3 Representation Theory

In this lecture we discuss the representation theory of $U_q(\mathfrak{sl}_2)$. As we shall see shortly, when q is not a root of unity, the theory closely mirrors the classical case. We also briefly discuss the root of unity situation, which turns out to be more involved. Explicit proofs of the results stated here can be found in [1].

3.1 q-Integers

For any fixed value of q, the set of *q-integers* is composed of elements

$$[a]_q = \frac{q^a - q^{-a}}{(q - q^{-1})} = q^{a-1} + q^{a-3} + \cdots + q^{-a+1}, \qquad (a \in \mathbf{Z}).$$

Some easily verifiable results about the q-integers are:

$$[-a]_q = -[a]_q, \qquad [a+b] = q^b[a]_q + q^{-a}[b]_q = q^{-b}[a]_q + q^a[b]_q,$$

and, for q not a root of unity, we have $[a]_q \neq 0$. Moreover, we can build upon the definition of a q-integer to produce q-analogues of other familiar integer functions:

$$[a]_q! := [1]_q[2]_q \cdots [a-1]_q[a]_q, \qquad \binom{a}{b}_q := \frac{[a]_q!}{[b]_q![a-b]_q!}.$$

While q-integers are ubiquitous in the theory of quantum groups, they are a much older idea, going back at least to the work of Boole.

3.2 The Representations $T_{\omega,l}$

Now that we have defined the q-integers, we can introduce a distinguished class of representations for $U_q(\mathfrak{sl}_2)$:

Definition 3.1. For $l \in \frac{1}{2}\mathbf{N}_0$, and $\omega \in \{-1, 1\}$, let V_l be the $(2l+1)$-dimensional vector space with basis $\{e_m \,|\, m = -l, -l+1, \cdots, l-1, l\}$. We define operators

$T_{\omega,l}(E)$, $T_{\omega,l}(F)$, and $T_{\omega,l}(K)$ acting on V_l by

$$T_{\omega,l}(K)e_m = \omega q^{2m} e_m, \qquad T_{\omega,l}(E)e_m = \sqrt{[l-m]_q[l+m+1]_q)}e_{m+1},$$

$$T_{\omega,l}(F)e_m = \omega\sqrt{[l+m]_q[l-m+1]_q)}e_{m+1}.$$

This gives a representation of $U_q(\mathfrak{sl}_2)$, which we denote by $T_{\omega,l}$.

We note that the representations of $U_q(\mathfrak{sl}_2)$ have an extra parameter than the representations of the classical enveloping algebra $U(\mathfrak{sl}_2)$. In can be explained (in the $q = 1$ case at least) by the fact $1sl2$ needs to be quotiented by the ideal $\langle K - 1\rangle$ in order to arrive at the classical enveloping algebra $U(\mathfrak{sl}_2)$.

For C_q the quantum Casimir defined in the previous lecture, a simple calculation will demonstrate that

$$T_{\omega,l}(C_q) = \omega(q^{l+1} + q^{-l-1})(q - q^{-1})^{-2}\mathrm{id}.$$

Thus, we see that just as in the classical case, the image of the Casimir under $T_{\omega,l}$ acts through scalar multiplication.

The following two propositions demonstrate the importance of these representations, and recalls the classical case.

Proposition 3.2 *For any $l \in \mathbf{N}_0$, and $\omega \in \{-1, 1\}$, the representation $T_{\omega,l}$ is irreducible. If $(\omega, l) \neq (\omega', l')$, then $T_{(\omega,l)}$ and $T_{(\omega',l')}$ are not equivalent, and the values of the operators $T_{(\omega,l)}(C_q)$ and $T_{(\omega',l')}(C_q)$ are different.*

Proposition 3.3 *Any irreducible finite-dimensional representation T of $U_q(\mathfrak{sl}_2)$ is equivalent to one of the representations $T_{\omega,l}$, for $l \in \frac{1}{2}\mathbf{N}_0$, and $\omega \in \{-1, 1\}$.*

3.3 The Generic Case

We this section we will assume that q is not a root of unity, which is to say we will work in the *generic* setting.

Let T be a finite dimensional representation of $U_q(\mathfrak{sl}_2)$. For any complex number $\lambda \in \mathbf{C}$, we set

$$V_\lambda := \{v \in V \mid T(K)v = \lambda v\}.$$

If $V_\lambda \neq \{0\}$, then we call V_λ a *weight space*, and λ a *weight*, of the representation T. The non-zero elements of V_λ are called *weight vectors*. A weight vector v for which it holds that

$$T(E)v = 0, \qquad \text{and} \qquad T(K)v = \mu v, \qquad (\mu \in \mathbf{C}),$$

is called a *highest weight vector* of T, while μ is called a *highest weight* of T. If T is the linear span of weight spaces of T, then T is called a *weight representation*.

Proposition 3.4 *Every finite dimensional representation of $U_q(\mathfrak{sl}_2)$ is a weight representation.*

We say that a representation of an $U_q(\mathfrak{sl}_2)$ is *completely reducible* if it decomposes as a direct sum of irreducible sub-representations.

Proposition 3.5 *Any finite dimensional representation T of $U_q(\mathfrak{sl}_2)$ is completely reducible.*

3.4 The Root of Unity Case

When q is a root of unity, things become more complicated. There can exist finite dimensional representations of $U_q(\mathfrak{sl}_2)$ which are *not* completely reducible. Indeed, for k an integer for which $q^k = 1$, and denoting $k' = k$ if p is odd, and $k/2$ is k is even, we have the following result:

Proposition 3.6 *Every irreducible representation of $U_q(\mathfrak{sl}_2)$ has dimension less than or equal to k'.*

References

[1] A. Klimyk, K. Schmüdgen, *Quantum Groups and their Representations*, Springer Verlag, Heidelberg–New York, 1997