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9 Quantum Equivariant Vector Bundles

Recall that classically, for $M = G/H$ a homogeneous space, we have a correspondence between the category of equivariant finite dimensional vector bundles over M, and the category of finite dimensional representations of H. In this lecture we will show how one can extend this correspondence to the noncommutative setting.

9.1 Quantum Homogeneous Spaces

From here on, G, and H will always be Hopf algebras over the complex numbers. We denote the comultiplication, counit, and antipode of each by Δ, ε, and S respectively. Throughout, we use Sweedler notation, as well as denoting $g^+ := g - \varepsilon(g)1$, for $g \in G$, and $V^+ = V \cap \ker(\varepsilon)$, for V a subspace of G

We will recall that a *(right) comodule* for H is a is a pair (V, Δ_R), where V is a vector space, and

$$\Delta_R : V \to V \otimes H,$$

is a linear map , called the *coaction*, for which it holds that

$$(\mathrm{id} \otimes \Delta) \circ \Delta_R = (\Delta_R \otimes \mathrm{id}) \circ \Delta_R, \qquad (\mathrm{id} \otimes \varepsilon) \circ \Delta_R = \mathrm{id}.$$

In an extension of Sweedler notation, we will usually denote

$$\Delta_R(v) = \sum_i v_i \otimes h_i =: v_{(0)} \otimes v_{(1)}.$$

We say that an element $v \in V$ is *coinvariant* if $\Delta_R(v) = v \otimes 1$. We denote the subspace of all coinvariant elements by V^G, and call it the *coinvariant subspace* of the coaction.

For H also a Hopf algebra, a *homogeneous* right H-coaction on G is a coaction of the form $(\mathrm{id} \otimes \pi) \circ \Delta$, where $\pi : G \to H$ is a Hopf algebra map. We call the coinvariant subspace $M := G^H$ of such a coaction a *quantum homogeneous space*. As is easy to see, M will always be a subalgebra of G. Moreover, it can be shown without difficulty that the coaction of G restricts to a right G-coaction on M, and that

$$\pi(m) = \varepsilon(m)1_H, \qquad (\text{for all } m \in M). \tag{1}$$

Note that G is itself a trivial example of a quantum homogeneous space, for the choice $\pi = \varepsilon$.

9.2 Takeuchi's Categorical Equivalence

and whose morphisms are both left M-module and right H-comodule maps. In what follows, for sake of clarity, we will denote the right M-action on an object in ${}^G_M\mathcal{M}_M$ by juxtaposition, while we will denote the right M-action on an object in $\mathcal{M}^H_M$ by $\triangleleft$.

Let us now move on to constructing an equivalence between these categories: For any object V in $\mathcal{M}^H_M$, we can associate to it a corresponding object in ${}^G_M\mathcal{M}_M$ as follows: Consider the coinvariant subspace $(G \otimes V)^H$, where $G \otimes V$ is endowed with the usual tensor product coaction. We can give $(G \otimes V)^H$ the structure of an object in ${}^G_M\mathcal{M}_M$ by defining right and left M-actions according to

$$m(\sum_i g^i \otimes v^i) = \sum_i mg^i \otimes v^i, \qquad (\sum_i g^i \otimes v^i)m = \sum_i g^i m_{(1)} \otimes (v^i \triangleleft m_{(2)}),$$

and defining a left G-coaction according to

$$\Delta_L(\sum_i g^i \otimes v^i) = \sum_i g^i_{(1)} \otimes g^i_{(2)} \otimes v^i.$$

The right M-module structure of $\mathcal{E}$ clearly restricts to a right M-module structure on $\mathcal{E}/(M^+\mathcal{E})$. Moreover, it can be shown using (1) that the left G-module structure of $\mathcal{E}$ induces a right H-comodule structure on $\mathcal{E}/(M^+\mathcal{E})$ defined by

$$\Delta_R(\overline{e}) = \overline{e_{(0)}} \otimes S(\pi(e_{(-1)})), \qquad (e \in \mathcal{E}), \tag{3}$$

where $\overline{e}$ denotes the coset of e in $\mathcal{E}/(M^+\mathcal{E})$. To show that these two structures are compatible in the sense of (2) is routine. Thus, we have given $\mathcal{E}/(M^+\mathcal{E})$ the structure of an object in $\mathcal{M}^H_M$. Consider now the functors

$$\begin{aligned} &\Phi_M : {}^G_M\mathcal{M}_M \to \mathcal{M}^H_M, && \Phi_M(\mathcal{E}) = \mathcal{E}/(M^+\mathcal{E}), \\ &\Psi_M : \mathcal{M}^H_M \to {}^G_M\mathcal{M}_M, && \Psi_M(V) = (G \otimes V)^H. \end{aligned}$$

Where for $f : \mathcal{E} \to \mathcal{F}$ a morphism in ${}^G_M\mathcal{M}_M$, the morphism $\Phi_M(f) : \Phi_M(\mathcal{E}) \to \Phi_M(\mathcal{F})$ is the function to which f descends on $\Phi_M(\mathcal{E})$. While for $\varphi : V \to W$ a morphism in $\mathcal{M}^H_M$, we define $\Psi_M(\varphi) := 1 \otimes \varphi$. To show that both morphisms are well-defined is routine. Moreover, using some basic linear algebra arguments, it can also be shown that, for $\mathcal{E}, \mathcal{F}$ two objects in ${}^G_M\mathcal{M}_M$, and V, W two objects in $\mathcal{M}^H_M$, we have

$$\Phi(\mathcal{E} \oplus \mathcal{F}) = \Phi(\mathcal{E}) \oplus \Phi(\mathcal{F}), \qquad \Psi(V \oplus W) = \Psi(V) \oplus \Psi(W), \tag{4}$$

and if we further assume that $\mathcal{E} \subseteq \mathcal{F}$, and $V \subseteq W$, then

$$\Phi(\mathcal{E}/\mathcal{F}) = \Phi(\mathcal{E})/\Phi(\mathcal{F}), \qquad \Psi(V/W) = \Psi(V)/\Phi(W). \tag{5}$$

What we should now ask is when this induces an equivalence of categories. This leads us to the notion of faithful flatness: We say that G is a *faithfully flat* module over M if the *tensor product functor* $G \otimes_M - : {}_M\mathcal{M} \to {}_{\mathbf{C}}\mathcal{M}$, from the category of left M-modules to the category of complex vector spaces, preserves and reflects exact sequences.

Theorem 9.1 (Takeuchi [2]) *Let $\pi : G \to H$ be a quantum homogeneous space for which G is a faithfully flat right module over $M = G^H$. A natural isomorphism between $\Psi_M \circ \Phi_M$ and the identity is determined by*

$$\mathrm{frame}_M : \mathcal{E} \to \Psi_M \circ \Phi_M(\mathcal{E}), \qquad e \mapsto e_{(0)} \otimes \overline{e_{(1)}}, \tag{6}$$

$$\mathrm{frame}_M^{\perp} : \Phi_M \circ \Psi_M(V) \to V, \qquad \sum_i \overline{g^i \otimes v^i} \mapsto \sum_i \varepsilon(g^i)v^i, \tag{7}$$

giving an equivalence of categories between ${}^G_M\mathcal{M}_M$ and $\mathcal{M}^H_M$.

9.3 The Quantum Projective Spaces

As an example of a quantum homogeneous space, we will take the quantum projective spaces. These are special cases of the quantum flag manifolds introduced in previous lectures. We will give an alternative presentation of these algebras. Our motivation here is two-fold: first we wish to provide a more concrete construction, and second we wish to give a first glimpse of the general Yang–Baxter construction of Hopf algebras.

9.3.1 The Quantum Special Unitary Group

For $q \in (0, 1]$ and $\nu := q - q^{-1}$, let $\mathbf{C}_q[M_N]$ be the quotient of the free noncommutative algebra $\mathbf{C}\left\langle u^i_j, \mid i, j = 1, \ldots, N \right\rangle$ by the ideal generated by the elements

$$u^i_k u^j_k - q u^j_k u^i_k, \qquad u^k_i u^k_j - q u^k_j u^k_i, \qquad (1 \le i < j \le N, 1 \le k \le N);$$
$$u^i_l u^j_k - u^j_k u^i_l, \quad u^i_k u^j_l - u^j_l u^i_k - \nu u^i_l u^j_k, \qquad (1 \le i < j \le N,\ 1 \le k < l \le N).$$

These generators can be more compactly presented as

$$\sum_{w,x=1}^{N} R^{ac}_{wx} u^w_b u^x_d - \sum_{y,z=1}^{N} R^{yz}_{bd} u^a_y u^c_z, \qquad (1 \le a,b,c,d \le N), \tag{8}$$

where, for H the Heaviside step function with $H(0) = 0$, we have denoted

$$R^{ik}_{jl} = q^{\delta_{ik}} \delta_{il} \delta_{kj} + \nu H(k-i) \delta_{ij} \delta_{kl}. \tag{9}$$

We can put a bialgebra structure on $\mathbf{C}_q[M_N]$ by introducing a coproduct Δ, and counit ε, uniquely defined by $\Delta(u^i_j) := \sum_{k=1}^N u^i_k \otimes u^k_j$, and $\varepsilon(u^i_j) := \delta_{ij}$. The *quantum determinant* of $\mathbf{C}_q[M_N]$ is the element

$$\det{}_N := \sum\nolimits_{\pi \in S_N} (-q)^{\ell(\pi)} u^1_{\pi(1)} u^2_{\pi(2)} \cdots u^N_{\pi(N)},$$

where summation is taken over all permutations π of the set of N elements, and $\ell(\pi)$ is the length of π. As is well-known, $\det_N$ is a central and grouplike element of the bialgebra. The centrality of $\det_N$ makes it easy to adjoin an inverse $\det_N^{-1}$. Both Δ and ε have extensions to this larger algebra, which are uniquely determined by $\Delta(\det_N^{-1}) = \det_N^{-1} \otimes \det_N^{-1}$, and $\varepsilon(\det_N^{-1}) = 1$. The result is a new bialgebra which we denote by $\mathbf{C}_q[GL_N]$. We can endow $\mathbf{C}_q[GL_N]$ with a Hopf algebra structure by defining

$$S(\det{}_N^{-1}) = \det{}_N, \qquad S(u^i_j) = (-q)^{i-j} \sum\nolimits_{\pi \in S_{N-1}} (-q)^{\ell(\pi)} u^{k_1}_{\pi(l_1)} u^{k_2}_{\pi(l_2)} \cdots u^{k_{N-1}}_{\pi(l_{N-1})} \det{}_N^{-1},$$

where $\{k_1, \ldots, k_{N-1}\} = \{1, \ldots, N\} \backslash \{j\}$, and $\{l_1, \ldots, l_{N-1}\} = \{1, \ldots, N\} \backslash \{i\}$ as ordered sets. Moreover, we can give $\mathbf{C}_q[GL_N]$ a Hopf $*$-algebra structure by setting $(\det_N^{-1})^* = \det_N$, and $(u^i_j)^* = S(u^j_i)$. We denote this Hopf $*$-algebra by $\mathbf{C}_q[U_N]$, and call it the *quantum unitary group of order N*. If we quotient $\mathbf{C}_q[U_N]$ by the ideal $\langle \det_N - 1 \rangle$, then the resulting algebra is again a Hopf $*$-algebra. We denote it by $\mathbf{C}_q[SU_N]$, and call it the *quantum special unitary group of order N*.

The algebra we have presented here the same as the one presented in previous lectures, however here we have constructed it from a so-called *R-matrix*. This construction holds true for any $R_1 \otimes R_2 \in M_N(\mathbf{C}) \otimes M_N(\mathbf{C})$ satisfying the *quantum Yang–Baxter equation*

$$\begin{aligned}(R_1 \otimes R_2 \otimes 1)(R_1 \otimes 1 \otimes R_2)(1 \otimes R_1 \otimes R_2) \\ = (1 \otimes R_1 \otimes R_2)(R_1 \otimes 1 \otimes R_2)(R_1 \otimes R_2 \otimes 1).\end{aligned}$$

This is a much more general construction than the Drinfeld–Jimbo quantised coordinate algebra approach.

9.3.2 The Quantum Projective Spaces $\mathbf{C}_q[\mathbf{C}P^{N-1}]$

We are now ready to reconstruct the quantum projective spaces as a quantum homogeneous space of $\mathbf{C}_q[SU_N]$:

Definition 9.2. Let $\alpha_N : \mathbf{C}_q[SU_N] \to \mathbf{C}_q[U_{N-1}]$ be the surjective Hopf algebra map defined by setting $\alpha_N(u_1^1) = \det_{N-1}^{-1}$; $\alpha_N(u_i^1) = \alpha_N(u_1^i) = 0$, for $i = 2, \cdots, N$; and $\alpha_N(u_j^i) = u_{j-1}^{i-1}$, for $i, j = 2, \ldots, N$. *Quantum projective* $(N-1)$*-space* $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ is defined to be the coinvariant subspace of the corresponding homogeneous coaction $\Delta_{SU_N,\alpha_N} = (\mathrm{id} \otimes \alpha_N) \circ \Delta$, that is,

$$\mathbf{C}_q[\mathbf{C}P^{N-1}] := \{f \in \mathbf{C}_q[SU_N] \,|\, \Delta_{SU_N,\alpha_N}(f) = f \otimes 1\}.$$

Now let us consider the element $z_{ij} := u_1^i S(u_j^1)$. From the following calculation, we can see that z_{ij} is contained in $\mathbf{C}_q[\mathbf{C}P^{N-1}]$, for all $i, j = 1, \ldots, N$:

$$\begin{aligned}\Delta_{SU_N}(z_{ij}) = \Delta_{SU_N}(u_1^i S(u_j^1)) &= (\mathrm{id} \otimes \alpha_N)(\sum_{a,b=1}^{N} u_a^i S(u_j^b) \otimes u_1^a S(u_b^1)) \\ &= \sum_{a,b=1}^{N} u_a^i S(u_j^b) \otimes \alpha_N(u_1^a S(u_b^1)) = u_1^i S(u_j^1) \otimes \alpha_N(u_1^1 S(u_1^1)) \\ &= u_1^i S(u_j^1) \otimes \det_N^{-1} \det_N = z_{ij} \otimes 1.\end{aligned}$$

Moreover, using representation theoretic methods, it can be shown that $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ is generated as a $\mathbf{C}$-algebra by the set $\{z_{ij} \,|\, i, j = 1, \ldots, N\}$. (See [1] for more details.)

As one would hope, $\mathbf{C}_q[SU_N]$ is a faithfully flat module over $\mathbf{C}_q[\mathbf{C}P^{N-1}]$. An important family of examples of objects in ${}^{SU_N}_{\mathbf{C}P^{N-1}}\mathcal{M}_{\mathbf{C}P^{N-1}}$ is the *quantum line bundles* $\mathcal{E}_p$, for $p \in \mathbf{Z}$: The module $\mathcal{E}_p$ is defined to be $\Psi_{\mathbf{C}P^{N-1}}(\mathbf{C})$, where $\mathbf{C}$ considered as an object in $\mathcal{M}^{U_{N-1}}_{\mathbf{C}P^{N-1}}$ according to the unique $\mathbf{C}[U_1]$-coaction for which $\lambda \mapsto \lambda \otimes \det_{N-1}^{-p}$, for $\lambda \in \mathbf{C}$. Clearly, we have that $\mathcal{E}_0 = \mathbf{C}_q[\mathbf{C}P^{N-1}]$. This generalises the classical construction of the line bundles over $\mathbf{C}P^{N-1}$.

References

[1] A. Klimyk, K. Schmüdgen, *Quantum Groups and their Representations*, Springer Verlag, Heidelberg–New York, 1997

[2] M. Takeuchi, Relative Hopf modules - equivalences and freeness conditions, *J. Algebra*, **60**, 452-471, (1979)