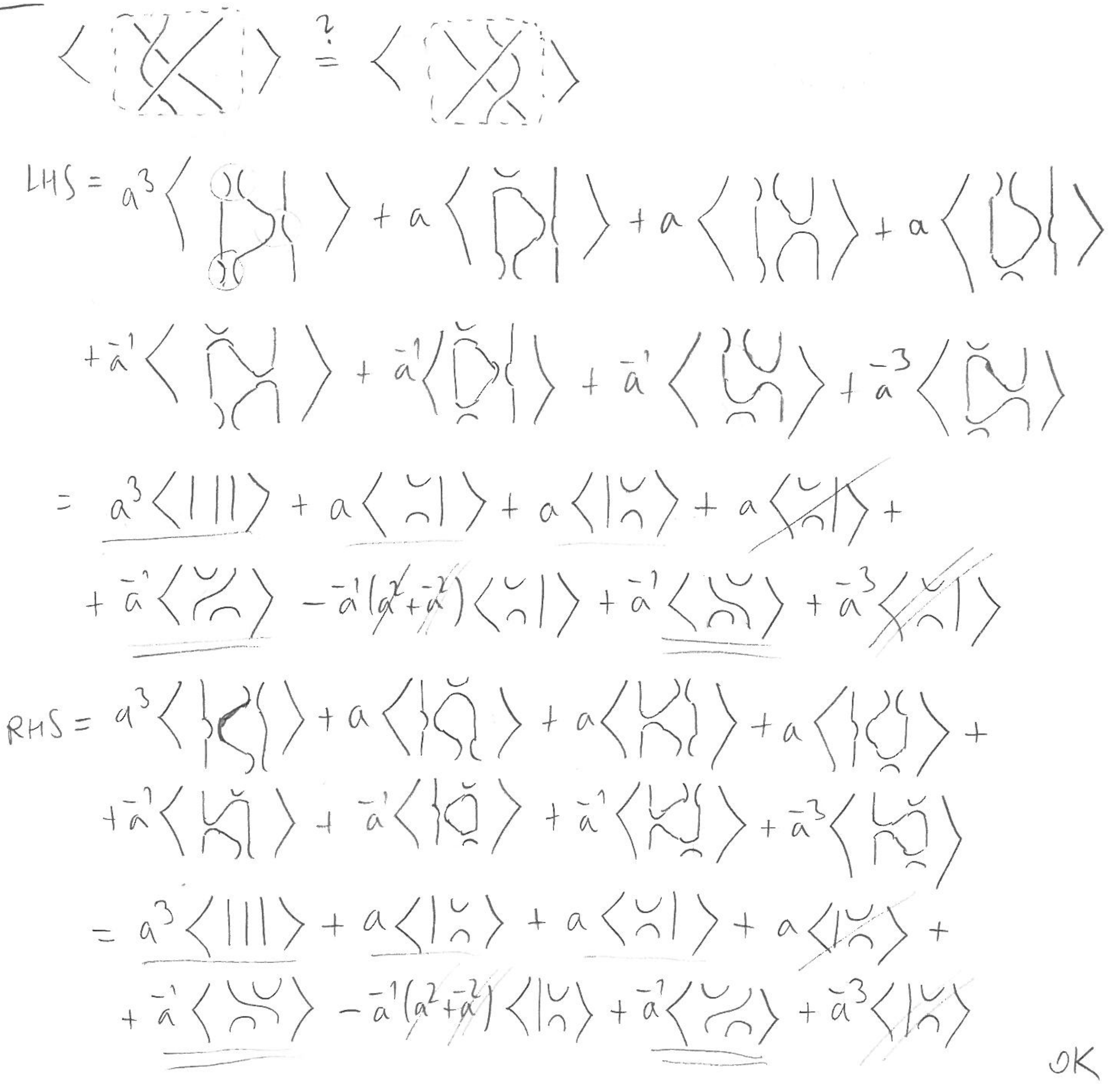
$= a^3(a^2+a^{-2})^2 + 3a + 3a^{-1}(a^2+a^{-2})^2 - a^{-3}(a^2+a^{-2})^3$

However, Kauffman br. is invariant under RM2, 3!

RM2:

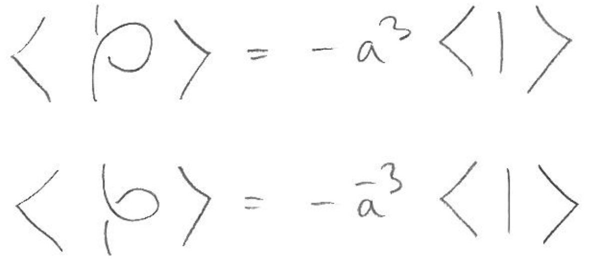



RM3:




OK

How can one make $\langle\ \rangle$ RM1-invariant



idea: multiply $\langle L \rangle$ by $-a^{-3}$ for each crossing "X" ; $-a^{3}$ ..... "X"

But these crossing are indistinguishable (link projection isotopic) unless one has an orientation on each branch of the crossing

[def] oriented link ... each component is oriented (an arrow in the link projection, ~~isotopy of oriented links~~)

reg. cross.

 positive crossing $\Rightarrow$ multiply by $-a^{-3}$

 negative $\qquad -a^{3}$

[def.] Let $L$ be an oriented link.

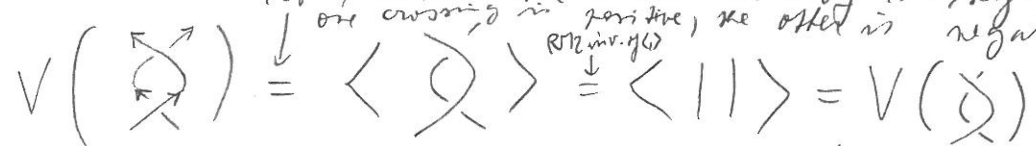
$$V(L) := (-a^{3})^{-(\#\text{ of pos. cr.} - \#\text{ of neg. cross.})} \langle L \rangle \in \mathbb{C}[a]$$

is called Jones polynomial of $L$.

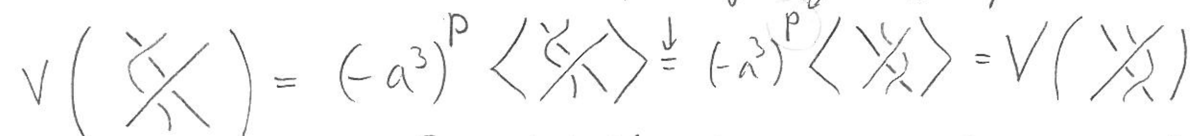
[Thm] Jones polynomial is invariant under isotopy of oriented links.

PF: By the constr., it's invariant under link proj. isotopies and under RM1

RM2: no matter the orientation of the segments one crossing is positive, the other is negative ; RM2 inv. of $\langle\rangle$



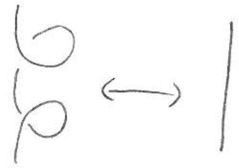
RM3: RM3 inv. of $\langle\rangle$ ; the same $p$ as on the left



$p \in \{-3,-1,1,3\}$ depends on the orient. of segments

<u>rem</u> There are <u>nonisotopic</u> knots $K_1, K_2$ s.t.
$V(K_1) = V(K_2)$ ( Jones pol. is <u>not</u> complete inv. )

## Framed links

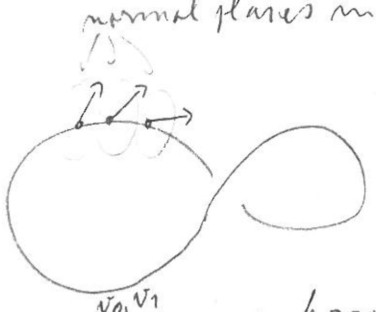
$\langle\,\rangle$ is invariant under RM0: 

a lot of theory works for objects that are only RM0, RM2, RM3 invariant. These have a natural geometric model

$\boxed{\text{def}}$ <u>Framed knot</u> (ribbon) is a knot $K$ equipped with a smooth normal nowhere vanishing vector field (called framing) $v: S^1 \to \mathbb{R}^3$

normal planes in $\mathbb{R}^3$

$v(s) \in N_{K(s)}(K(I))$ ← normal space



Two framings $v_0, v_1$ are homotopic iff $\exists$ homotopy
$h: S^1 \times I \longrightarrow \mathbb{R}^3$ s.t. $h(-,0) = v_0(-)$
$h(-,1) = v_1(-)$
$\forall t$ $h(-,t)$ is a framing

We identify homotopic framings

$\boxed{\text{observation}}$ Up to homotopy, the framing is given by an integer ( = # of times $v$ "winds around" $K(I)$)
[ $v$ is homotopic to <u>unital</u> framing $v': S^1 \to S^1$ and up to homotopy, this is classified by the homological degree ]

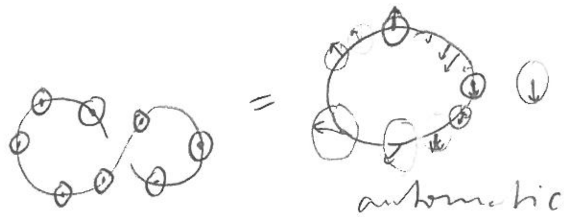
$\boxed{\text{remark}}$ • The framing sweeps a 2D "ribbon" whose one boundary is the knot 

isotopy of framed knots $\equiv$ isotopy of underl. knots & homotopy of framings, simultaneously in the obv. sense

framed links, their isotopy, ...

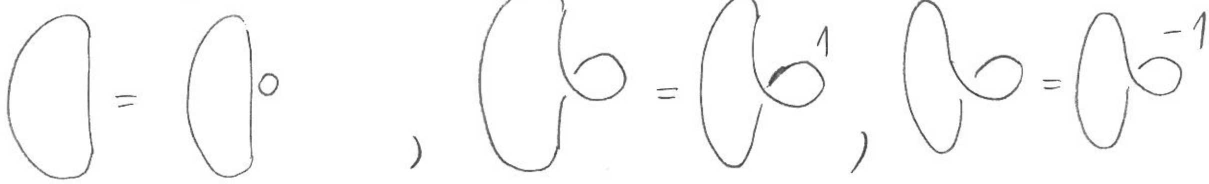
• <u>another</u> eq. def. of framed knot: it's smooth embedding $\underbrace{S^1 \times D^2}_{\text{solid torus}} \to \mathbb{R}^3$
(compare unfr. knot $S^1 \to \mathbb{R}^3$ or. embed.)

Link projection induces a canonical framing on the corr. link: the vector fields point upwards (from the blackboard towards the reader)



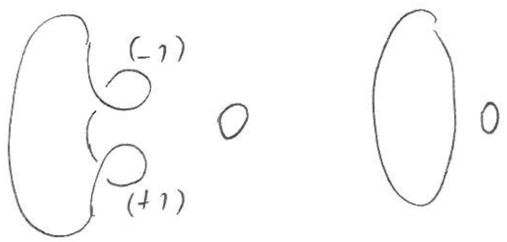
i.e. picture  automatically induces the framing  $^1$

$\boxed{\text{observe}}$ For framed links:

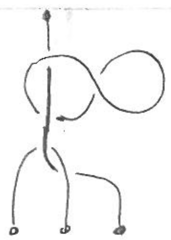



This shows why RM1 doesn't hold for framed links (knot is invariant but the framing is not)

But RM0 does hold:



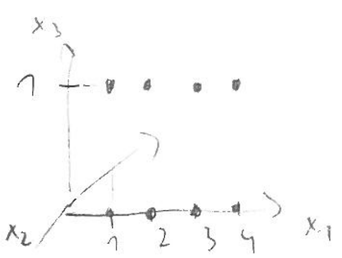

In the sequel, we'll be interested in invariants of framed links.



(3,1)-tangle

## Tangles

$\boxed{\text{def.}}$ (informally) (k,l)-tangle $(k,l \in \mathbb{N}_0)$.
set of disjoint smoothly embedded circles and arcs in $\mathbb{R}^2 \times [0,1]$ s.t. (a) the endpoints of arcs are the points $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix}, \dots, \begin{pmatrix}k\\0\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \dots, \begin{pmatrix}l\\0\\1\end{pmatrix}$




(b) the circles lie in $\mathbb{R}^2 \times (0,1)$

component of the tangle is one of the circles or arcs

framed tangle $\equiv$ each component is framed
(arcs are framed by vector fields equal to $\begin{pmatrix}0\\-1\\0\end{pmatrix}$ on endpoints)

oriented tangle $\equiv$ each comp. oriented isotopy of (framed) tangles $\equiv (I\times\dots\times S^1\times)\times \underset{\uparrow\ \text{dif. param.}}{I} \to \mathbb{R}^2\times I$ through tangles (9)

tangle projection (diagram) ...

RM 0,1,2,3 for (framed) tangles ...

(ex.) (0,0) tangle is a link

Goal: organize tangles into (isomorphism in) a category Tan

Tan is a (free, gen. by a single obj.) ribbon category.

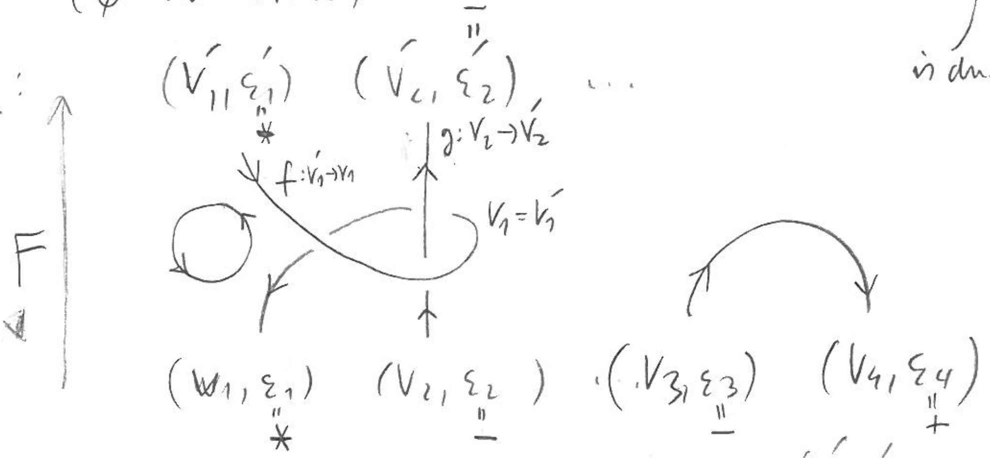
(monoidal c. w. some restriction on $\otimes$ and addit. str.)

def. Let $\mathcal{C}$ be a category. A category of $\mathcal{C}$-coloured framed oriented tangles $Tan_{\mathcal{C}}$ consists of:

objects: sequences $((V_1,\varepsilon_1), \dots, (V_n,\varepsilon_n))$, $V_i \in \mathrm{obj}\,\mathcal{C}$, $\varepsilon_i \in \{+,-\}$ "dual marker" ($-$ is dual, $+$ is NOT dual)

($\emptyset$ admitted)

morphs:




between $((V_1,\varepsilon_1), \quad (V_n,\varepsilon_n)) \to ((V_1',\varepsilon_1'), \dots (V_{n'}',\varepsilon_{n'}'))$ is an oriented framed $(n,n')$-tangle (iso class of ...)

s.t. • if $i$-th output is connected to the $j$-th input (by an arc directly), then either a) $V_i' \xleftarrow{f} V_j$ and $\varepsilon_i = \varepsilon_j = -$, the arc is oriented $V_j \to V_i'$
or b) $V_i' \xrightarrow{f} V_j$ and $\varepsilon_i = \varepsilon_j = +$ ... $V_j \leftarrow V_i'$

universal rule for $\varepsilon$'s: if the orientation at an endpoint $(V,\varepsilon)$ of the arc coincides with the orientation of the morph. F — then $\varepsilon = -$, otherwise $\varepsilon =-$

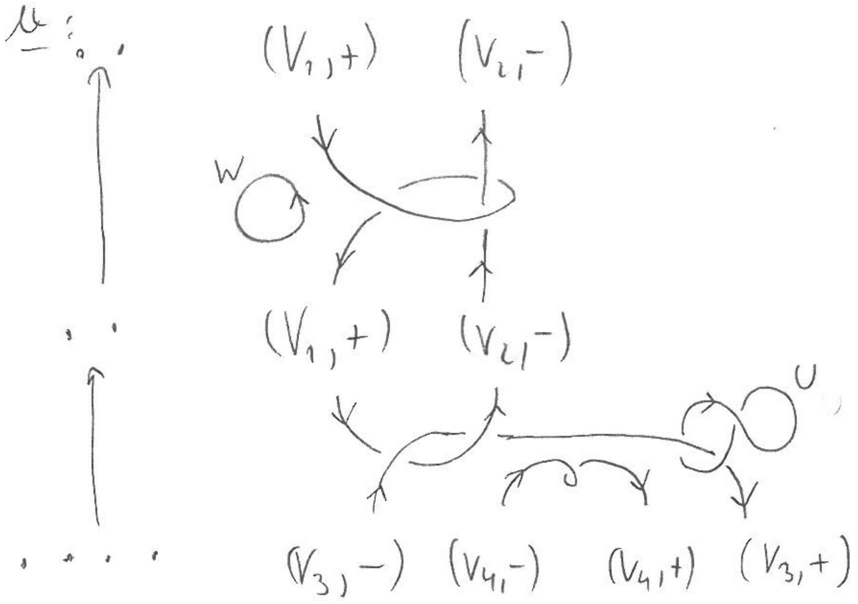
• if $i$-th outp. is connected to the $j$-th output ($\cup$) and the orientation is $i \to j$, then $\varepsilon_i = +$, $\varepsilon_j = -$

• if $i$-th input is connected to the $j$-th input ($\cap$) and the orientation is $i \to j$, then $\varepsilon_i = -$, $\varepsilon_j = +$

• circular components are labelled by endomorphs in $\mathcal{C}$ MODULO "trace equivalence" $fg \sim gf$.

Why the trace equivalence? see p. 9,5 (15,5)

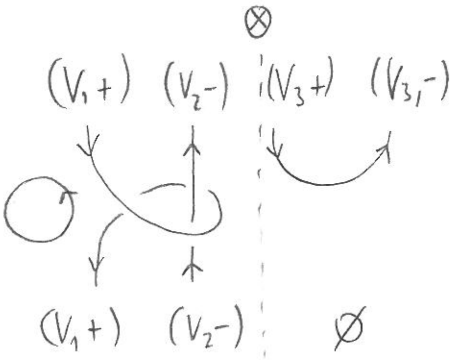
**composition** of morphs ≡ write the tangles one **above** the other and compress into $\mathbb{R}^2\times[0,1]$:




**monoidal str.** $\otimes : Tan_{\mathcal{C}} \times Tan_{\mathcal{C}} \longrightarrow Tan_{\mathcal{C}}$

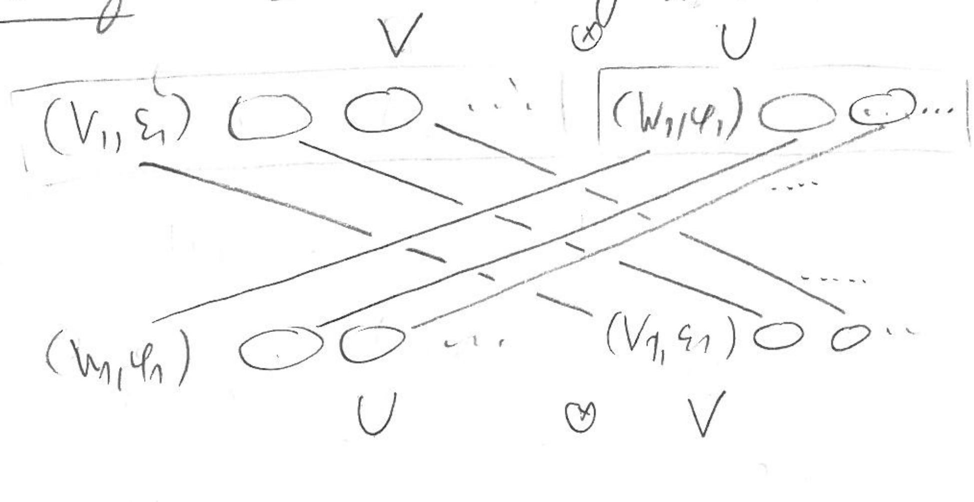
≡ write the sequences and tangles one **next** to the other

ex.:



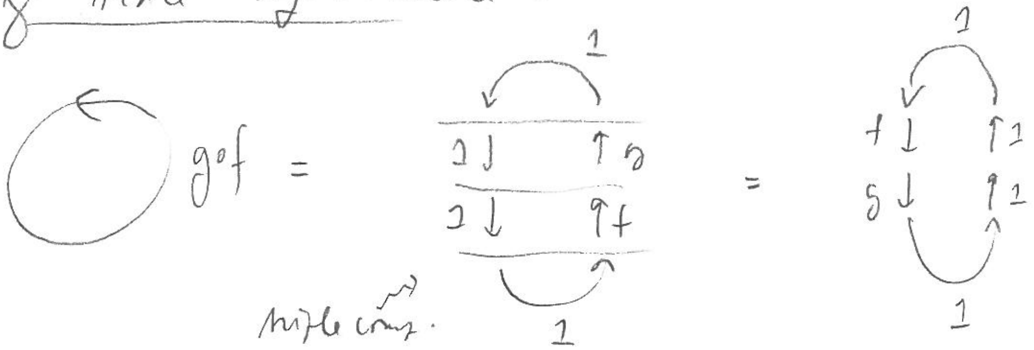

**monoidal unit** $\mathbb{1}$ = the empty tangle

**braiding str.** ≡ nat. transf. $c_{UV} : U \otimes V \longrightarrow V \otimes U$




orientation is uniquely determ. by $\varepsilon$'s and $\varphi$'s

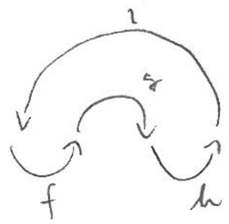
## Why trace equivalence?




subtle comp. in $\mathrm{Tan}_\mathcal{C}$

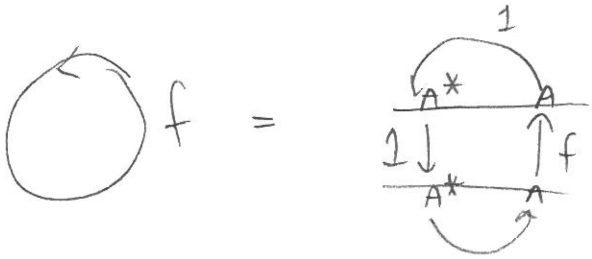
so e.g. writing the bottom map left doesn't make sense!
LHS: f is at bottom, RHS: f at top

also e.g.




What order to choose?

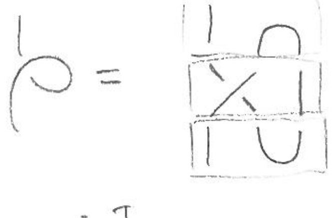
## Why the name "trace equivalence"?



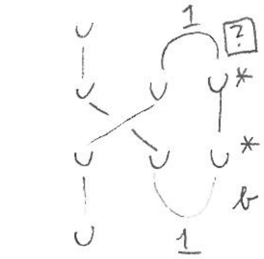

$= d_A \circ (1 \otimes f) \circ b_A$

in vector spaces: $d_A \circ (1 \otimes f) \circ b_A = \sum_{i=1}^{\dim A} e^i(f(e_i)) = \mathrm{Tr}\, f$

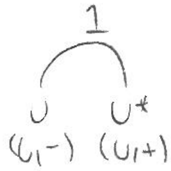
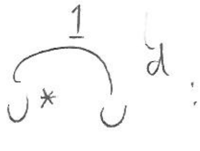
What is $\tilde{d}_{U^*} : U \otimes U^* \longrightarrow \mathbb{1}$ ? (recall $d : U^* \otimes U \to \mathbb{1}$)

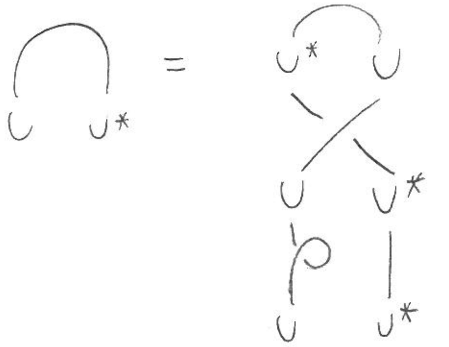


in Tan

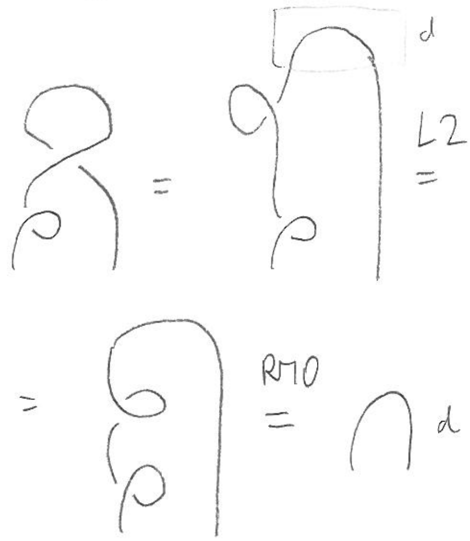



in R

Express  in Tan using  :



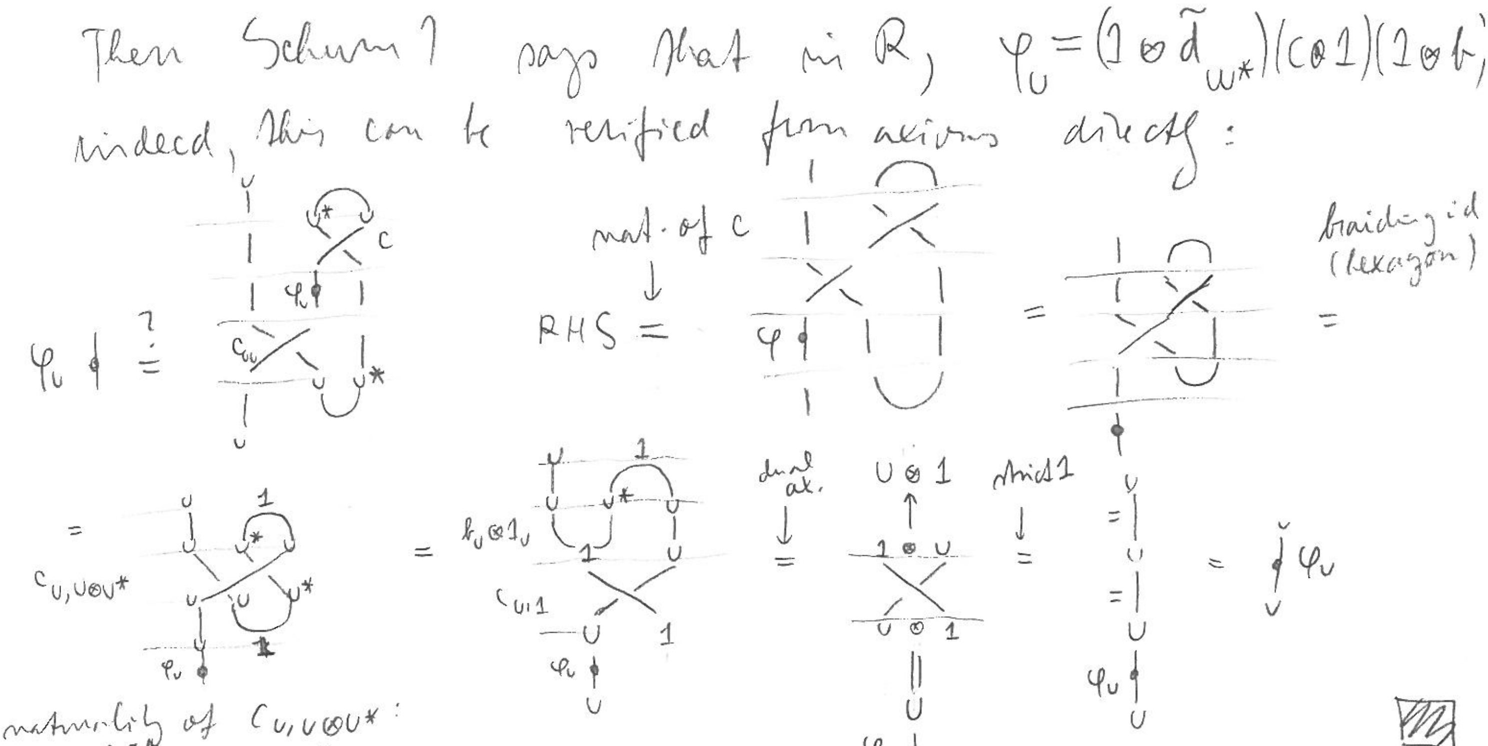
indeed:




Thus we define $\tilde{d}_{UU^*} := d_{U^*U}\, c_{UU^*} (\varphi_U \otimes 1_{U^*})$

Then Schur 7 says that in R, $\varphi_U = (1 \otimes \tilde{d}_{UU^*})(c \otimes 1)(1 \otimes b)$;

indeed, this can be verified from axioms directly:




naturality of $c_{U,U\otimes U^*}$:

$$\begin{tikzcd}
U = U \otimes \mathbb{1} \arrow[r, "1_U \otimes b_U"] \arrow[d, "c_{U,1}"'] & U \otimes U \otimes U^* \arrow[d, "c_{U,U\otimes U^*}"] \\
\mathbb{1} \otimes U \arrow[r, "b_U \otimes 1_U"'] & U \otimes U^* \otimes U
\end{tikzcd}$$

There is still more structure, but first we prove: (11)

$\boxed{\text{Thm}}$ $Tang_{\mathcal{C}}$ is a strict braided (monoidal) category (a.k.a. tortile cat.)

PF: recall that we should verify: $\forall\, U,V,W \in \operatorname{obj} Tang_{\mathcal{C}}$

(definition) (0) $Tang_{\mathcal{C}}$ is strict monoidal: $\otimes : Tang \times Tang \to Tang$ is a functor, $(U\otimes V)\otimes W = U\otimes(V\otimes W)$, $U\otimes 1 = U = 1\otimes U$, $(f\otimes g)\otimes h = f\otimes(g\otimes h)$; $f\otimes 1 = f = 1\otimes f$ – trivial

(a) $c_{U\otimes V,W} = (c_{U,W}\otimes 1_V)(1_U \otimes c_{V,W})$

(b) $c_{U,V\otimes W} = (1_V \otimes c_{UW})(c_{UV}\otimes 1_W)$

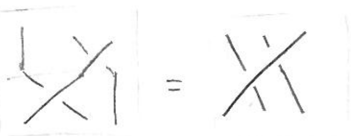
(hexagons – here it is simplified by assuming $\otimes$ strictly ass.)

$$\begin{tikzcd} U\otimes V\otimes W \arrow[d, "c_{UV}\otimes 1_W"'] \arrow[dr, "c_{U,V\otimes W}"] & \\ V\otimes U\otimes W \arrow[r, "1_V\otimes c_{UW}"'] & V\otimes W\otimes U \end{tikzcd} \qquad \begin{tikzcd} U\otimes V\otimes W \arrow[d, "1_U\otimes c_{VW}"'] \arrow[dr, "c_{U\otimes V,W}"] & \\ U\otimes W\otimes V \arrow[r, "c_{UW}\otimes 1_V"'] & W\otimes U\otimes V \end{tikzcd}$$

$\Updownarrow$

$1_V\otimes c_{UW}$

$c_{UV}\otimes 1_W$



$c_{U,V\otimes W}$

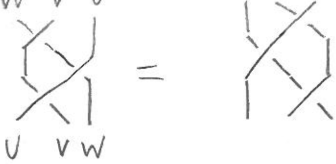
$\boxed{a}$

(axioms for str. monoidal cat. are also easy...)

digression on braided monoidal categories

it holds: $(c_{UW}\otimes 1)(1_U\otimes c_{VW})(c_{U,V}\otimes 1_W) = (1_W\otimes c_{UV})(c_{UW}\otimes 1_V)(1_U\otimes c_{VW})$

(categorical) Yang-Baxter eq.

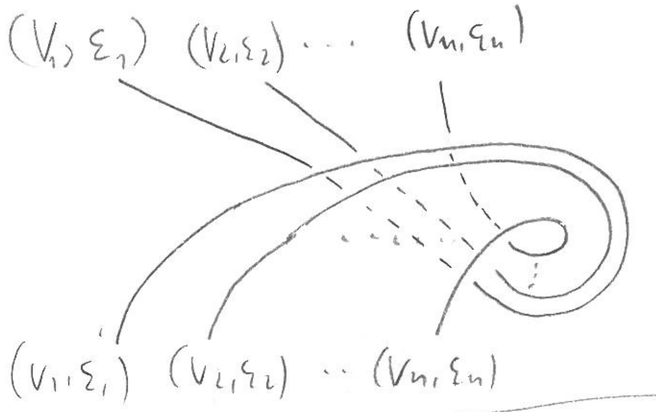


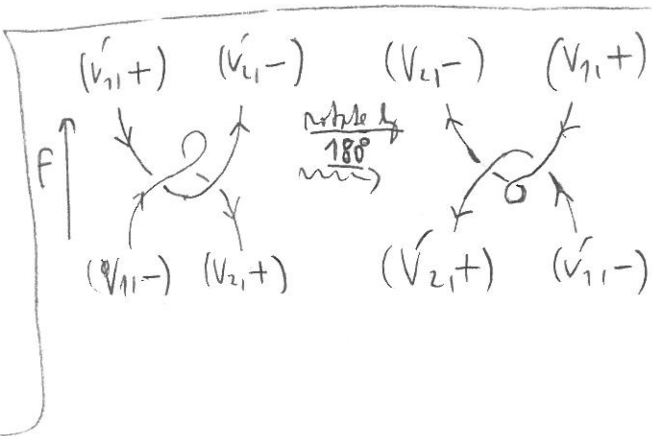

Y-B. appears in other contexts – statistical physics (→ Baxter), Hopf algebras, ...

another $\boxed{\text{example}}$ of braided mon. cat:
vector spaces w. usual $\otimes$ (NOT strict, but we ignore it)
$c_{U,V} : U\otimes V \cong V\otimes U$ satisfies $c_{VU}c_{UV} = 1_{UV}$ → symmetric monoidal cat. is stronger than Y.-B.

further structure in $\mathrm{Tang}_{\mathcal{C}}$ :

<u>twist</u> $\theta_U : U \to U$ natural tr. (check!)



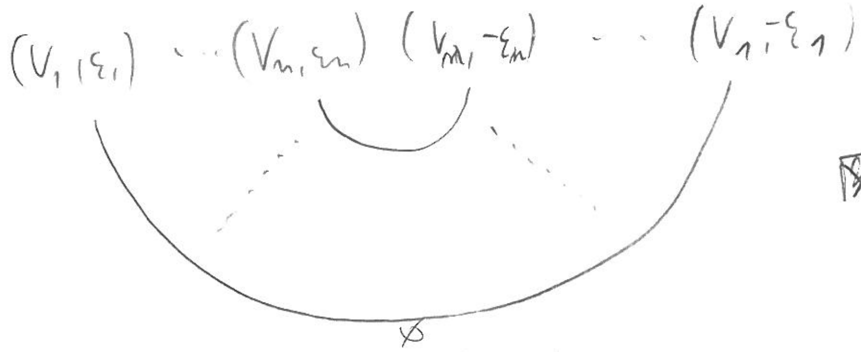



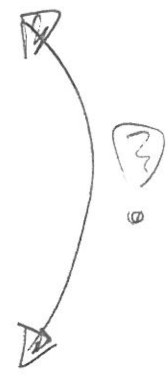

<u>duality</u> : $U \mapsto U^*$ : $\big((V_1,\varepsilon_1), \ldots, (V_n,\varepsilon_n)\big) \mapsto \big((V_n,-\varepsilon_n), \ldots, (V_1,-\varepsilon_1)\big)$

on morph.: "double mirror image" (contravariant ftor)

<u>evaluation</u> $d_V : U^* \otimes U \longrightarrow 1$ ... unit of the monoidal cat.

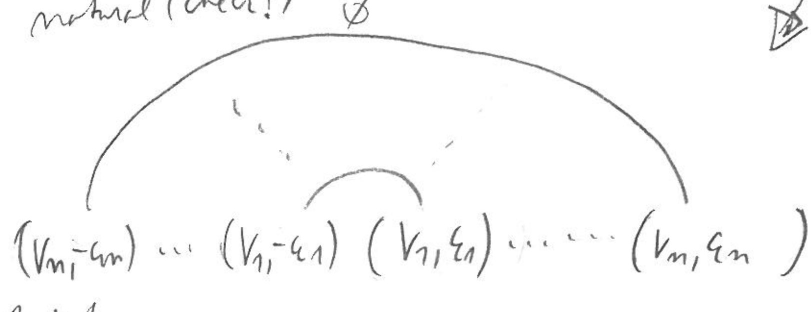
natural (check!) in $\mathrm{Tang}_{\mathcal{C}}$, $1 = \emptyset$




<u>coevaluation</u> $b_V : 1 \longrightarrow U \otimes U^*$

natural (check!)



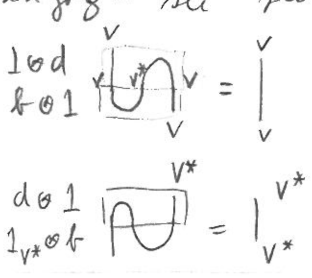

$\boxed{\text{Thm}}$ $\mathrm{Tang}_{\mathcal{C}}$ is a <u>strict ribbon category</u>

Pf. (definition): we should verify:

(0) $(\mathrm{Tang}_{\mathcal{C}}, \theta, 1)$ is a strict braided category - see prev. Thm

(a) $(1_V \otimes d_V)(b_V \otimes 1_V) = 1_V$

(b) $(d_V \otimes 1_{V^*})(1_{V^*} \otimes b_V) = 1_{V^*}$

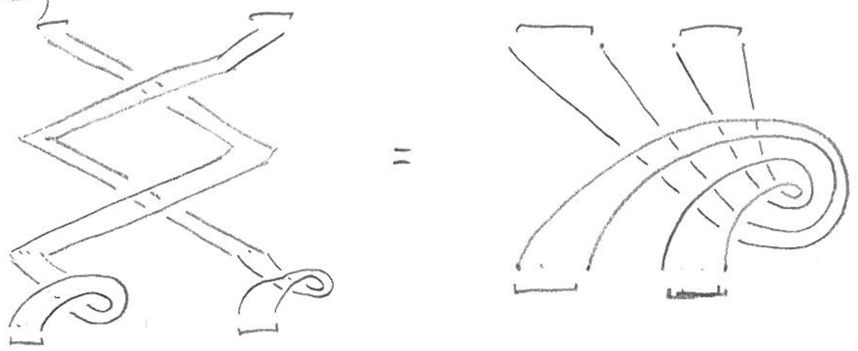


(c) $\theta_{V\otimes W} = c_{WV}\, c_{VW} (\theta_V \otimes \theta_W)$

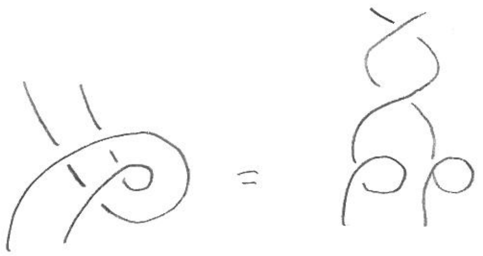
(d) $(\theta_V \otimes 1_{V^*})\, b_V = (1_V \otimes \theta_{V^*})\, b_V$

for $\mathrm{Tang}_{\mathcal{C}}$, (a), (b) are obvious

(c)



for simplicity, we prove just:

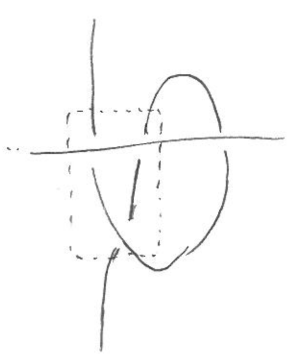


Lemma 1

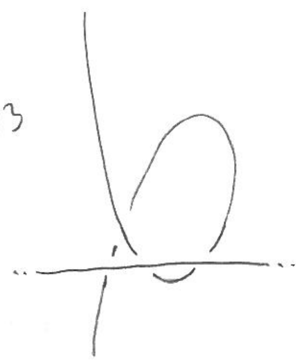
(obv. for tangles but we have framed tangles so we must avoid RM1)

(obv. by directly isotopy of tangles, but we want RM1)

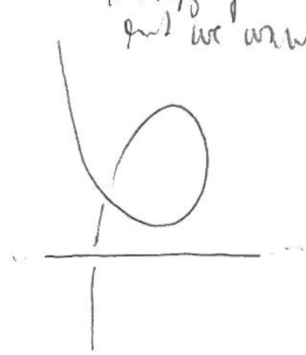
Pf: LHS $\overset{RM2}{=}$



$\overset{RM3}{=}$

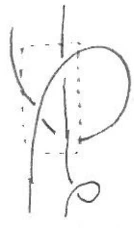


$\overset{RM2}{=}$



$\square$

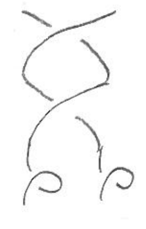
$\overset{L1}{=}$



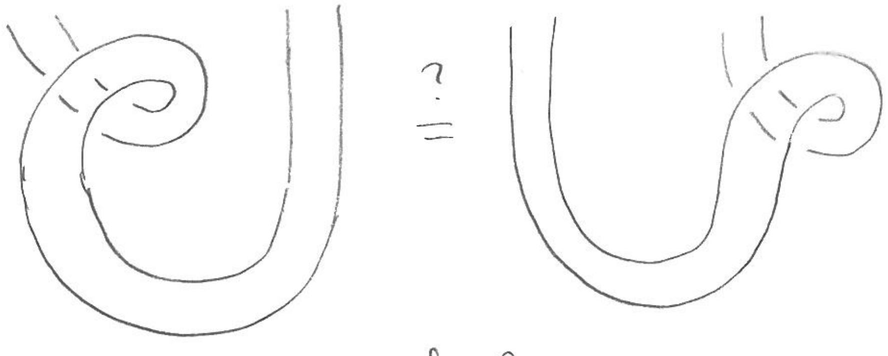
$\overset{RM3}{=}$

$\overset{L1}{=}$

$\overset{iso}{=}$



(d)



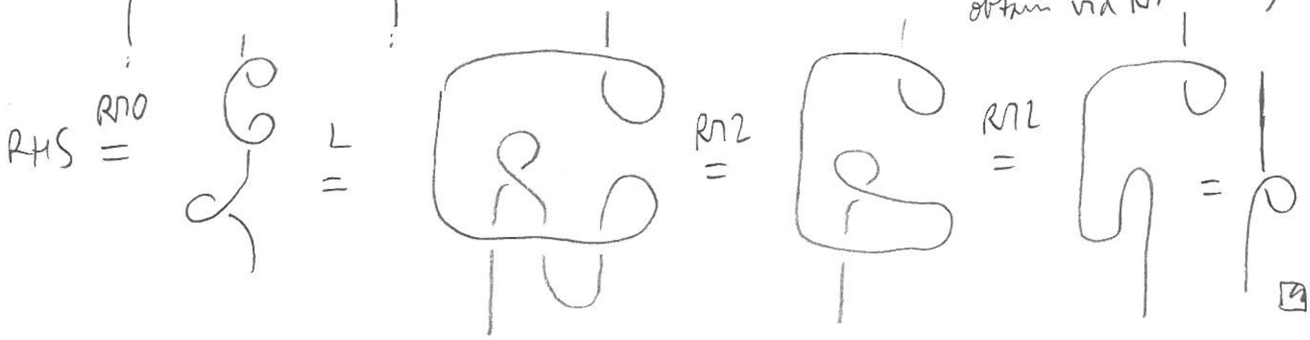

(obv. by isotopy of tangles, but we want to use RM)

for simplicity assume only 2 strands as on the picture

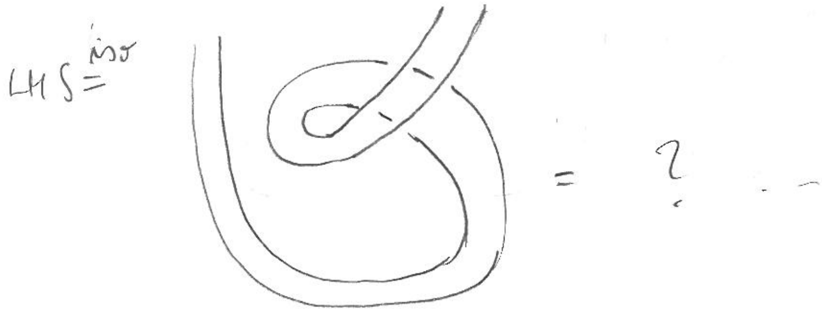
<u>lemma 2</u>

(fixed endpoints) (obv. directly by isotopy of tangles, but instructive to obtain via RM)

PF:




$\square$



$\boxed{\text{Ex.}}$ Another ribbon category in nature (<u>NOT STRICT</u>) (15)
Finite dimensional vector spaces:  (axioms more complicated)

$\otimes$ as usual
$1 = k$ the field, $\cong$ 1dim v.s.
$c_{UV} : U \otimes V \xrightarrow{\cong} V \otimes U$ symmetric braiding: $c_{VU} c_{UV} = 1_{U \otimes V}$

$U^*$ the usual dual

$U^* \otimes U \to 1$ ... the evaluation $\underset{U^*}{\varphi} \otimes \underset{U}{v} := \underset{1 = k}{\varphi(v)}$

$1 \to U \otimes U^*$ ... $1 \mapsto \sum_{k=1}^{\dim U} e_i \otimes e^i$ ← dual basis

$\theta_U : U \to U$ ... $\theta_U = 1_U$

---

Set $\mathcal{C}_0$ be the category w. single obj and single mor. Then $\mathrm{Tan}_{\mathcal{C}_0}$ are essentially unlabelled oriented framed tangles.

$\boxed{\text{Thm}}$ (<u>Schum</u>'89)

$\mathrm{Tan}_{\mathcal{C}_0}$ is equivalent as a (strict?) ribbon category to the <u>free</u> (strict?) ribbon category generated by $\mathcal{C}_0$.

- unsure about strict

<u>Explanations</u>:
- free ribbon category gen. by a cat. $\mathcal{C}$:

$$\mathrm{RibCat} \underset{F}{\overset{\text{forget}}{\rightleftarrows}} \mathrm{Cat} \quad F \text{ ... left adjoint}$$

$$\begin{tikzcd}
\text{ord. cat. } \mathcal{C} \arrow[r, "\text{ord. ftor}", "(\text{part of } F(\mathcal{C}))"'] \arrow[dr, "\forall"', "\text{ord. ftor}"] & \mathbb{F}_{\text{RIBBON}}(\mathcal{C}) \text{ ribbon} \arrow[d, "\exists! \text{ ribbon ftor}"] \\
& R \text{ .. ribbon}
\end{tikzcd}$$

- ribbon ftor $\equiv$ ftor preserving all the structure.

<u>idea of $\mathbb{F}(\mathcal{C}_0)$</u>: a) structure "ops" for ribbon cat:
$\otimes, 1, c, *, ev, coev$

each application of str. op. to an obj. of $\mathcal{C}$ yields a distinct obj. in $F(\mathcal{C}_0)$
iterate; the iterates are subject to only those relations implied by axioms of ribbon cat.

or b) every relation amongst the str. ops which holds in $\mathbb{F}(\mathcal{C}_0)$ holds in <u>any</u> ribbon category!

$\boxed{\text{Ex.}}$ since $\mathrm{Tan}_{\mathcal{C}_0} \simeq \mathbb{F}(\mathcal{C}_0)$ Tan: $\varphi =$  thus in any $R$: $\varphi = (1 \otimes \tilde{d})(c \otimes 1)(1 \otimes b)$

$\boxed{\text{Thm}}$ (Shum 2) Let $\mathcal{C}$ be any cat.

$Tang_{\mathcal{C}}$ is equivalent as a ~~(strict?)~~ ribbon category to the free ~~strict~~ ribbon category generated by $\mathcal{C}$.

$\boxed{\text{Corr.}}$ For any (ord.) ftor $\mathcal{C} \xrightarrow{F} R$, $\mathcal{C}$ ord. cat, $R$ ribbon, There is a unique ribbon functor $\widetilde{F}$ s.t.

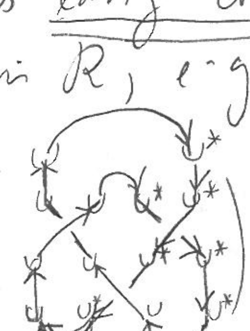
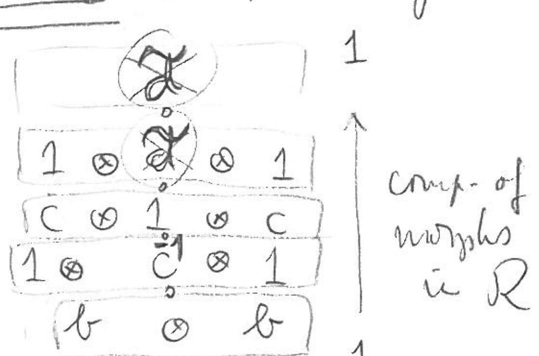
$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[dr, "F"'] & \mathcal{F}_{\text{RIBBON}}(\mathcal{C}) \simeq Tang_{\mathcal{C}} \arrow[d, "\widetilde{F}"] \\
& R
\end{tikzcd}$$

In particular, any ribbon category yields an isotopy invariant of tangles (by choosing $\mathcal{C} = \mathcal{C}_0$ to get unlabelled tangles $Tang_{\mathcal{C}_0}$)

In particular, any (isotopy class of) link is a morphism $L \in Hom_{Tang_{\mathcal{C}_0}}(\emptyset, \emptyset)$

Thus $\widetilde{F}(L) \in Hom_R(1, 1)$ is an isotopy invariant of $L$.

Notice that $\widetilde{F}(L)$ is <u>easily computable</u> in terms of the ribbon str. in $R$, e.g.

$\widetilde{F}\Big($  $\Big) = \widetilde{F}\Big($  $\Big) =$ 


applications

a) $R$ = ribbon cat. of modules over a ribbon Hopf algebra (= quasitriangular & ...) $H$, in particular $H$ a quantum group (at roots of unity? of $sl(n)$)

b) Dehn surgery of 3-manifold – constructing the man. out of a $\mathcal{C}$-coloured link $\Rightarrow$ invariants of the man.

c) Kauffman bracket via Shum's thm 

(artificial ...)

$\boxed{\text{def.}}$ The ribbon cat. of "skein tangles" $\mathrm{SkTan}_{\mathcal{C}_0}$:

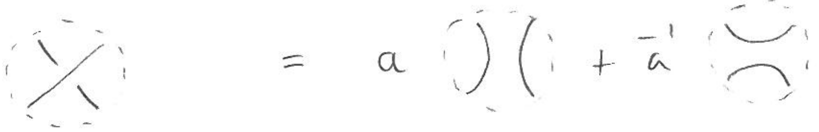
"idea: $\mathbb{C}$-linearization of $\mathrm{Tan}_{\mathcal{C}_0}$, morphisms are factorized by the skein relations used to define K-bracket."

objects: as in $\mathrm{Tan}_{\mathcal{C}_0}$, i.e. $((*, \varepsilon_1), (*, \varepsilon_2), \dots)$ sequences of obj. of $\mathcal{C}$ w. $\varepsilon$'s

morphs: $\mathrm{SkTan}_{\mathcal{C}_0}(X, Y) := \mathrm{FreeMod}_{\mathbb{C}[a, a^{-1}]} \mathrm{Tan}_{\mathcal{C}}(X, Y) \Big/ \dots$

$$\bigcirc = 1$$

$$T \sqcup \bigcirc = -(a^2 + a^{-2})\, T$$



$a \in \mathbb{C}$ fixed

claim: $\mathrm{SkTan}_{\mathcal{C}_0}$ inherits the ribbon cat. str. from $\mathrm{Tan}_{\mathcal{C}_0}$.

Obviously, the ftor $\tilde{F} : \mathrm{Tan}_{\mathcal{C}_0} \longrightarrow \mathrm{SkTan}_{\mathcal{C}_0}$

restricted to $\tilde{F}(f) \in \mathrm{SkTan}_{\mathcal{C}_0}(\emptyset, \emptyset) \cong \mathbb{C}[a, a^{-1}]$

$f \in \mathrm{Hom}_{\mathrm{Tan}_{\mathcal{C}_0}}(\emptyset, \emptyset)$

is the Kauffman bracket.

$\boxed{?}$ modification for Jones polynomial: $c'_{U,V} = -a^{-3}\, c_{U,V}$

redefine the braiding (check axioms!) $c'^{-1}_{U,V} = -a^{3}\, c^{-1}_{U,V}$

probably has to alter more structure!