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$\gamma(t) = \left(\frac{1}{2}(3 - e^{2\pi i t}), \frac{1}{2}(3 + e^{2\pi i t})\right) \qquad t \in [0;1]$

$\gamma(0) = (\underline{1}, \underline{2}) = \gamma(1)$. $\qquad \gamma(\frac{1}{2}) = (\underline{2}, \underline{1})$

a gen. of $\pi_1(\; _2, x_0)$

$f(\gamma(t)) = e^{2\pi i t \frac{\Omega_{12}}{K}} v \xrightarrow{t \to 1} e^{2\pi i \frac{\Omega_{12}}{K}} v$

$\xrightarrow{t \to 0} v$

$\Rightarrow \rho^{KZ_2}(\underset{PB_2}{(\gamma)}) = e^{2\pi i \frac{\Omega_{12}}{K}} \in GL(V^{\otimes 2})$

in the monodr. representation

$\gamma(t) = \left(\frac{1}{2}(3 - e^{\pi i t}), \frac{1}{2}(3 + e^{\pi i t})\right) \qquad \gamma(0) = (\underline{1}, \underline{2}) \qquad \gamma(1) = (\underline{2}, \underline{1})$

a gen. of $\pi_1(Y_2, x_0)$

$f(\gamma(t)) = e^{\pi i t \frac{\Omega_{12}}{K}} v \xrightarrow{t \to 0} v$

$\xrightarrow{t \to 1} e^{\pi i \frac{\Omega_{12}}{K}} v$

$((\underline{1}, \underline{2}), v_1 \otimes v_2) = ((\underline{2}, \underline{1}), v_2 \otimes v_1)$

$\cap$

$Y_2 \times_{S_2} V^{\otimes 2}$

$P: V^{\otimes 2} \to V^{\otimes 2}$

$v_1 \otimes v_2 \mapsto v_2 \otimes v_1$

$\rho^{KZ_2}([\gamma]) = P(e^{\pi i \frac{\Omega_{12}}{K}})$

$\cap$

B_2

AFFINE KAC-MOODY ALGEBRAS AND CONFORMAL BLOCKS

$\mathfrak{g}$ a cpx simple Lie alg., e.g. $\mathfrak{g} = sl(2, \mathbb{C})$

$\mathbb{C}((t))$ ... the quot. field of formal power series $\mathbb{C}[[t]]$

exp. finite in negative powers, infinite in positive numbers

$L\mathfrak{g} \equiv \mathfrak{g}((t)) := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((t))$ ... the loop alg. of $\mathfrak{g}$

$[X \otimes f(t), Y \otimes g(t)] := [X, Y] \otimes f(t) g(t)$

central extension of a Lie alg. $\mathfrak{g}$ by a comm. Lie alg. $\mathfrak{a}$ is a SES

$0 \to \mathfrak{a} \xrightarrow{i} \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$ of Lie algs s.t. $i(\mathfrak{a}) \subset Z(\hat{\mathfrak{g}})$

the center of $\hat{\mathfrak{g}}$

central extensions are classified by $H^2(\mathfrak{g}, \mathfrak{a})$ ... Lie alg. cohom.

Lemma For a simple Lie alg $\mathfrak{g}$,
$H^2(\mathfrak{g}((t)), \mathbb{C})$ is 1-dimensional and is identified with the space of invariant bilin. forms on $\mathfrak{g}$.

K .. an inv. bilin. form
the corresponding cocycle in H^2 is
$$c(X\otimes f(t), Y\otimes g(t)) = -K(X,Y)\cdot \operatorname{Res}_{t=0}(f(t)\,g'(t))$$
the coeff. at t^{-1}

def The central extension $0 \to \mathbb{C}c \to \hat{\mathfrak{g}}_K \to \mathfrak{g}((t)) \to 0$ corresponding to a nonzero inv. bilin. form K is called <u>aff. Kac-Moody algebra</u>, K is called <u>level</u>

$\hat{\mathfrak{g}}_K \cong \mathbb{C}c \oplus \mathfrak{g}((t))$ as vector spaces

derivative

$$[X\otimes f(t), Y\otimes g(t)] = [X,Y]\otimes f(t)g(t) - K(X,Y)\cdot \operatorname{Res}_{t=0}(f(t)g'(t))\cdot c$$
$c \in Z(\hat{\mathfrak{g}}_K)$

e.g.
$$[X\otimes t^{n}, Y\otimes t^{m}] = [X,Y]\otimes t^{n+m} - K(X,Y)\cdot \operatorname{Res}_{t=0}(t^{n} m t^{m-1})\cdot c$$
$$= [X,Y]\otimes t^{n+m} - m\,K(X,Y)\,\delta_{n,-m}\cdot c$$

K_{Kill} .. the Cartan-Killing form
$K_c = -\frac{1}{2}K_{Kill}$ .. the "critical" form (level)
K_0 ... normalized (-Killing) $\quad K_0(\Theta,\Theta)=2$ for Θ the highest (longest) root.
the sum of simple roots

$$\begin{cases} K_0 : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C} \\ \Theta \in \mathfrak{h}^* \subset \mathfrak{g}^* \end{cases} \qquad K_0^*: \mathfrak{g}^*\otimes\mathfrak{g}^* \to \mathbb{C}$$

ex. $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ then $K_0(a,b) = \operatorname{Tr}(ab)$
$K_{Kill}(a,b) = 2n\operatorname{Tr}(ab)$

$K, K' \neq 0 \overset{\text{easy}}{\Longrightarrow} \hat{\mathfrak{g}}_K \cong \hat{\mathfrak{g}}_{K'}$

$\hat{\mathfrak{g}} := \hat{\mathfrak{g}}_{K_0}$

$K = k\cdot K_0$ any inv. bilin. form
$\uparrow$ "level"

$K_{kill} = 2h^\vee K_0$, $h^\vee$ .. "dual Coxeter number"
(ex. $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ ... $h^\vee = n$)

$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_- \oplus \mathfrak{g} \oplus \mathbb{C}c \oplus \hat{\mathfrak{g}}_+$

where $\hat{\mathfrak{g}}_- = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$
$\hat{\mathfrak{g}}_+ = \mathfrak{g} \otimes t\,\mathbb{C}[[t]]$

$\hat{\mathfrak{p}}_+ = \mathfrak{g} \oplus \mathbb{C}c \oplus \hat{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbb{C}[[t]] \oplus \mathbb{C}c$ ... the standard parabolic subalgebra of

$P_+ = \{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha^\vee) \in \mathbb{N}_0 \ \forall \alpha \in \Delta_+\}$ .. integral dominant weights

where $\mathfrak{h}$ is Cartan subalg. of $\mathfrak{g}$ (ex. $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{h} = \{\mathrm{diag}(a_1 \ldots a_n) \mid \sum a_i = 0\}$

Δ_+ positive roots of $\mathfrak{g}$

$\check{\alpha}$ coroot

$$P_+ = \bigoplus_{i=1}^{\mathrm{rank}\,\mathfrak{g}} \mathbb{N}_0 \omega_i$$

For $\lambda \in P_+$, $\exists V_\lambda$ irred. finite dim. representation of $\mathfrak{g}$ w. highest weight λ

<u>Verma modules</u>

$\lambda \in P_+$, $k \in \mathbb{C}$

$M_{k,\lambda} := \mathrm{Ind}_{\hat{\mathfrak{p}}_+}^{\hat{\mathfrak{g}}} V_{k,\lambda}$ .. the induced representation of $\hat{\mathfrak{g}}$ from $\hat{\mathfrak{p}}_+$ on V_λ
(recall $\hat{\mathfrak{p}}_+ = \mathfrak{g}[[t]] \oplus \mathbb{C}c$)

$V_{k,\lambda}$ ... a $\mathfrak{g}$-mod ~~[illegible]~~

$\begin{cases} c \cdot v := kv & \forall v \in V_{k,\lambda} \\ a \cdot v := 0 & \forall a \in \mathfrak{g}[[t]] \end{cases}$

gives V_λ a $(\mathbb{C}[[t]] \oplus \mathbb{C}c)$-mod

$= \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathbb{C}[[t]] \oplus \mathbb{C}c)} V_\lambda \overset{\text{as v.s.}}{\cong} \mathcal{U}(\hat{\mathfrak{g}}_-)$

In general $M_{k,\lambda}$ is <u>not</u> irreducible and contains the maximum proper submodule $J_{k,\lambda}$.

$H_{k,\lambda} := M_{k,\lambda} / J_{k,\lambda}$

$\boxed{\text{Thm}}$ (Kac) Let $k \in \mathbb{N}$ and $\lambda \in P_k = \{\lambda \in P_+ \mid 0 \le (\theta, \lambda) \le k\}$
Then the left $\hat{\mathfrak{g}}$-mod $H_{k,\lambda}$ is the unique left $\hat{\mathfrak{g}}$-mod satisfying
(1) $V_{k,\lambda} \cong \{v \in H_{k,\lambda} \mid a \cdot v = 0 \ \forall a \in \hat{\mathfrak{g}}_+\}$ as $\mathfrak{g}$-mod
is the irred. highest weight λ $\mathfrak{g}$-module
(2) c acts on $H_{k,\lambda}$ as $k \cdot \mathrm{id}$

(3) $H_{k,\lambda}$ is generated by $V_{k,\lambda}$ over $\mathfrak{g}_-$ with only 1 relation

$$(X_\Theta \otimes t^{-1})^{k-(\Theta,\lambda)+1} v_\lambda = 0$$

where $X_\Theta \in \mathfrak{g}_\Theta$

$v_\lambda \in V_\lambda$ is the highest weight vector

Such a $H_{k,\lambda}$ is called integrable highest weight $\hat{\mathfrak{g}}$-mod of level k.

CONFORMAL BLOCKS

$\mathbb{C}P^1$ Riem. surface of genus 0 (can be generalized)

$x_1, \dots, x_n \in \mathbb{C}P^1$ distinct

$t_1, \dots, t_n$ local coord. around $x_1, \dots, x_n$

$t_i : \underset{\underset{\mathbb{C}P^1}{\cap \text{ open}}}{U_i^{\ni x_i}} \longrightarrow \mathbb{C}$ holom. s.t. $t_i(x_i) = 0$

$x = \infty = (1:0) \in \mathbb{C}P^1$ $\quad t_x : (z_0, z_1) \mapsto \frac{z_1}{z_0}$

$x \neq \infty$ $\quad t_x : (z_0, z_1) \mapsto \frac{z_0}{z_1} - \frac{z_0(x)}{z_1(x)}$

" $(z_0(x) : z_1(x))$

$\mathfrak{g}(x_1, \dots, x_n) :=$ the Lie alg. of $\mathfrak{g}$-valued meromorphic functions on $\mathbb{C}P^1$ with poles in $x_1, \dots, x_n$ at most.

$[x \otimes f, y \otimes g] := [x, y] \otimes fg$

$\tau_j : \mathfrak{g}(x_1, \dots, x_n) \hookrightarrow \mathfrak{g}((t_j))$

$f \longmapsto$ Laurent expansion of f at x_j.

$\Rightarrow \mathfrak{g}(x_1, \dots, x_n) \hookrightarrow \bigoplus_{i=1}^{n} \mathfrak{g}((t_i))$ and we take diagonal central ext.:

$$0 \to \mathbb{C}c \to \bigoplus_{i=1}^{n} \mathfrak{g}((t_i)) \oplus \mathbb{C}c \to \bigoplus_{i=1}^{n} \mathfrak{g}((t_i)) \to 0$$

$$[((X_1 \otimes f_1), \dots, (X_n \otimes f_n)), ((Y_1 \otimes g_1), \dots, (Y_n \otimes g_n))] := ([X_1, Y_1] \otimes f_1 g_1, \dots, [X_n, Y_n] \otimes f_n g_n)$$
$$- \sum_{i=1}^{n} K_0(X_i, Y_i) \operatorname{Res}_{t_i=0}(f_i(t_i) g_i'(t_i)) \cdot c$$

The embedding $\mathfrak{g}(x_1, \dots, x_n) \hookrightarrow \bigoplus_{i=1}^{n} \mathfrak{g}((t_i))$ can be lifted to $\bigoplus_{i=1}^{n} \mathfrak{g}((t_i)) \oplus \mathbb{C}c$

Pf: the map is obv., to prove it's mor. of Lie algs., use Residue theorem:

$$\varphi([X\otimes f, Y\otimes g]) = \varphi([X,Y]\otimes fg) = ([X,Y]\otimes \tau_1(fg), \dots, [X,Y]\otimes \tau_n(fg))$$ 11

$$\overset{?}{\|}$$

$$[\varphi(X\otimes f), \varphi(Y\otimes g)] = [(X\otimes\tau_1 f, \dots, X\otimes \tau_n f), (Y\otimes\tau_1 g, \dots, Y\otimes \tau_n g)] =$$

$$= ([X,Y]\otimes \tau_1 f\tau_1 g, \dots, [X,Y]\otimes \tau_n f \tau_n g) - \sum_{i=1}^{n} \kappa_0(X,Y)\, \mathrm{Res}_{t_i=0}(\tau_i f\,(\tau_i g)')$$

BUT $\sum_{i=1}^{n} \mathrm{Res}_{t_i=0}(\tau_i f \cdot (\tau_i g)') = 0$ on $\mathbb{CP}^1$!
by the Residue Thm.

$k\in\mathbb{N}$, $\lambda_1,\dots,\lambda_n \in P_k$ $\Rightarrow$ H_{k,λ_i} $\hat{\mathfrak{g}}$-mod

$\mathfrak{g}(x_1,\dots,x_n)$ acts on $\bigotimes_{i=1}^{n} H_{k,\lambda_i}$ by the embedding $\mathfrak{g}(x_1\dots x_n) \hookrightarrow \bigoplus_{i=1}^{n}\mathfrak{g}((t_i)) \oplus \mathbb{C}$

$\boxed{\text{def}}$ The space of conformal blocks $\mathcal{H}(x_1,\dots,x_n,\lambda_1,\dots,\lambda_n)$ is the space of linear forms

$$\varphi : \bigotimes_{i=1}^{n} H_{k,\lambda_i} \longrightarrow \mathbb{C}$$

invariant w. respect to $\mathfrak{g}(x_1,\dots,x_n)$: $\varphi(\gamma\cdot v) = 0$ $\forall \gamma \in \mathfrak{g}(x_1,\dots,x_n)$
$\forall v \in \bigotimes_{i=1}^{n} H_{k,\lambda_i}$

next time: • $\mathcal{H}(\dots)$ is finite dimensional

• Verlinde formula $\Rightarrow$ dim of $\mathcal{H}(\dots)$