2 q-Deforming the Hopf Algebra $U(\mathfrak{sl}_2)$

In this lecture we extend our discussion of Hopf algebras to include our first q-deformed example. As is well known, deformations of a Lie algebra $\mathfrak{g}$, in the category of Lie algebras, can exist only if the second cohomology group $H^2(\mathfrak{g},\mathfrak{g})$ is non-zero. As a direct consequence of this fact, all semi-simple Lie algebras must be rigid.

This caused some to guess that there could exist no non-trivial deformations of $U(\mathfrak{sl}_2)$ in the category of Hopf algebras. Thus, when $U_q(\mathfrak{sl}_2)$ first appeared in the early 1980's, it came as quite a surprise.

We will adopt the same conventions here as outlined at the beginning of the first lecture, with the additional assumption that q denotes a fixed complex number such that, unless otherwise stated, $q \neq -1, 0, 1$.

2.1 The Hopf Algebra $U_q(\mathfrak{sl}_2)$

Definition 2.1. We define $U_q(\mathfrak{sl}_2)$ to be $\mathbf{C}\left\langle E, F, K, K^{-1}\right\rangle / I_{U_q(\mathfrak{sl}_2)}$, where $I_{U_q(\mathfrak{sl}_2)}$ is the two-sided ideal generated by the elements

$$KK^{-1} - 1, \quad K^{-1}K - 1, \qquad KEK^{-1} - q^2E, \quad KFK^{-1} - q^{-2}F, \tag{1}$$

$$[E,F] - \frac{K - K^{-1}}{q - q^{-1}}.$$

While it is not immediately clear how, or even if, this is a deformation of $U(\mathfrak{sl}_2)$, we do have the following two familiar looking results. (The proof of each result is a simple exercise in linear algebra and can be found in [1].)

Lemma 2.2 *The following two sets are vector space bases for $U_q(\mathfrak{sl}_2)$:*

$$\{F^lK^mE^n \,|\, m \in \mathbf{Z};\, l,n \in \mathbf{N}_0\}, \qquad \{E^lK^mF^n \,|\, m \in \mathbf{Z}; l,n \in \mathbf{N}_0\},$$

which we call the PBW *bases.*

Lemma 2.3 *The* quantum Casimir element

$$C_q := EF + \frac{Kq^{-1} + K^{-1}q}{(q-q^{-1})^2} = FE + \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2}$$

lies in the centre of $U_q(\mathfrak{sl}_2)$*. If* q *is not a root of unity, then the centre of* $U_q(\mathfrak{sl}_2)$ *is generated by* C_q*.*

We should note that the requirement on q not to be a root of unity is a common feature in many results about quantised enveloping algebras. In many ways, the root of unity case and the non-root of unity cases can be quite distinct.

Now in addition to the PBW bases, and the quantum Caisimir, $U(\mathfrak{sl}_2)$ has the following all-important additional structure:

Lemma 2.4 *There exists a Hopf algebra structure on* $U_q(\mathfrak{sl}_2)$ *with comultiplication* Δ*, counit* ε*, and antipode* S*, uniquely determined by*

$$\begin{aligned} \Delta(E) &= 1 \otimes E + E \otimes K, & \Delta(F) &= K^{-1} \otimes F + F \otimes 1, \\ \Delta(K) &= K \otimes K, & \Delta(K^{-1}) &= K^{-1} \otimes K^{-1}, \\ \varepsilon(E) &= \varepsilon(F) = 0, & \varepsilon(K) &= \varepsilon(K^{-1}) = 1. \end{aligned}$$

$$S(E) = -EK^{-1}, \qquad S(F) = -KF, \qquad S(K) = K^{-1}, \qquad S(K^{-1}) = K.$$

Proof. Just as for the classical example of $U(\mathfrak{sl}_2)$, the proof amounts to showing that the maps Δ, ε, and S, vanish on the generators of the ideal, and that they satisfy the axioms of a Hopf algebra on the generators of the algebra. For example, we have

$$\begin{aligned} (\Delta \otimes \mathrm{id}) \circ \Delta(E) &= (\Delta \otimes \mathrm{id})(1 \otimes E + E \otimes K) \\ &= 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K, \end{aligned}$$

and

$$\begin{aligned} (\mathrm{id} \otimes \Delta) \circ \Delta(E) &= (\mathrm{id} \otimes \Delta)(1 \otimes E + E \otimes K) \\ &= 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K. \end{aligned}$$

Hence, coassociativity holds for the generator E. The other calculations are left as an excercise. □

2.2 The Classical $(q = 1)$-Limit of $U_q(\mathfrak{sl}_2)$

It is now time to address the question of how $U_q(\mathfrak{sl}_2)$ q-deforms the classical Hopf algebra $U(\mathfrak{sl}_2)$. The obvious problem with setting $q = 1$ is that $U_q(\mathfrak{sl}_2)$ is no longer well-defined. To get around this problem we will need to consider the following reformulation of $U_q(\mathfrak{sl}_2)$: Define $\widetilde{U}_q(\mathfrak{sl}_2)$ to be the algebra $\mathbf{C}\langle E, F, K, K^{-1}, G\rangle / \widetilde{I}_{U_q(\mathfrak{sl}_2)}$, where $\widetilde{I}_{U_q(\mathfrak{sl}_2)}$ is the ideal generated by the elements (1), and the additional generators

$$[G, E] = E(qK + q^{-1}K^{-1}), \qquad [G, F] = -(qK + q^{-1}K^{-1})F,$$
$$[E, F] = G, \qquad (q - q^{-1})G = K - K^{-1}.$$

Lemma 2.5 *We have an algebra isomorphism* $\alpha : \widetilde{U}_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$*, uniquely determined by*

$$\alpha(E) = E, \qquad \alpha(F) = F, \qquad \alpha(K) = K, \qquad \alpha(G) = \frac{(K - K^{-1})}{q - q^{-1}}.$$

With respect to the induced Hopf algebra structure on $\widetilde{U}_q(\mathfrak{sl}_2)$*, we have*

$$\Delta(G) = G \otimes K + K^{-1} \otimes G, \qquad \varepsilon(G) = 0, \qquad S(G) = -G.$$

Proof. The proof is another basic exercise in generators and relations, and as such, we leave it to the reader. □

Now for $q = 1$, it is clear that $\widetilde{U}_q(\mathfrak{sl}_2)$ is well-defined. Indeed, in $\widetilde{U}_1(\mathfrak{sl}_2)$, we have that $K^2 = 1$, and moreover that K is an element of the centre of the algebra. The other relations reduce to

$$[E, F] = G, \qquad [G, E] = 2EK, \qquad [G, F] = -2FK.$$

Hence, we have the following result:

Lemma 2.6 *There exists an isomorphism*

$$\beta : \widetilde{U}_1(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2) \otimes (\mathbf{C}[K]/\left\langle K^2 - 1\right\rangle),$$

uniquely defined by

$$\beta(E) = E \otimes K, \qquad F \to F \otimes 1, \qquad G \to H \otimes K, \qquad K \to 1 \otimes K.$$

As a few basic checks will confirm, the quotient $\widetilde{U}_1(\mathfrak{sl}_2)/\langle K - 1\rangle$ is still well-defined as a Hopf algebra, and as such, it is isomorphic to the Hopf algebra $U(\mathfrak{sl}_2)$.

References

[1] A. Klimyk, K. Schmüdgen, *Quantum Groups and their Representations*, Springer Verlag, Heidelberg–New York, 1997

[2] C. Kassel, *Quantum Groups*, Springer–Verlag, New York–Heidelberg-Berlin, 1995