$\boxed{\text{def}}$ on V, there is a Frobenius alg. str. given by

$m: V\otimes V \longrightarrow V$ $\qquad 1\cdot 1 = 1$, $1\cdot x = x = x\cdot 1$, $x\cdot x = 0$

$\Delta: V \longrightarrow V\otimes V$ $\qquad \Delta(1) = 1\otimes x + x\otimes 1$, $\Delta(x) = x\otimes x$

notice that $|m| = -1 = |\Delta|$ (q-degrees).

$d^i: C^{i,*}(L) \to C^{i+1,*}(L)$

Then define $d^i(v) := \sum_{\substack{\alpha' \\ r_{\alpha'} = r_\alpha + 1}} d_{\alpha\alpha'}(v)\cdot(-1)^{\#\text{ of 1's to the left of } * \text{ in } \alpha \xrightarrow{\xi} \alpha'}$

for $v \in V_\alpha \subset C^{i,*}(L)$

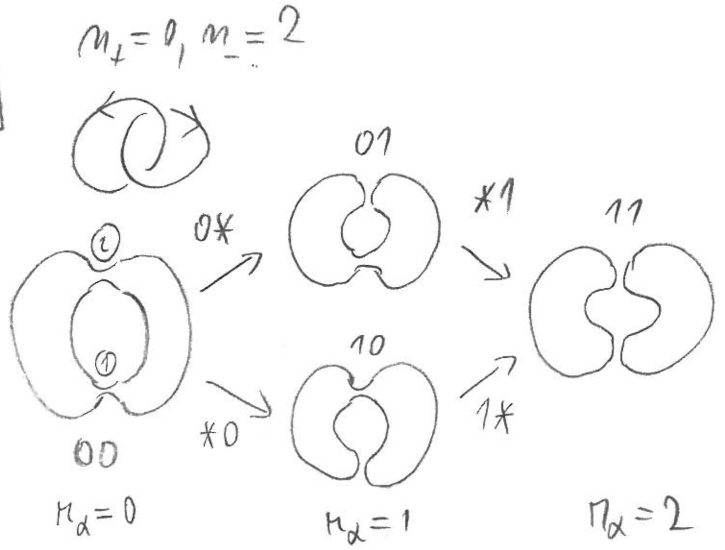
$$\alpha = (\dots \alpha_{p-1}, 0, \alpha_{p+1}, \dots)$$
$$\downarrow (\dots, \alpha_{p-1}, *, \alpha_{p+1}, \dots)$$
$$\alpha' = (\dots, \alpha'_{p-1}, 1, \alpha'_{p+1}, \dots)$$

$\alpha_m = \alpha'_m \Leftarrow m \neq p$

then $(-1)^{\#\dots} = (-1)^{|\{\alpha_m \mid \alpha_m = 1 \text{ and } m<p\}|}$

$= (-1)^{\alpha_1 + \dots + \alpha_{p-1}}$

$\boxed{\text{Ex.}}$ $n_+ = 0$, $n_- = 2$



$$(V\otimes V)[-4] \xrightarrow{m[1]} V[-3] \oplus V[-3] \xrightarrow{\Delta(1),\ -\Delta[1]} V\otimes V[-2]$$

$$0 \to C^{-2,*} \longrightarrow C^{-1,*} \longrightarrow C^{0,*} \longrightarrow 0$$

d^{-2}		d^{-1}	
$1\otimes 1 \longmapsto$	$1 \oplus 1$	$1 \oplus 0 \longmapsto$	$1\otimes x + x\otimes 1$
$1\otimes x \longmapsto$	$x \oplus x$	$x \oplus 0 \longmapsto$	$x\otimes x$
$x\otimes 1 \longmapsto$	$x \oplus x$	$0 \oplus 1 \longmapsto$	$-1\otimes x - x\otimes 1$
$x\otimes x \longmapsto$	$0 \oplus 0$	$0 \oplus x \longmapsto$	$-x\otimes x$

up to [?]

$H^{i,j}$

$i \backslash j$	-6	-5	-4	-3	-2	-1	0
0					$\Bbbk$		$\Bbbk$
-1							
-2	$\Bbbk$		$\Bbbk$				

$\chi_q(C^{**}(L)) = +(1+q^{-2}) + (q^{-4}+q^{-6})$ ok

$H^{-2,*}_{KH}(L) \cong \Bbbk\{1\otimes x - x\otimes 1, x\otimes x\}[-4]$

$H^{-1,*}_{KH}(L) \cong \dfrac{\Bbbk\{1\oplus 1, x\oplus x\}}{\Bbbk\{1\oplus 1, x\oplus x\}[-3]} \cong 0$

$H^{0,*}_{KH}(L) \cong \dfrac{\Bbbk\{1\otimes 1, 1\otimes x, x\otimes 1, x\otimes x\}[-2]}{\Bbbk\{x\otimes x, 1\otimes x + x\otimes 1\}[-2]} \cong \Bbbk\{1\otimes 1, 1\otimes x\}[-2]$

$\boxed{\text{Thm}}$ $d^{i+1} d^{i} = 0$

PF: $v \in V_\alpha \subset C^{i,*}(L)$

$$d^{i+1} d^{i}(v_\alpha) = d^{i+1} \sum_{\substack{\alpha' \\ \alpha \xrightarrow{\xi} \alpha'}} d_{\alpha\alpha'}(v)\,(-1)^{\xi} =$$

$$= \sum_{\alpha \xrightarrow[\xi]{} \alpha' \xrightarrow[\xi']{} \alpha''} d_{\alpha'\alpha''}\, d_{\alpha\alpha'}(v)\,(-1)^{\xi+\xi'} = \sum_{\substack{\alpha \to \alpha_1' \to \alpha'' \\ \alpha \to \alpha_2' \to \alpha''}} \left(d_{\alpha_1'\alpha''}\, d_{\alpha\alpha_1'}(v)(-1)^{k+(l+1)} + d_{\alpha_2'\alpha''}\, d_{\alpha_2'\alpha}(v)(-1)^{k+l} \right),$$

$$\alpha = (\underbrace{\underbrace{\ldots}_{k\ 1's} 0 \ldots}_{l\ 1's} 0 \ldots) \qquad \alpha' = \begin{cases} (\ldots 1 \ldots 0 \ldots) = \alpha_1' \\ \text{or} \\ (\ldots 0 \ldots 1 \ldots) = \alpha_2' \end{cases} \qquad \alpha'' = (\ldots 1 \ldots 1$$

$$\begin{tikzcd} & \alpha_1' \arrow[dr] & \\ \alpha \arrow[ur] \arrow[dr] & & \alpha'' \\ & \alpha_2' \arrow[ur] & \end{tikzcd}$$ face of the cube $\{0,1\}^n$

suffices: $\forall$ face, $d_{\alpha_1'\alpha''}\, d_{\alpha\alpha_1'}(v) = d_{\alpha_2'\alpha''}\, d_{\alpha\alpha_2'}(v)$.

This can be computed directly, but there is a more enlightening way:

Correspondence between cm Frobenius algebras and 2D TQFT's.

$\boxed{\text{def}}$ $Bord_2$ is the foll. symmetric monoidal cat:

objects = disjoint union of oriented circles

morph. = $A \to B$ is an $\boxed{\text{homeo}}$ class of oriented 2-manifold C

w. boundary $\partial C = A \sqcup \overline{B}$

← reversed or.

i.e. a bordism between A and B.

$A \xrightarrow{1_A} A$ is a tube; composition is gluing of bordisms

$\otimes$ = disjoint union, the symmetry is

$\emptyset$ = unit w.r.t. $\otimes$

$\boxed{\text{def.}}$ 2D TQFT is a sym. monoidal ftr $\mathcal{F}: Bord_2 \to Vec$

(Vect-$\otimes$, $\Bbbk$)

swap

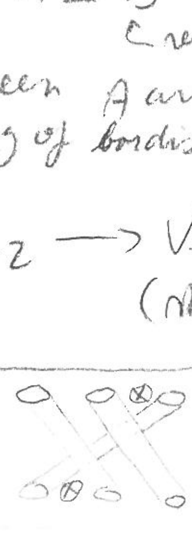
part of the data

i.e. $\mathcal{F}(A \sqcup B) \cong \mathcal{F}(A) \otimes \mathcal{F}(B)$ (nat. iso)

$\mathcal{F}(\emptyset) \cong \Bbbk$

$\mathcal{F}(\text{sym.}) = \text{sym.}$

symmetry (braiding) e.g.



Thm Every bordism in $Bord_2$ can be composed of "elementary pieces":

Consequently, 2D TQFT $\mathcal{F}$ is determined by

(1) $\mathcal{F}(\text{circle}) =: A$ (exerc.: what about the orient.?)

(2) $\cdot\ \mathcal{F}($  $)$: $\mathcal{F}(\circ\ \circ) \longrightarrow \mathcal{F}(\circ)$

$$\mathcal{F}(\circ\ \circ) = \mathcal{F}(\circ \sqcup \circ) \cong \mathcal{F}(\circ)\otimes\mathcal{F}(\circ) = A\otimes A \xrightarrow[m]{} A \cong \mathcal{F}(\circ)$$

mult.

$\cdot\ \mathcal{F}($  $)$: $A \xrightarrow{\Delta} A\otimes A$ comult

$\cdot\ \mathcal{F}($  $)$: $\Bbbk \xrightarrow{e} A$ unit

$\cdot\ \mathcal{F}($  $)$: $A \xrightarrow{\varepsilon} \Bbbk$ counit

($\cdot\ \mathcal{F}($  $)$: $A \xrightarrow{1_A} A$)

($\cdot\ \mathcal{F}($  $)$: $A\otimes A \longrightarrow A\otimes A$ the symmetry)

(PF: easy if you know 2D cpt orientable manifolds w. boundary being disj. union of S^1's "easy")

def. Com. Frobenius algebra is a f.d. vect. space A with

$m: A\otimes A \to A \qquad e: \Bbbk \to A$

$\Delta: A \to A\otimes A \qquad \varepsilon: A \to \Bbbk$

s.t. $\cdot\ (A, m, e)$ is unital comm. ass. alg.

$\cdot\ (A, \Delta, \varepsilon)$ counital cocomm. coass. alg.

$\cdot$
$$\begin{array}{ccc} A\otimes A & \xrightarrow{\Delta\otimes 1} & A\otimes A\otimes A \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle 1\otimes m} \\ A & \xrightarrow[\Delta]{} & A\otimes A \end{array}$$

$\cdot$
$$\begin{array}{ccc} A\otimes A & \xrightarrow{1\otimes\Delta} & A\otimes A\otimes A \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m\otimes 1} \\ A & \xrightarrow[\Delta]{} & A\otimes A \end{array}$$

$\cdot\ \varepsilon m: A\otimes A \to \Bbbk$ is nondeg.

Ex. $V = \Bbbk\{1, x\}$, $\varepsilon(1) = 0$, $\varepsilon(x) = 1$

$\boxed{\text{Thm}}$ There is a bijection between (iso classes) comm. Frob algs and (iso cl. of) 2D TQFTs.

$$A, m, \Delta, e, \varepsilon \quad \longleftrightarrow \quad \mathcal{F}(\circ) = A$$

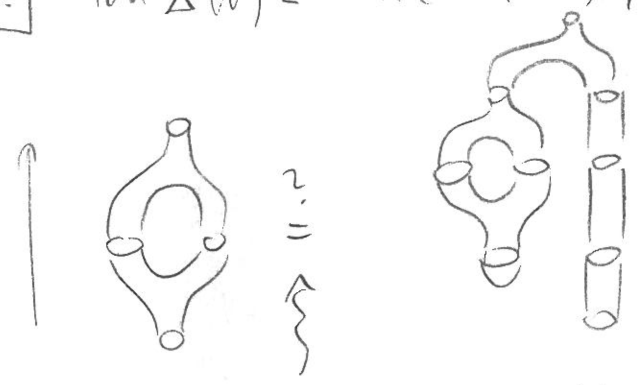
$\mathcal{F}($  $) = \Delta$ etc.

$\vdots$

[PF: $\leftarrow$ easy (nice exerc.), $\rightarrow$ difficult]

Consequence: algebraic identities in com. Fr. algs can be proved topologically:

$\boxed{\text{Ex.}}$ $m\Delta(v) = m(m(\Delta(1)), v)$



homeo surfaces!

(SHOW IN DETAILS!)

rem. analogous to the relation between tangles and alg. identities in a ribbon category.

back to proving $d^{i+1} d^i = 0$:

V is com. Frob. alg., hence defines a 2D TQFT: $\mathrm{Bord}_2 \to \mathrm{Vec}$

$$\begin{tikzcd} & \alpha_1' \arrow[dr] & \\ \alpha \arrow[ur] \arrow[dr] & & \alpha \\ & \alpha_2' \arrow[ur] & \end{tikzcd}$$

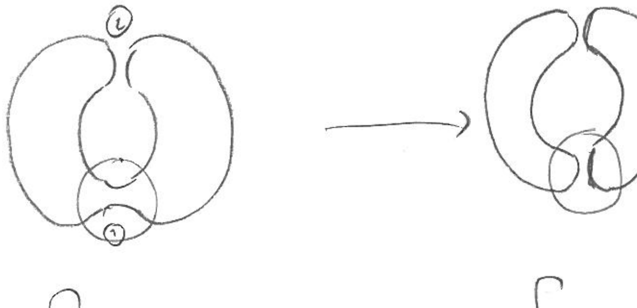
- Γ_α's are disjoint union of circles
- we want to find bordisms in place of dotted arrows of the form  or 

$$\begin{tikzcd} & \Gamma_{\alpha_1'} \arrow[dr, dotted] & \\ \Gamma_\alpha \arrow[ur, dotted] \arrow[dr, dotted] & & \Gamma_\alpha \\ & \Gamma_{\alpha_2'} \arrow[ur, dotted] & \end{tikzcd}$$

- Then $\mathcal{F}$ evaluated at these bordisms would be $\Delta \otimes 1 \otimes \cdots \otimes 1$ or $m \otimes 1 \otimes \cdots \otimes 1$ and $d_{\alpha\alpha'} = \mathcal{F}(\Gamma_\alpha \to \Gamma_{\alpha'})$

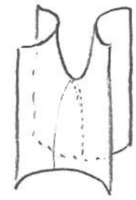
To prove $d_{\alpha_1'\alpha''}\, d_{\alpha\alpha_1'} = d_{\alpha_2'\alpha''}\, d_{\alpha\alpha_2'}$, it is then enough to verify $\Gamma_\alpha \longrightarrow \Gamma_{\alpha_1'} \longrightarrow \Gamma_{\alpha''} = \Gamma_\alpha \longrightarrow \Gamma_{\alpha_2'} \longrightarrow \Gamma_{\alpha''}$

definition of the bordism by example:



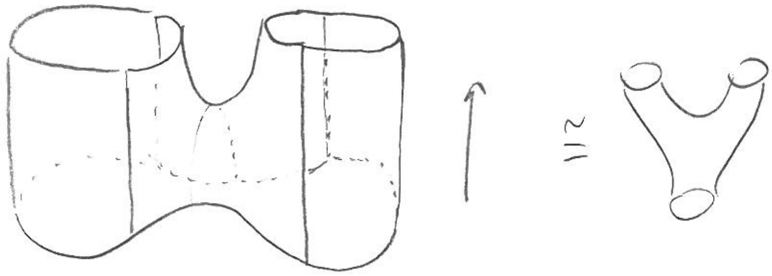
$\Gamma_{(01)}$ $\qquad\qquad$ $\Gamma_{(11)}$

- outside the changing disc, take $\Gamma_\alpha \times [0;1]$
- the missing tube is filled by the saddle



$\Gamma_{(11)}$

thus



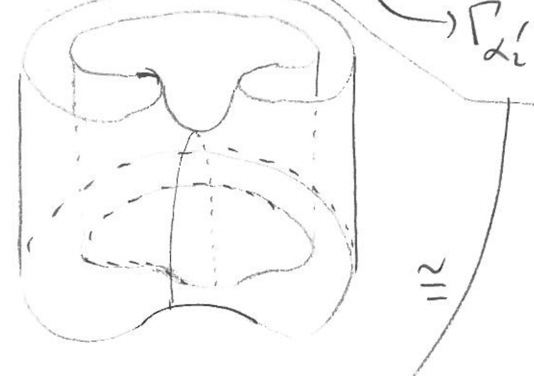
as desired!

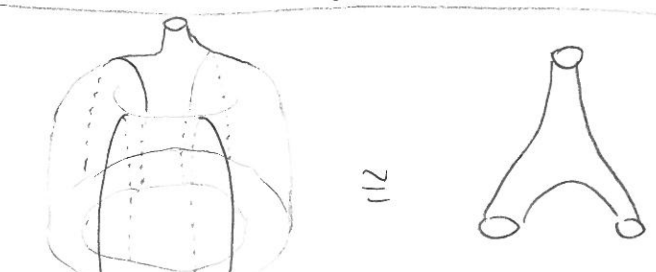
$\Gamma_{(01)}$

to check
$$\begin{tikzcd} & \Gamma_{\alpha_1'} \arrow[dr, bend left] \arrow[d, phantom, "\parallel"] & \\ \Gamma_\alpha \arrow[ur, bend left] \arrow[r, phantom, ""] & \, & \Gamma_{\alpha''} \\ & \Gamma_{\alpha_2'} \arrow[ur, bend right] & \end{tikzcd}$$

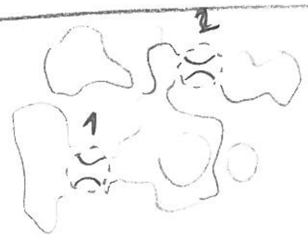
one has to distinguish several cases, but it's just playing w. pictures

rem





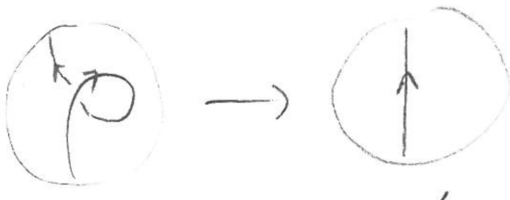
upper arrows - glue a saddle first at 1, then at 2:
lower - 2 1.



Isotopy invariance of $KH^{*,*}(L)$

Thm If L and L' are isotopic links, then
$KH^{*,*}(L) \cong KH^{*,*}(L')$.

We verify invariance under RM1 positive only:



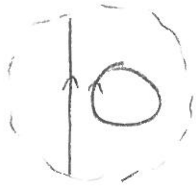
L $\qquad$ L'

For a smoothing α of L, let the displayed crossing be the first one from the left in α

We construct a homotopy equivalence of cplxes:

$$C^{*,*}(L) \underset{G}{\overset{F}{\leftrightarrows}} C^{*,*}(L') \quad \text{s.t.} \quad \begin{array}{l} GF = 1 \\ FG - 1 = dH + Hd \\ \text{for some } H^i : C^{i,*}(L) \to C^{i-1,*}(L) \end{array}$$

Observation $\quad C^{i,*}(L) \cong C^{i,*-1}(L_0) \oplus C^{i-1,*-2}(L_1)$
(as gr. vec. spaces), where

$L_0 =$  $\qquad L_1 =$ 

(0-sm.) $\qquad$ (1-sm.)

$$C^{i,*}(L) = \bigoplus_{\substack{\alpha \in \{0,1\}^n \\ r_\alpha = i + n_-}} \underbrace{V^{\otimes k_\alpha}[r_\alpha + n_+ - 2n_-]}_{=: V_\alpha}$$

$\alpha = (0,\dots)$ induces a smoothing α^0 of L_0 by omitting the first slot (0).
$\alpha = (1,\dots)$ $\quad$ — " — $\quad$ α^1 $\quad$ — " — $\quad$ (1)

$$C^{i,*}(L_0) = \bigoplus_{\substack{\alpha = (0,\dots) \in \{0,1\}^n \iff \alpha^0 \in \{0,1\}^{n-1} \\ r_{\alpha^0} = i + n^0_-}} V^{\otimes k_{\alpha^0}}[r_{\alpha^0} + n^0_+ - 2n^0_-]$$

$r_{\alpha^0} = r_\alpha \quad n^0_- = n_- \quad k_{\alpha^0} = k_\alpha \quad n^0_+ = n_+ - 1 \Rightarrow$ the deg. shift $* - 1$

analogously: $n_{\alpha^1} = n_\alpha - 1 \Longrightarrow$ the deg shift $i-1$

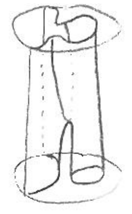
$n_-^1 = n_-$

$n_+^1 = n_+ - 1$ $\Bigg\} \Longrightarrow$ the deg shift $*-2$

$k_{\alpha^1} = k_\alpha$

The Khovanov differential $d^i : C^{i,*}(L) \longrightarrow C^{i+1,*}(L)$ becomes

$$\begin{array}{ccc} C^{i,*-1}(L_0) & \oplus & C^{i-1,*-2}(L_1) \\ d_{00} := d^i(L_0)[-1] \big\downarrow & \searrow^{d_{10}} & \big\downarrow d^{i-1}(L_1)[-2] =: d_{11} \\ C^{i+1,*-1}(L_0) & \oplus & C^{i,*-2}(L_1) \end{array}$$



where d_{10} corresponds to the bordism $\Gamma_{(0,\dots)} \to \Gamma_{(1,\dots)}$

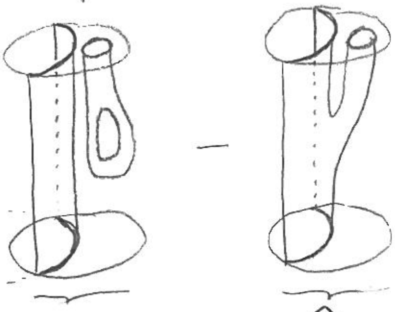
(ad hoc) construction of F, G, H:

1) $F : C^{*,*}(L') \xrightarrow{(F_0, F_1)} C^{*,*-1}(L_0) \oplus C^{\hat{*},*-2}(L_1)$

- $F_1 = 0$
- $F_0 : C^{*,*}(L') \longrightarrow C^{*,*-1}(L_0)$

is dfd on each $V_{\alpha'}$ by the bordism

$\alpha^0 = (0, \alpha')$ Γ_{α^0}



α' $\Gamma_{\alpha'}$

$\underbrace{\qquad}_{\xi_{\alpha'}}$ $-$ $\underbrace{\qquad}_{\hat{\xi}_{\alpha'}}$

i.e. ξ is $\Gamma_{\alpha'} \times I$ except for the changing disc $\times I$, where it is displaced on the right

$\hat{\xi}$ sim.

the TQFT corr. to V

in detail: $F_0\big|_{V_{\alpha'}} := \mathcal{F}(\xi_{\alpha'}) - \mathcal{F}(\hat{\xi}_{\alpha'})$

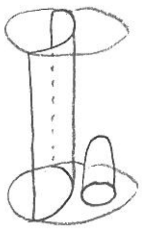
- $Fd = dF$ because d doesn't affect the changing circle

INSERT OBS. 1)

2) $G : C^{*,*-1}(L_0) \oplus C^{\hat{*},*-2}(L_1) \xrightarrow{G_0 + G_1} C^{*,*}(L')$

- $G_1 = 0$

• $G_0 : C^{*,*-1}(L_0) \longrightarrow C^{*,*}(L')$ is dfd by the bordism

α'  on each V_{α^0}, i.e. $G_0|_{V_{\alpha^0}} := \mathcal{F}(\xi_{\alpha^0})$

$\alpha^0 = (0,\alpha')$, ξ_{α^0}

• $Gd = dG$ as before

observation: 1) $C^{*,*}(L') \xrightarrow{F_0} C^{*,*-1}(L_0)$

$$\begin{array}{ccc} \cup & & \cup \\ V_{\alpha'} = V' \otimes V & & V_{\alpha^0 = (0,\alpha')} = V_{\alpha'} \otimes V = V' \otimes V \otimes V \end{array}$$

↑ corr. to the circle in the changing disc

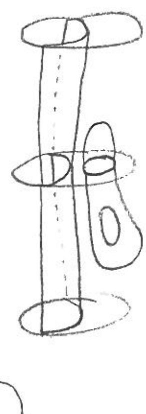
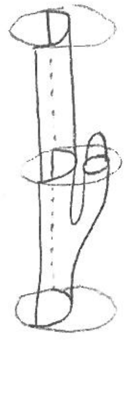
$$v' \otimes v \longmapsto v' \otimes v \otimes 2x - v' \otimes \Delta(v)$$

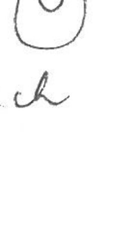
2) $C^{*,*-1}(L_0) \longrightarrow C^{*,*}(L')$

$$\begin{array}{ccc} \cup & & \cup \\ \underset{\substack{\| \\ (0,\alpha')}}{V_{\alpha^0}} = V' \otimes V \otimes V & & V_{\alpha'} = V' \otimes V \end{array}$$

$$v' \otimes v \otimes 1 \longmapsto 0$$
$$v' \otimes v \otimes x \longmapsto v' \otimes v$$

observation $GF = 1$

sufficient: $G_0 F_0 \overset{?}{=} 1$ $\qquad G_0 F_0 =$  $-$  $=$

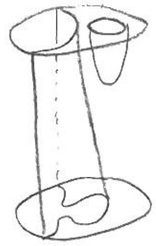
$= 2$  $-$  $=$  (because  $= 2 \iff \varepsilon\, m\, \Delta\, e \overset{\text{easy}}{=} 2$)

[formally - write F in front of each bordism]

3) the homotopy $H: C^{*,*-1}(L_0)\oplus C^{*-1,*-2}(L_1) \longrightarrow C^{*-1,*-1}(L_0)\oplus C^{*-2,*-2}(L_1)$ (13)

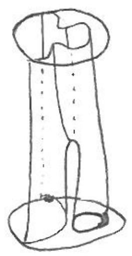
is given by the matrix $\begin{pmatrix}0 & h\\ 0 & 0\end{pmatrix}$, $h: C^{\cdot\cdot}(L_1)\longrightarrow C^{\cdot\cdot}(L_0)$

$h = -F$



recall: d has matrix $\begin{pmatrix}d_{00} & 0\\ d_{10} & d_{11}\end{pmatrix}$

$d_{10} = F$

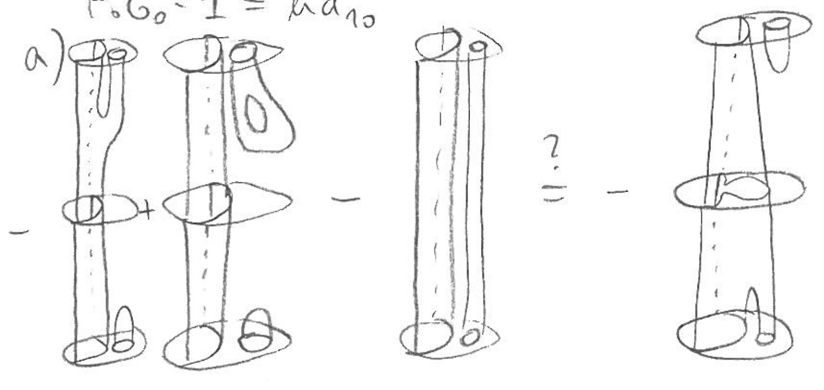


$$FG - 1 \stackrel{?}{=} dH + Hd$$

$$LHS = \begin{pmatrix}F_0G_0 - 1 & 0\\ 0 & -1\end{pmatrix} \qquad RHS = \begin{pmatrix}hd_{10} & d_{00}h + hd_{11}\\ 0 & d_{10}h\end{pmatrix}$$

$F_0G_0 - 1 \stackrel{?}{=} hd_{10}$

a)



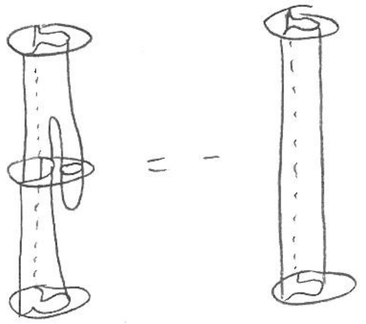
algebraically: LHS:

$$\begin{array}{lllll}
1\otimes 1 \longmapsto & 0 & +\ 0 & -\ 11 & = -11\\
1x & -1x - x1 & +\ 1\otimes 2x & -\ 1x & = -x1\\
x1 & 0 & +\ 0 & -\ x1 & = -x1\\
xx & -xx & +\ x\otimes 2x & -\ xx & = 0
\end{array}$$

RHS:

$$\begin{array}{ll}
11 \longmapsto & -11\\
1x & -x1\\
x1 & -x1\\
xx & 0
\end{array}$$

b) $-1 \stackrel{?}{=} d_{10} h$

RHS = $-$  = LHS

c) exercise to make d_{00} and d_{11} explicit (signs!)

<u>remarks</u> - There are links L, L' s.t. $J(L) = J(L')$
but $KH^{*,*}(L) \not\cong KH^{*,*}(L')$

- the TQFT $\mathcal{F}$ used can be changed, but we have to require $\dim V = 2$
$\varepsilon(1) = 0,\ \varepsilon(x) = 1$

to make the construction work

There are only 2 such nonisomorphic TQFT's $\Rightarrow$ "Lee theory"

- <u>references</u>