Summary Overview of the Method Applied to the Determination of the Orbits of the Two New Planets*

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The determination of the orbit of a heavenly body from a short series of observations, independent of the circular and parabolic hypotheses, depends on two requirements: I. There must be some means to find the orbit which satisfies three given complete observations. II. There must be a way to adjust the orbit so found, such that the difference between the calculation and the entire supply of observations becomes as small as possible.

The most convenient means to satisfy the second condition appears to be to reduce it to the first. For the times t, t', t'', etc., the observed positions would be m, m', m'', etc. (each of which will have two components); the positions p, p', p'', etc. calculated according to the known elements e (considered as having six components), and finally, the positions q, q', q'', etc. calculated according to the (still considered as undetermined) elements f. The differences of the [positions calculated according to the] elements e are thus

$$p - m,\ p' - m',\ p'' - m'',\ \text{etc.}$$

*When I had the pleasure of making the personal acquaintance of Herr Professor Gauss some time back, I saw among his papers the following essay, already outlined many years ago and yet nowhere published, which contained the earlier method of the author for determining the orbit. In my cursory reading of this summary overview I was soon convinced that the method developed here by the author, for making a first approximation of two distances of the planets from the Earth, was essentially different from that which the author has now publically expounded upon in his larger work. So, I asked him for permission that I might make this treatise known, with the assumption that it would be interesting to all connoisseurs to know the way in which the author succeeded at arriving at a complete solution, which differed from that of which an overview had been communicated to our readers in earlier issues. I originally had the goal of accompanying the essay with some remarks for the purpose of making a comparison of the earlier and later methods of the author; but these, had they actually been explained, would be somewhat lengthy, and without reference to the work itself, would remain ever unclear. It thus appeared advisable to me to communicate the entire essay without further addenda (which is more intended for connoisseurs who have the work itself at hand) to the astronomical readers of this periodical, just as it was set down by the author in writing six years ago. - von Lindenau

On the the other hand, the differences of the [positions calculated according to the] elements f are

$$(p-m)+(q-p),\ (p'-m')+(q'-p'),\ (p''-m'')+(q''-p''),\ \text{etc.}$$

These latter should thus become as small as possible, and maintain no regularity. The differences $q-p$, $q'-p'$, etc., are, as long as the elements f are considered as constant, functions of time and, since, properly considered, they will be small by the nature things, it may thus be assumed from the short duration of the observations that they [the differences] are found for the intermediate times with sufficient precision through interpolation, if two outer and one inner [observation] are taken as given. They are denoted for those three times by x, y, and z (each considered as having two components), thus, by the [well] known first principles of interpolation theory, they will have a linear form $\alpha x+\beta y+\gamma z$, where the coefficients α, β, γ are dependent on time. These differences of the elements f will thus have, for the times t, t', t'', etc. the form:

$$\begin{aligned}
&p-m+\alpha x+\beta y+\gamma z\\
&p'-m'+\alpha' x+\beta' y+\gamma' z\\
&p''-m''+\alpha'' x+\beta'' y+\gamma'' z\\
&\text{etc.}
\end{aligned}$$

where everything besides x, y, z is known. It will then be possible to easily judge which values for x, y, and z are the most suitable. *This can indeed yield a completely methodical process for finding these values through calculation*: however, a certain instinct will always be just as sure a guide. [1]

It is therefore apparent, that as soon as x, y, and z are determined, the second condition is reduced to the first, and thus are we able to limit ourselves solely to this.

Determination of the Orbit from Three Complete Observations

2.

It would indeed not be difficult to represent the relation of the six unknown magnitudes to the given ones in six equations. Only, this would turn out to be far too unhelpful; in order to be in the least bit useful, we must be satisfied with reaching the orbit which exactly represents the three observations, stepwise. Obviously all methods at all useful to this [purpose] ultimately yield the same result; the quality of the end results is therefore no measure for the value of the method, but rather only for the acuity of the observations on which it is based. The value of the method can only be appreciated for the number and convenience of the steps, and a method whereby a representation of the three observations could not always be arrived at if so desired would not be a worse one, but rather it would be no method at all. The investigation is thus split into two parts, a

[1] [insert translators footnote on gewisser Tact]

first approximation, and *a method of adjustment*. The former will be based upon certain, almost true relations drawn from the nature of the problem, and these are of such a type that the closer the approximations lie to one another, the smaller they err, and mathematically speaking, they are rigorously exact for observations which are infinitely close to one another. In any event, the influence of their deviation from the truth thus diminishes the closer the observations are on which they are based, whereby the method of adjustment is made easier or wholly superfluous. Only it has to be considered that with close observations small errors in the observations become very serious, and sometimes affect the elements enormously, and hence the subsequent adjustment according to the entire set of observations, which we above called the second requirement, turns out to be all the more difficult. General rules cannot be given for the most appropriate selection of the observations. In order to determine the value, or more or less advantageous positions, of the observations, it is necessary to take the type of orbit into consideration. For Ceres, the outermost observations, distant by 41 days, were sufficiently well applied with great success in the first approximation, and the calculations for the adjustment were quite simple. Also, with the calculation of the second Pallas orbit, the preceding approximation was not used, but rather the first method of approximation is well enough applied anew to the 27 day observations. For orbits which were to come nearer to a parabola, and in which the geocentric motion is very fast, the calculation would preferably be begun with somewhat shorter time intervals. Here, a judgement developed by experience is the best guide.

Principle Points of the First Approximation

First Point

Approximate determination of the distances from the Earth for the two outer observations.

3.

In order to simplify the overview with respect to the great number of symbols necessary for the following investigation, analogous things for the Earth, P, and for the observed planet, p, should be denoted by the same characters, only the former with capital letters, and the latter with lowercase. If the same letters appear without an apostrophe as well as with two or three, it must therefore be assumed, that the second and third have a similar relation to a second and third observation for the times τ', τ'', as the first has to the observation for time τ. Moreover, in and of itself it is not necessary that the time τ' fall between the time τ and the time τ''; nevertheless, the use of the following prescriptions is most advantageous if τ' lies approximately halfway between τ and τ''.

S, is the position of the Sun (in space) considered as fixed. p is the position of the planet p at time τ. Similarly p', p'', P, P', P'.

$\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ are three arbitrary fixed planes, which intersect one another perpendicularly at the center of the Sun.

x, y, z are the perpendicular distances of the planet p from these three planes at time τ. Similarly x', y', z'; x'', y'', z''; X, Y, Z; X', Y', Z'; X'', Y'', Z''.

$$\left.\begin{array}{ccc} \xi & = & x - X \\ \eta & = & y - Y \\ \zeta & = & z - Z \end{array}\right| \text{ hence } \xi', \eta', \zeta', \xi'', \eta'', \zeta''$$

Thus, ξ, η, ζ are the perpendicular distances of the planet p from three moveable planes laid through P and parallel with $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$.

$$\begin{array}{c|c|c|c|c|l} r & & \text{of } p & & S & \\ R & \text{distance} & \text{of } P & \text{from} & S & \text{all positive. Similarly } r' \text{, etc.} \\ \rho & & \text{of } p & & P & \end{array}$$

$$\begin{array}{c|c|c|c|l} b & & Sp & & \mathfrak{Z} \\ B & \text{angle of the line} & SP & \text{with the plane} & \mathfrak{Z} \\ \beta & & Pp & & \text{which is parallel to } \mathfrak{Z} \end{array}$$

$$\left.\begin{array}{ccc} d & = & r\cos b \\ D & = & R\cos B \\ \delta & = & \rho\cos\beta \end{array}\right| \begin{array}{l} \\ \text{i.e. the projected distance on the plane } \mathfrak{Z}, \\ \text{and that parallel to it.} \end{array}$$

$$\begin{array}{c|c|l} l & & \mathfrak{Y} \\ L & \text{angle of the projection with the plane} & \mathfrak{Y} \\ \lambda & & \text{which is parallel to } \mathfrak{Y} \end{array}$$

The angles b and l are to be taken as positive on the same side of $\mathfrak{Z}$ and $\mathfrak{Y}$ as z and y are considered to be positive. The angle b can always be taken between the limits -90, +90 (so that d, etc. always remain positive); whereas the angle l can always be allowed to grow from 0 to 360^o, and indeed so that it is set $= \left\{\begin{array}{c} 0^o \\ 180^o \end{array}\right\}$ where x is $\left\{\begin{array}{c} \text{positive} \\ \text{negative} \end{array}\right\}$. In this way is obtained

$$x = r\cos b\cos l = d\cos l$$
$$y = r\cos b\sin l = d\sin l$$
$$z = r\sin b = d\tan b$$

and similar equations for x etc., X etc., ξ etc. We assume the orbits of p and P to be in planes, and in so doing we abstract them from any outside influences which might affect them. We set the longitude of p in the orbit at time τ, $= v$ (similarly v', v''; V, V', V''); and make $\frac{1}{2}r'r''\sin(v''-v') = f$, $\frac{1}{2}r''r\sin(v-v'') = f'$, $\frac{1}{2}rr'\sin(v'-v) = f''$. Thus f, $-f'$, f'', the areas of the triangles $p'Sp''$, pSp'', pSp', are positive (assuming that p runs in its proper direction, and τ' lies between τ and τ''; the arrangement of the symbols for other cases presents no difficulty). Similarly F, F', F''. By g, $-g$, g, G, $-G'$, G'' we denote the areas of the sectors of the whole orbit, to which these three triangles correspond, whose signs we assume as equal to those of f, $-f'$, f'', F, $-F'$, F''. Hence, g, g', g'' and G, G', G'' are proportional to the time intervals $\tau''-\tau'$, $\tau-\tau''$, $\tau'-\tau$.

The orbits of p and P are conic sections, whose semimajor axes we denote by a, A. The eccentricity of the orbit p we set $= e = \sin\phi$ (for an ellipse); hence $a\cos\phi^2 = k$ will be the half parameter. Longitude of the aphelion of p, in its orbit is $= \pi$. The mean longitude is $= m$ (similarly m', m'', M, M', M''). The other symbols will be indicated in the course of the investigation itself.

4.

Since p, p', p'' are in a plane with S, it follows from a known theorem that

$$0 = xy'z'' + x'y''z + x''yz' - xy''z' - x'yz'' - x''y'z$$

and hence, that the upper three of the following nine magnitudes

$$\begin{array}{lll} y'z''-y''z' & y''z-yz'' & yz'-y'z \\ z'x''-z''x' & z''x-zx'' & zx'-z'x \\ x'y''-x''y' & x''y-xy'' & xy'-x'y \end{array}$$

are proportional to the three middle ones and to the three lower, respectively.

It is easily concluded,

I. that these very magnitudes are also proportional to f, f', f'', since the three upper, middle, and lower simply represent the doubled area of the projection of the triangles whose areas are f, f', f'', onto the fundamental planes $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, and thus are proportional to them as the doubled cosine of the inclination of the orbit of p with respect to these planes is to unity. (In a complete treatment, further remarks regarding the signs would be necessary, which can, however, also be easily circumvented by a mere assessment).[2]

II. that if the three upper, the three middle, or the three lower are multiplied by x, x', x'', y, y', y'' or z, z', z'', the sum of the products will $= 0$. From this can easily be concluded

$$\left.\begin{array}{r} 0 = fx + f'x' + f''x'' \\ 0 = fy + f'y' + f''y'' \\ 0 = fz + f'z' + f''z'' \end{array}\right| \begin{array}{c} \text{through completely} \\ \text{analagous conclusions,} \\ \text{is obtained} \end{array} \left|\begin{array}{l} 0 = FX + F'X' + F''X'' \\ 0 = FY + F'Y' + F''Y'' \\ 0 = FZ + F'Z' + F''Z'' \end{array}\right.$$

[2]durch blossen Calcul

From this, the following three equations can easily be derived

$$\begin{aligned}
&(F+F'')(f\xi+f'\xi'+f''\xi'') \\
&\qquad = (Ff''-F''f)(X-X'')+[F'(f+f'')-(F+F'')f']X' \\
&(F+F'')(f\eta+f'\eta'+f''\eta'') \\
&\qquad = (Ff''-F''f)(Y-Y'')+[F'(f+f'')-(F+F'')f']Y' \\
&(F+F'')(f\zeta+f'\zeta'+f''\zeta'') \\
&\qquad = (Ff''-F''f)(Z-Z'')+[F'(f+f'')-(F+F'')f']Z'
\end{aligned}$$

From these three equations we derive four others, by multiplying them

first with	then with	then with	and finally, with	and
$\eta\zeta''-\eta''\zeta$	$\eta Z'-Y'\zeta$	$\eta' Z'-Y'\zeta'$	$\eta'' Z'-Y'\zeta''$	adding
$\zeta\xi''-\zeta''\xi$	$\zeta X'-Z'\xi$	$\zeta' X'-Z'\xi'$	$\zeta'' Z'-Y'\xi''$	the
$\xi\eta''-\xi''\eta$	$\xi Y'-X'\eta$	$\xi' Y'-X'\eta'$	$\xi'' Y'-X'\eta''$	products

For a convenient overview, we denote the sum of the products which arise.

					after multiplication of these factors with
	*	*	$\delta'\delta D'[\pi'\pi P']$	$\delta''\delta D'[\pi''\pi P']$	ξ, η, ζ
$[\pi\pi'\pi'']$	$\delta\delta'\delta''\times[\pi\pi'\pi'']$	$\delta\delta' D'[\pi\pi' P']$	*	$\delta''\delta' D'[\pi''\pi' P']$	ξ', η', ζ'
	*	$\delta\delta'' D'[\pi\pi'' P']$	$\delta'\delta'' D'[\pi'\pi'' P']$	*	ξ'', η'', ζ''
$[\pi P\pi'']$	$\delta D\delta''[\pi P\pi'']$	$\delta DD'[\pi PP']$	$\delta' DD'[\pi' PP']$	$\delta' DD'[\pi'' PP']$	X, Y, Z
$[\pi P'\pi'']$	$\delta D'\delta''[\pi P'\pi'']$	*	*	*	X', Y', Z'
	$\delta D''\delta''[\pi P''\pi'']$	$\delta D'' D'[\pi P'' P']$	$\delta' D'' D'[\pi' P'' P']$	$\delta'' D'' D'[\pi'' P'' P']$	X'', Y'', Z''

It is clear, that in the spots here filled with *, there must be a 0, and that all magnitudes indicated by bracketed symbols are given. That is to say

$$[\pi\pi'\pi''] = \tan\beta\cdot\sin(\lambda'-\lambda'')+\tan\beta'\cdot\sin(\lambda''-\lambda)+\tan\beta''\cdot\sin(\lambda-\lambda')$$
$$[\pi P\pi''] = \tan\beta\cdot\sin(L-\lambda'')+\tan B\cdot\sin(\lambda''-\lambda)+\tan\beta''\cdot\sin(\lambda-L)$$
etc.

It is not necessary to put down all 16 equations here, since they can all be derived in an analogous way from the first, by simply exchanging β with β', β'' B, B', B'' and λ with λ', λ'', L, L', L'' if in the place π of stands π', π'', P, P', P'' respectively, and so on. At the same time, it is seen that the 16 magnitudes are reduced to 12, since

$$\begin{aligned}
+\quad [\pi P'\pi''] &= -[\pi\pi'' P'] = +[\pi''\pi P'] \\
[\pi\pi' P'] &= -[\pi'\pi P'] \\
[\pi''\pi' P'] &= -[\pi'\pi'' P']
\end{aligned}$$

Further, it is easily recognized that the expression $[\pi\pi'\pi'']$, multiplied by the product of the three cosines of the latitudes which appear in it, is the sixfold volume of a pyramid, whose apex falls in the center, and whose three base angle points fall on the surface of a sphere described with radius 1, such that they correspond to three geocentric positions of p, and they will give positive or negative signs to the sixfold value, according as those three geocentric positions lie on the sphere in an opposite or the same sense as the positive poles[3] of the planes $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, respectively. And the remaining signs express entirely similar things. In this way, the following four equations appear:

$$(F+F'')f'\delta'[\pi\pi'\pi''] = \\ (Ff''-F''f)(D[\pi P\pi'']-D''[\pi P''\pi'']) + (F'(f+f'')-(F+F'')f')D'[\pi P'\pi''] \tag{1}$$

$$(F+F'')(f'\delta'[\pi\pi'P'] + f''\delta''[\pi\pi''P']) = (Ff''-F''f)(D[\pi PP']-D''[\pi P''P']) \tag{2}$$

$$(F+F'')(f\delta[\pi'\pi P'] + f''\delta''[\pi'\pi''P']) = (Ff''-F''f)(D[\pi' PP']-D''[\pi' P''P']) \tag{3}$$

$$(F+F'')(f\delta[\pi''\pi P'] + f'\delta'[\pi''\pi'P']) = (Ff''-F''f)(D[\pi'' PP']-D''[\pi'' P''P']) \tag{4}$$

5.

We want to examine more closely [*näher betrachten*] these four equations, which are rigorously correct, in order to base our first approximation upon them. If we conceive of the time intervals as infinitely small magnitudes of the first order, then f, f', f'', G, G', G'' and all bracketed [quantities] will also be of the first order, with the exception of $[\pi\pi'\pi'']$ which is of the third order. I omit the proofs, as well as the easily offered [*sich leicht darbietenden*] remarks on special exceptions. Should the inclination of the orbits of p and P relative to one another be considered as magnitudes of the first order, all bracketed magnitudes would stand one order higher. Further

$$Ff''-F''f = \frac{F}{G}\cdot\frac{f''}{g''}Gg'' - \frac{F''}{G''}\cdot\frac{f}{g}G''g$$

or (because $Gg''=G''g$)

$$= \left(\frac{F}{G}\cdot\frac{f''}{g''} - \frac{F''}{G''}\cdot\frac{f}{g}\right)Gg''$$

Now $G-F$ is a magnitude of the third order, hence $1-\frac{F}{G}$ is one of the second, etc., and therefore $\frac{F}{G}\cdot\frac{f''}{g''}-\frac{F''}{G''}\cdot\frac{f}{g}$ is also of the second, and consequently $Ff''-F''f$ is of the fourth (it would actually be of the fifth, were τ' to fall halfway between τ and τ''). Hence, what stands on the right hand side in the second, third, and fourth equations is of the fifth order, and of that which is on

[3] I permit myself this easily understandable expression on account of brevity. The positive pole of $\mathfrak{X}$ lies on the side of this plane where x is considered positive, etc.

the left hand side, both the first and the second part are of the third. Thus, for the first approximation, we can set

from 2) $\quad f'\delta'[\pi\pi'P'] = -f''\delta''[\pi\pi''P']$

from 4) $\quad f\delta[\pi\pi''P'] = -f'\delta'[\pi'\pi''P']$

What follows from 3), is identical with these results. To achieve a further reduction we could set $\frac{f}{g} = \frac{f'}{g'}$, both of which magnitudes differ from unity by only the second order, and differ from each other by just as much (if τ' falls halfway between τ and τ'', the difference is only of the third order). Since therefore: $g : -g' : g'' = \tau'' - \tau' : \tau'' - \tau : \tau' - \tau$, we will have

$$\delta = \frac{g}{f} \cdot \frac{f'}{g'} \cdot \frac{\tau'' - \tau}{\tau'' - \tau'} \cdot \frac{[\pi'\pi''P']}{[\pi\pi''P']} \cdot \delta' \tag{5}$$

$$\delta'' = \frac{g''}{f''} \cdot \frac{f'}{g'} \cdot \frac{\tau'' - \tau}{\tau' - \tau} \cdot \frac{[\pi\pi'P']}{[\pi\pi''P']} \cdot \delta' \tag{6}$$

These formulas give δ and δ'' from δ', up to and including the second order, if τ' lies halfway between τ and τ'', otherwise, the second order is excluded. In the latter case we can set $\frac{f'}{g'} = 1$, since the difference is only of the second order; on the other hand, in the first case it would not be unworth the effort to set

$$f' = g' + \frac{4}{3}(f + f' + f'') \text{ or } \frac{f'}{g'} = 1 + \frac{4}{3}\frac{f + f' + f''}{f'}$$

which will soon be more precisely determined and is more exact by one order. (It is easily seen that $f + f' + f''$ is equal to the triangle between the three positions p, p', p''; thus, by a known approximation $= \frac{3}{4}\times$section of the curved surface between the chord pp'' and the arc.) Moreover it follows from the above formula

$$\frac{\delta''}{\delta} = \frac{[\pi\pi'P']}{[\pi'\pi''P']} \cdot \frac{\tau'' - \tau'}{\tau' - \tau}$$

which, if $\mathfrak{Z}$ is taken for the ecliptic or B, B', $B'' = 0$, transforms itself into

$$\frac{\delta''}{\delta} = \frac{\tan\beta \sin(\lambda' - L') - \tan\beta' \sin(\lambda - L')}{\tan\beta' \sin(\lambda'' - L') - \tan\beta'' \sin(\lambda' - L')} \cdot \frac{\tau'' - \tau'}{\tau' - \tau}$$

that is, the known, Olbersian formula.[4]

6.

Now that we have derived tractable approximations from formulas 2, 3, and 4, we can undertake the first in a similar way. As is known

$$\frac{1}{r} = \frac{1}{k}(1 - e\cos(v - \pi))$$
$$\frac{1}{r'} = \frac{1}{k}(1 - e\cos(v' - \pi))$$
$$\frac{1}{r''} = \frac{1}{k}(1 - e\cos(v'' - \pi))$$

[4] Short and Easy Method, to find the Approximate Partial Determination of the Path of a Comet p.45

It follows from this, that if these are multiplied with $\sin(v''-v')$, $\sin(v-v'')$, and $\sin(v'-v)$ and added, then

$$\begin{aligned}\tfrac{f+f'+f''}{rr'r''} &= \tfrac{1}{k}(\sin(v''-v')+\sin(v-v'')+\sin(v'-v)) \\ &= -\tfrac{4}{k}\sin\tfrac{1}{2}(v''-v')\sin\tfrac{1}{2}(v-v'')\sin\tfrac{1}{2}(v'-v)\end{aligned}$$

or

$$\frac{f+f'+f''}{f'} = -\frac{2r'}{k}\cdot\frac{\sin\frac{1}{2}(v''-v')\sin\frac{1}{2}(v'-v)}{\cos\frac{1}{2}(v''-v)}$$

According to a known theorem of theoretical astronomy

$$\frac{\text{traversed space}}{\text{mean motion}} = \frac{a^{\frac{3}{2}}\cdot\sqrt{k}}{2}$$

Consequently

$$k = \frac{4gg'}{a^3(m'-m)(m''-m')} = \frac{4gg'}{A^3(M'-M)(M''-M')}$$

Therefore

$$\begin{aligned}\tfrac{f+f'+f''}{f'} &= -\tfrac{A^3(M'-M)(M''-M')}{2\cos\frac{1}{2}(v''-v)}\cdot\tfrac{1}{r'^3}\cdot\tfrac{r'r'}{rr''}\cdot\tfrac{r'r''\sin\frac{1}{2}(v''-v')}{g}\cdot\tfrac{rr'\sin\frac{1}{2}(v'-v)}{g''} \\ \tfrac{f+f'+f''}{g'} &= -\tfrac{r'r''\sin\frac{1}{2}(v''-v')}{g}\cdot\tfrac{rr''\sin\frac{1}{2}(v''-v)}{g'}\cdot\tfrac{rr'\sin\frac{1}{2}(v'-v)}{g''}\cdot\tfrac{A^3(M'-M)(M''-M')}{2r'^3}\cdot\tfrac{r'r'}{rr''}\end{aligned}$$

It is easily concluded from this, since

$$\frac{1}{\cos\frac{1}{2}(v''-v)},\ \frac{r'r''\sin\frac{1}{2}(v''-v')}{g},\ \frac{rr'\sin\frac{1}{2}(v'-v)}{g''}$$

will only fall short of unity by a magnitude of the second order, as will $\frac{r'r'}{rr''}$ if either falls τ' halfway between τ and τ'', or the difference between the path of p and a circle can be considered as being of the first order, that we may set approximately

$$\frac{f+f'+f''}{f'} = -\frac{A^3}{2r'^3}(M'-M)(M''-M')$$

In the same manner is approximately

$$\frac{F+F'+F''}{F'} = -\frac{A^3}{2R'^3}(M'-M)(M''-M')$$

If it is so desired, the latter magnitude can also be exactly calculated, since everything relevant to it is given. Both are of the second order, and are determined up to, but excluding the fourth. Thus, we have

$$\begin{aligned}F'(f+f'')-(F+F'')f' &= F'(f+f'+f'')-(F+F'+F'')f' \\ &= F'f'\tfrac{1}{2}A^3(M'-M)(M''-M')(\tfrac{1}{R'^3}-\tfrac{1}{r'^3})\end{aligned}$$

which are magnitudes of the fourth order determined up to, and excluding the sixth. In equation 1 above, the part on the left hand side is of the fifth order; in the part on the right hand side, the first term is of the sixth or seventh order, that is to say $Ff'' = F''f$ is of the fourth or fifth, and $D[\pi P\pi''] - D''[\pi P''\pi'']$ is of the second.[5] The second term is of the fifth order. We omit the former, and thereby obtain

$$\frac{[\pi\pi'\pi'']}{[\pi P'\pi'']} \cdot \frac{2}{A^3(M'-M)(M''-M')} = (\frac{1}{R'^3} - \frac{1}{r'^3})\frac{R}{\delta'} \tag{7}$$

accurate up to and excluding the second [order], if τ' falls halfway between τ and τ''; otherwise, it is incorrect only by a magnitude of the first order. This formula, which receives the following form if we take the ecliptic for $\mathfrak{Z}$, is the most important part of the entire method and is its first foundation.

$$\left\{1 - \left(\frac{R'}{r'}\right)^3\right\} \cdot \frac{R'}{\delta'} = \frac{-2}{A^3(M'-M)(M''-M')} \cdot \frac{\tan\beta' \sin(\lambda''-\lambda) - \tan\beta \sin(\lambda''-\lambda') - \tan\beta'' \sin(\lambda'-\lambda)}{\tan\beta \sin(L'-\lambda'') - \tan\beta'' \sin(L'-\lambda)}$$

where L is the longitude of the Sun $+180^o$.

Since that which stands here on the right hand side is given, one sees that from the combination of this equation with the following

$$\frac{\frac{R'}{\delta'}}{\frac{R'}{r'}} = \sqrt{(1 + \tan\beta'^2 + \frac{R'R'}{\delta'\delta'} + 2\frac{R'}{\delta'}\cos(\lambda'-L'))}$$

r' can easily be found. The indirect method is here by far the most convenient. After a few trials [*Versuchen*], for which suitable prescriptions can easily be given, the goal is very quickly arrived at. It can also always be seen whether there is more than one value for r', and thus more than one orbit that can represent the observations, which can indeed sometimes be the case.

Otherwise it is further to be remarked, that here the longitudes ought not be computed from the mobile equinoctial point, but rather from a fixed point; in practice, however, this difference is of no meaning. If the time is expressed in days, we obtain

$$\log(M'-M)(M''-M') = \log(\tau'-\tau) + \log(\tau''-\tau') + 6.4711352(-10)$$

(where, M, etc. must be expressed not in degrees, but rather in parts of the radius).

If we have δ' and r', then $\frac{f+f'+f''}{f'}$ can also be determined, and thus also δ and δ''. Moreover, from the consideration of formula 7), yet more interesting corollaries can be derived, which must be omitted here.

[5] That is to say, that this is actually subtraction, since $[\pi P\pi''],[\pi P''\pi'']$ have the same sign; this is not the case with the coefficients of $Ff'' - F''f$ in equations 2, 3, and 4, but rather the parts will there actually be added. A deeper investigation would be too lengthy here.

Second Point
Approximate Determination of the Elements
7.

We leave out entirely the middle observation for the time τ', and use instead the distances δ and δ'', which were approximately determined in the preceding point. It is clear, that from this the heliocentric longitude, latitude and distance, can thus be derived; and hence, the longitude of Ω[ascending node symbol] and the inclination of the orbit, and the longitude in the orbit. Thus the problem still remains to determine the remaining elements, namely a, e, π, and the epoch,

from the two longitudes in the orbit . . .v v''
the distances from the Sun . . .r r''
and the corresponding times . . .τ τ''

Since the relations of these magnitudes to the given ones are transcendental, we must again depend upon the indirect method. We will consider three here.

First Method.
If π is taken as given,

$$\frac{r''+r}{r''-r}\tan\frac{1}{2}(v''-v)=\tan\zeta$$

so

$$e=\frac{\cos\zeta}{\cos\frac{1}{2}(v''-v)\cos\left[\pi-\frac{1}{2}(v+v'')-\zeta\right]}$$

It is therefore most advisable to calculate k in a twofold way:

$$k=r[1-e\cos(v-\pi)]=r''[1-e\cos(v''-\pi)]$$

which will also serve to check the calculation. If $e=\sin\phi$, then $a=\frac{k}{\cos\phi^2}$. The eccentric and mean anomalies and longitude can then be calculated from the true anomaly either by the usual method or more conveniently by the indirect method; from this, and from the mean motion given by a, is obtained a twofold determination of the mean longitude for any epoch whatsoever. If both agree, the correct value for π has been found; if not, the calculation must be repeated with a somewhat modified value for π, and the true value found through interpolation. It is advisable regarding this to seek the remaining elements *not* through interpolation, but rather from a new calculation from the corrected value of π, and not stop until both values for the epoch come into complete agreement.

Second Method.
If e is taken as given.
Here the calculation is exactly the same, only the true value must be approximately known, since here π must be sought for by the equation

$$\cos\left[\pi-\frac{1}{2}(v+v'')-\zeta\right]=\frac{\cos\zeta}{e\cos\frac{1}{2}(v''-v)}$$

and two different values belong to the cosine. Moreover II is prefered to I, and in general both methods are only appropriate if the traversed arc is indeed very large, and the elements are already approximately known. For the first approximation from a short series of observations, we must always depend on the following

Third Method.

If k is taken as given.

Here,

$$\frac{\frac{1}{r''}-\frac{1}{r}}{2\sin\frac{1}{2}(v''-v)} = \frac{e}{k}\sin\left(\frac{1}{2}(v+v'')-\pi\right)$$

$$\frac{\frac{2}{k}-\frac{1}{r''}-\frac{1}{r}}{2\cos\frac{1}{2}(v''-v)} = \frac{e}{k}\cos\left(\tfrac{1}{2}(v+v'')-\pi\right)$$

Division thus gives $\tan\left[\frac{1}{2}(v+v'')-\pi\right]$, hence π, and afterwards e from one of the two equations. The rest is entirely the same as in the foregoing methods. The advantage of this third method consists in the fact that a very approximate value for k can be found directly, if the arc $v''-v$ is not too large. Namely, the sector between the two radii vectors is

$$g' = \frac{a^{\frac{3}{2}}\sqrt{k}}{2}(m''-m) = \tfrac{1}{2}A^{\frac{3}{2}}(M''-M)\sqrt{k}$$

Thus $2g' = \int rrdw$ from $w=v$ to $w=v''$.

Now, however, according to the known approximation-integration formula of Cotes, $\int \phi w\cdot dw$ from $w=v$ to $w=v''$ is

$$= (\tfrac{1}{2}\phi v + \tfrac{1}{2}\phi v'')(v''-v)$$

and still more exact

$$= (\tfrac{1}{6}\phi v + \tfrac{2}{3}\phi\tfrac{1}{2}(v+v'') + \tfrac{1}{6}\phi v'')(v''-v)$$

still more exact

$$= (\tfrac{1}{8}\phi v + \tfrac{3}{8}\phi(\tfrac{2}{3}v+\tfrac{1}{3}v'') + \tfrac{3}{8}\phi(\tfrac{1}{3}v+\tfrac{2}{3}v'') + \tfrac{1}{8}\phi v'')(v''-v) \text{ etc.}$$

It is sufficient to stick with the first two.

From the first, we thus have

$$2g' = \tfrac{1}{2}(rr+r''r'')(v''-v) \text{ and } \sqrt{k} = \frac{\frac{1}{2}(rr+r''r'')}{A^{\frac{3}{2}}}\cdot\frac{v''-v}{M''-M}$$

A is usually made $=1$; $v''-v$ and $M''-M$ are expressed in seconds, so $\log(M''-M) = \log(\tau''-\tau) + 3.5500073$. In order to simplify the calculation, we set $\frac{r'}{r} = \tan(45^o \pm \psi)$ whereby $\frac{1}{2}(rr+r''r'') = \frac{1}{\cos 2\psi}$. From the second integration formula, the radius which belongs to the longitude $\frac{1}{2}(v''+v)$ is set $=r^*$ so

$$\frac{1}{r^*} = \tfrac{1}{2}(\frac{1}{r}+\frac{1}{r''}) + \left[\tfrac{1}{2}(\frac{1}{r}+\frac{1}{r''}) - \frac{1}{k}\right]\frac{2\sin\frac{1}{4}(v''-v)^2}{\cos\frac{1}{2}(v''-v)}$$

By means of this equation, r^* can be determined by means of the first value of k. Then

$$2g' = \tfrac{1}{6}(rr + r''r'') + \tfrac{2}{3}r^*r^*$$

thus the new value of k is

$$= k\left\{1 + \frac{\frac{2}{3}(r^*r^* - \frac{1}{2}(rr + r''r''))}{rr + r''r''}\right\}^2$$

In practice, it is usually exact enough, and more convenient, to search for the logarithms of the new values of k thereby, so that the logarithm of the first will be increased by $\frac{4}{3}\log\frac{r^*r^*}{\frac{1}{2}(rr+r''r'')}$. If it is desired to use the new values of k to determine the value of $\frac{1}{r^*}$ still more exactly according to the above equation, and accordingly to correct the value of k, then the twofold determination of the epoch will almost always agree so well, that absolutely no new assumptions will be necessary. With Ceres and Pallas, since $\tau'' - \tau$ was of course 41 and 42 days, they always agreed to within a couple of hundredths of a second.

Adjustment Methods
8.

If the position for time τ' is calculated according to the approximated elements found through the foregoing method, and found to be in agreement with observation, the work is then complete. Usually the agreement will be very close (the difference often amounts to only a few seconds in my calculations) but rarely complete, partly *because it is to a degree based on only approximate assumptions, and partly because the heliocentric positions which are used in it are not elliptical but rather they include small perturbations*. The values of the small magnitudes of higher order which were discarded above could indeed be very closely determined from the approximated elements, and so the above formulas and the values of δ and δ'' could thereby be improved; but I am of the opinion, that these calculations would be far more difficult than one of the following methods.

The most simple first method of adjustment, which I first chanced upon at the inducement of Pallas, and which, because the intervals were small enough, I applied with the happiest success, is the following.

According to the approximated elements, *which were found in the above manner*, let us set the calculated position for the time τ at longitude $= \lambda' + \mathfrak{L}$ and at latitude $= \beta' + \mathfrak{B}$, since those observed are λ' and β', so that all of the small inaccuracies in the assumptions conspire such that the longitude turns out to be too large by $\mathfrak{L}$, and the latitude by $\mathfrak{B}$; thus the orbit is calculated entirely anew, and *in exactly the same way*, by using the observations

$$\lambda,\ \lambda' - L,\ \lambda''$$
$$\beta,\ \beta' - B,\ \beta'$$

as a basis. The result will be that the positions calculated from the new elements which follow from that will be so little different from λ' and β'(in my experience, only in parts of seconds) that it will require no further adjustment.

9.

Since the determination of the orbit will be based upon the *same* observations, the just indicated operation applies only in the case that it is applied to the first approximation. If it is desired afterwards to adjust the elements by means of complete [*lauter*] or partial [*zum Teil*] observations, I have, after many other tests, discovered the following two methods to be the most useful.

I. The heliocentric locations are calculated from two outer geocentric observations according to three hypotheses, by first assuming the approximate distances for these observations, and afterwards slightly adjusting first the one, then the others. The location for the middle observation is calculated according to the elements which were found in all three hypotheses, and is compared with the observed location. The corrected distances are then found through interpolation and, if desired, the corrected elements as well, though it is better not to avoid the effort of calculating these from the new distances through particular calculations, especially if the variations of the elements are still very large.

II^a. Entirely the same procedure is made use of, only with the difference that instead of the approximate distances in the outer observations, the approximate determination of the inclination and of the ascending nodes are used, and each of these is somewhat adjusted.

II^b. The heliocentric positions are calculated from the three geocentric positions, partly with approximated and partly with somewhat adjusted determination of the inclination and of the ascending node; from the two outer heliocentric positions the elements are calculated, and from these elements the middle heliocentric observation is calculated, which is then compared with the positions derived from the observed geocentric positions, and then the improved inclination and ascending node are sought through interpolation, etc.

In II^b the ellipse could also be determined from the *three heliocentric* positions according to the known formula, without simultaneously utilizing the times; the two intervals can be calculated from the dimensions of the ellipse and compared with the true, and then the corrected inclination and ascending node could be sought through interpolation, just as before. Only I had to reject this procedure in my experience. In this way an exact representation of the observations would only be attained only after repeated operations with far more trouble. To investigate the causes of this in detail would be too extensive.[6] I will merely remark that in this way the second differential, which we did away with immediately through the manipulations detailed in articles 5 and 6, is again brought about, and that these delicate second differentials can be *massively* distorted through a change in the inclination and ascending node which is itself not very large. It can easily happen here that a change of a few minutes in the ascending node or the inclination can bring about an ellipse

[6]cf. Theoria motus corporum coelest. art. 93.

which has almost no similarity at all with the previous one, hence, therefore the interpolation can understandably no longer be trusted. This is not the case with our method, which is always based on only two observations. *Sapienti sat.*– From my repeated experience, I find the first method to be the most suitable and the most general. Moreover, all of these methods are only applicable as long as the arc remains sufficiently large. If observations are taken of one or many years, other [methods] will again be necessary, upon which I am not able to expound more extensively here. In these cases it is generally not advisable to base the elements on three complete observations, but rather it is far more suitable to use four longitudes and two latitudes.– If the observations encompass still more years, and if the elements are determined *to within a small amount of error [bis auf kleinigkeiten]*, I consider the method of the differential variations [*Differential-Änderungen*] whereby an arbitrary number of observations can be used as a basis, to be the best method.