(1) $U_q(\mathfrak{g})$, $\mathfrak{g} = sl(2)$, $q^n \neq 1$ $\forall n \in \mathbb{N}$, $U_q(\mathfrak{g}) = \langle E, F, K, K^{-1} \rangle / I$

K ... a field, $\dim_K Z(U_q(\mathfrak{g})) = 1$, generated by quadratic Casimir

Z ← centrum of $U_q(\mathfrak{g})$

$$Cas := \pm \left(FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2} \right)$$

± ← character on $\mathbb{Z}_2$

Let V be a represent. for $U_q(\mathfrak{g})$, $\dim_K V = 1$

$V = K\langle u \rangle$, $u \neq 0$ $\quad \exists c \in K^*$: $Ku = cu$

$\Downarrow$

$K(Eu) = q^2 c (Eu)$ => Eu is lin. indep. of u

=> $Eu = 0$

$Fu = 0$

Then $0 = [e, f] u = \dfrac{K - K^{-1}}{q - q^{-1}} u = \dfrac{c - c^{-1}}{q - q^{-1}} u \underset{u \neq 0}{\Rightarrow} c^2 = 1 \Rightarrow c = \pm 1$

$\exists$ 2-non isom. classes of 1-dim repr. :

1/ V_+ $\quad K\langle v_+ \rangle \quad Kv_+ = v_+$

$Fv_+ = 0 = Ev_+$

2/ V_- $\quad K\langle v_- \rangle \quad Kv_- = -v_-$

$Fv_- = 0 = Ev_-$

For $\mathrm{char}(K) = 2$, $V_+ \simeq V_-$.

Weyl group ⇝ Hecke algebra

$q \in \mathbb{C}$, $\mathcal{H}_n = K\langle \sigma_1, \dots, \sigma_{n-1} \rangle / I$

An-series

$I =$
$\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}$ $\quad 1 \le k \le n-2$

$(\sigma_k - 1)(\sigma_k + q^2) = 0$

$\sigma_k \sigma_e = \sigma_e \sigma_k$ $\quad |k - e| \geq 2$

$P_n := \sum_{\sigma \in S_n} q^{-2\ell(\sigma)} \sigma \in \mathcal{H}_n$, $\ell(\sigma)$ ... minimal length of σ

$(\sigma_k - 1) P_n = 0$, symmetric tensors when acting on $V^{\otimes n}$

$\forall k = 1, \dots, n-1$

(2)

Bruhat - Poisson bracket on compact connected Lie groups and flag manifolds

$\mathfrak{g}$ ... complex simple, $\mathfrak{h} \subseteq \mathfrak{g}$
G

$R \subseteq \mathfrak{h}^*$ root system of $(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{g}_\alpha$ root space
α

Δ ... simple roots $\alpha_1, \dots, \alpha_r$
$R^\pm$

$\alpha_i^\vee := \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$ dual coroot of α_i

$GL(\mathfrak{h}^*) \supset W \ni s_i$ simple refl.

$\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$

$H_\alpha \mapsto \alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ $\quad [X_\alpha, X_{-\alpha}] = H_\alpha$

$B(X_\alpha, X_{-\alpha}) = \frac{2}{\langle \alpha, \alpha \rangle}$

real span of H_α's

$[X_\alpha, X_\beta] = C_{\alpha,\beta} X_{\alpha+\beta}$ for $\alpha+\beta \in R$

$\mathfrak{h}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{h}$

$\mathfrak{u} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{g}$ $\quad \mathfrak{u} := \bigoplus_{\alpha \in R^+} \mathbb{R}\langle X_\alpha - X_{-\alpha} \rangle \oplus \bigoplus_{\alpha \in R^+} i\mathbb{R}\langle X_\alpha + X_{-\alpha} \rangle \oplus i\mathfrak{h}_0$

a compact real form. (skew herm. matrices, $(\ ,\ )$ is negative definite)

Def (Manin triple): $(\mathfrak{g}, \mathfrak{g}_1, \mathfrak{g}_2)$

$\mathfrak{g}$ - L. alg. with non-deg inv. symm. bil. form

$\mathfrak{g}_1 \subseteq \mathfrak{g} \supseteq \mathfrak{g}_2$ two isotropic subalgebras, $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \cong \mathfrak{g}$

Manin triples $\leftrightarrow$ Lie bialgebras

Ex: $\tilde{\mathfrak{g}}$ ... Lie alg. (complex semisimpl), $(\ \ )$ K.-C. form

$\mathfrak{g} \simeq \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}^* \simeq \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}$ is a Manin triple

$\langle (v_1, v_2), (v_1', v_2') \rangle = (v_1, v_1') - (v_2, v_2')$

(diagonal) $\mathfrak{g}_1 = \mathrm{diag}(\tilde{\mathfrak{g}}) \hookrightarrow \mathfrak{g}$

(two opposite Borel's) $\mathfrak{g}_2 = (x, y)$, $x \in \tilde{\mathfrak{b}} \subseteq \tilde{\mathfrak{g}}$, $y \in \tilde{\mathfrak{b}}^{opp} \subseteq \tilde{\mathfrak{g}}$ $\quad x|_{\tilde{\mathfrak{h}}} = y|_{\tilde{\mathfrak{h}}}$

$\mathfrak{b} = \mathfrak{h}_{\mathbb{R}} \oplus \mathfrak{n}_+$ $\quad \mathfrak{n}_+ := \sum_{\alpha \in R^+} \mathfrak{g}_\alpha \subseteq \mathfrak{g}$

then $(\mathfrak{g}, \mathfrak{u}, \mathfrak{b})$ is Manin triple (w. r. to imaginary part of K.-C. form)

③ So $(\mathfrak{g}, \mathfrak{b}, \mathfrak{u})$ is a Lie bialgebra, $\mathfrak{u}^* = \mathrm{Hom}(\mathfrak{u}, \mathbb{R}) \underset{(\,,\,)}{\simeq} \mathfrak{b}$,

the co-commutator $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is the coboundary:

$$\delta(X) = (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X)(r) \quad , \quad r \in \mathfrak{g} \wedge \mathfrak{g}$$
$X \in \mathfrak{g}$ — solution of class. Y-B. equation

$$r = i \sum_{\alpha \in R^+} d_\alpha \,(X_{-\alpha} \otimes X_\alpha - X_\alpha \otimes X_{-\alpha}) \in \mathfrak{u} \wedge \mathfrak{u}.$$

Poisson structure on $U \subseteq G$:

$$\Omega_g = L_g^{\otimes 2} r - R_g^{\otimes 2} r \quad , \quad g \in U$$

and its symplectic foliation closely related to the Bruhat decomp.

$B \subseteq G$ connected subgroup, $\mathrm{Lie}(B) = \mathfrak{b}$, $T \subseteq U$ maximal torus $\mathrm{Lie}(T) = i\mathfrak{h}_{\mathbb{R}}$

$B_+ := TB$ (Borel subgroup of G), $N_U(T)/T \simeq W$

$$\exp\left(\tfrac{\pi}{2}(X_{\alpha_i} - X_{-\alpha_i})\right)/T \leftarrow\!\shortmid s_i$$

Bruhat decomp.: $$G = \bigsqcup_{w \in W} B_+ w B_+ .$$

A refinement:

$$G = \bigsqcup_{m \in N_U(T)} BmB \qquad N_U(T) = W \ltimes T$$

Set $\Sigma_m := U \cap BmB$, $m \in N_U(T)$. Then $\Sigma_m \neq \emptyset$ $\forall m \in N_U(T)$, and

$$U = \bigsqcup_{m \in N_U(T)} \Sigma_m .$$

Another realization: $\underset{K}{\underset{\wr}{U}} \times \underset{AN}{\underset{\wr}{B}} \to G$ is global diff. (Iwasawa decomp. of G)

$$u \times b \mapsto u^{b^{-1}} \qquad u^b \in U : \; bu \in u^b B$$

right action of B on U, the orbits of B on U correspond to the decomp. $U = \bigcup_{m \in N_U(T)} \Sigma_m$

<u>Ex</u>: $G = SL(2, \mathbb{C})$, $U = SU(2)$

2-dim sympl. leaves

$$S_t := \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \in SU(2) \;\middle|\; \arg(\beta) = \arg(t) \right\}, \; t \in \mathbb{T}.$$

(4)

Poisson bracket on gen. flag manifold:

$S \subseteq \Delta$ simple roots, $P_S \subseteq G$ parab. subgrp, $\mathfrak{p}_S := \mathrm{Lie}(P_S)$

$$\mathfrak{p}_S = \Big(\mathfrak{h} \oplus \bigoplus_{\alpha \in (\langle S\rangle \cup R_+)} \mathfrak{g}_\alpha\Big)$$

Levi factor of $\mathfrak{p}_S$

$$\langle S\rangle = R^+ \cup \{\alpha \in R \mid \alpha \in \langle S\rangle\}$$

$$\mathfrak{l}_S := \Big(\mathfrak{h} \oplus \bigoplus_{\alpha \in \langle S\rangle} \mathfrak{g}_\alpha\Big), \quad \mathfrak{k}_S := \mathfrak{l}_S \cap \mathfrak{u} = \mathfrak{p}_S \cap \mathfrak{u} \text{ (compact real form of } \mathfrak{l}_S)$$

$$K_S := U \cap P_S, \quad \mathrm{Lie}(K_S) = \mathfrak{k}_S.$$

Poisson-Lie subgroup of U

$\Rightarrow \exists!$ Poisson bracket on U/K_S such that $\pi: U \to U/K_S$ is Poisson map

and $U \times U/K_S \to U/K_S$ is Poisson map

Def: (Poisson-Lie group) Poisson-Lie group G is a Poisson manifold and a Lie group, where group mult. $\mu: G\times G \to G$ is compatible with the Poisson algebra structure. The Lie algebra of a Poisson-Lie group is a Lie bialgebra.

$\mu: G \times G \to G$

$(g_1, g_2) \mapsto g_1 g_2$ is Poisson alg. map:

$$\{f_1, f_2\}(gg') = \{f_1 \circ L_g, f_2 \circ L_g\}(g')$$

$$+ \{f_1 \circ R_{g'}, f_2 \circ R_{g'}\}(g)$$

$f_1, f_2 \in C^\infty(G, \mathbb{R})$

L, R - left, right G-action

Poisson str. is not comp. with inverse map $G \to G$, $g \mapsto g^{-1}$

For $g, g' = e \in G \Rightarrow \{f, g\}(e) = 0$ ($\Rightarrow$ neversymplectic, constant rank)

Poisson map $\varphi: G_1 \to G_2$ L. group homom. + $\{f_1, f_2\}_{G_2} \circ \varphi = \{f_1\circ\varphi, f_2\circ\varphi\}$

W_S ... Weyl group generated by $\langle S\rangle$

$P_S = B_+ W_S B_+$, and so double cosets $B_+ x P_S$, $x \in G$ are parametrized by $W^S := W/W_S$.

$$U/K_S \simeq G/P_S \simeq \bigsqcup_{\bar{w} \in W/W_S} X_{\bar{w}}, \quad X_{\bar{w}} = (U \cap B_+ w P_S)/K_S \simeq B_+ w / P_S.$$

$\bar{w}$ ... right coset in W/W_S, containing $w \in W$

(5) B preserves K_S $\Rightarrow$ descends to U/K_S, the orbits = Schubert cells

<u>Th</u>: Sympl. leaves of Poisson man. U/K_S = orbits of B on U/K_S

[Sympl. leaf of $(X, \{\ ,\ \})$ Poiss. man. : $Y \hookrightarrow X$ a maximal submanifold, Poisson brack. restricts to a sympl. structure)

X_{f_1}, X_{f_2} Hamilt. vect. fields gener. a sympl. leaf passing through a given point $p \in X$]

W^S, $\ell : W^S \to \mathbb{N}_0$

$\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$, $w_1, w_2 \in W^S$

length function: $\ell(w) := \#\{R^+ \cap wR^-\}$

$$\pi : U \to U/K_S$$

$$\pi|_{\Sigma_m} = \pi_m \quad \Sigma_m \to \pi(\Sigma_m) = X_{\overline{w(m)}} \subseteq U/K_S, \quad w(m) = m/T \in W$$

$m \in N_U(T)$, $\Sigma_m \subseteq U$

Poisson surjective map

<u>Prop</u>: $m \in N_U(T)$; then $\pi_m : \Sigma_m \to X_{\overline{w(m)}}$ is a symplectic automorphism iff $w(m) \in W^S$.

$W^S = \{ w \in W \mid \ell(w s_\alpha) > \ell(w)\ \forall \alpha \in S\}$

min. length representative

A description of sympl. leaves for U/K_S:

<u>Th</u>: $m \in N_U(T)$, $w := m/T \in W$, $w_1 \in W^S$, $w_2 \in W_S$ $\}$ $w = w_1 w_2$, reduced expression:

$w_1 = s_{i_1} \cdots s_{i_p}$

$w_2 = s_{i_{p+1}} \cdots s_{i_\ell}$

$\jmath_i : SU(2) \to U$

$\alpha \mapsto \alpha_i \in S$

Poisson-Lie group map

Then

$$(g_1, \ldots, g_\ell) \mapsto \jmath_{i_1}(g_1) \cdots \jmath_{i_\ell}(g_\ell)/K_S$$

is surj Poiss. map $\times^\ell S_1 \to X_{\overline{w}}$ and it

$$\times^\ell S_1 = \times^p S_1 \times \times^{(\ell-p)} S_1$$

quotient through $\times^p S_1 \to X_{\overline{w}}$ (this being sympl. autom.)